Combinatorics of singular cardinals

Mirna Džamonja

University of East Anglia, UK

April 2014

Is strange and singular the same thing?

Is strange and singular the same thing?

"Everybody knows" that the combinatorics of singular cardinals is "strange"!

Is strange and singular the same thing?

“Everybody knows” that the combinatorics of singular cardinals is “strange”!

Let us explore if we do really and if it is really strange.

Is strange and singular the same thing?

"Everybody knows" that the combinatorics of singular cardinals is "strange"!

Let us explore if we do really and if it is really strange.

We shall be interested in singular cardinals and their successors.

Is strange and singular the same thing?

"Everybody knows" that the combinatorics of singular cardinals is "strange"!

Let us explore if we do really and if it is really strange.

We shall be interested in singular cardinals and their successors. We put forward a thesis that they are in fact nicer than the successors of regulars.

Some well known properties of singular cardinals κ

Some well known properties of singular cardinals κ

Singular cardinal compactness (Shelah) : every almost-free structure on κ (e.g. group) is free.

Some well known properties of singular cardinals κ

Singular cardinal compactness (Shelah) : every almost-free structure on κ (e.g. group) is free.

Inner model theory: in the absence of large cardinals $\square_\kappa$ holds for all singular κ.

Some well known properties of singular cardinals κ

Singular cardinal compactness (Shelah) : every almost-free structure on κ (e.g. group) is free.

Inner model theory: in the absence of large cardinals $\Box_\kappa$ holds for all singular κ. Need a Woodin cardinal to kill $\Box^*_\kappa$.

Some well known properties of singular cardinals κ

Singular cardinal compactness (Shelah) : every almost-free structure on κ (e.g. group) is free.

Inner model theory: in the absence of large cardinals $\square_\kappa$ holds for all singular κ. Need a Woodin cardinal to kill $\square^*_\kappa$.

SCH: for every singular κ, if $2^{\mathrm{cf}(\kappa)} < \kappa$ then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.

Some well known properties of singular cardinals κ

Singular cardinal compactness (Shelah) : every almost-free structure on κ (e.g. group) is free.

Inner model theory: in the absence of large cardinals $\Box_\kappa$ holds for all singular κ. Need a Woodin cardinal to kill $\Box^*_\kappa$.

SCH: for every singular κ, if $2^{\mathrm{cf}(\kappa)} < \kappa$ then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.

SCH is not true in ZFC + large cardinals (Magidor),

Some well known properties of singular cardinals κ

Singular cardinal compactness (Shelah) : every almost-free structure on κ (e.g. group) is free.

Inner model theory: in the absence of large cardinals $\square_\kappa$ holds for all singular κ. Need a Woodin cardinal to kill $\square^*_\kappa$.

SCH: for every singular κ, if $2^{\mathrm{cf}(\kappa)} < \kappa$ then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.

SCH is not true in ZFC + large cardinals (Magidor), its failure is equiconsistent with $\exists\kappa o(\kappa) = \kappa^{++}$ (Gitik)

Some well known properties of singular cardinals κ

Singular cardinal compactness (Shelah) : every almost-free structure on κ (e.g. group) is free.

Inner model theory: in the absence of large cardinals $\Box_\kappa$ holds for all singular κ. Need a Woodin cardinal to kill $\Box^*_\kappa$.

SCH: for every singular κ, if $2^{\mathrm{cf}(\kappa)} < \kappa$ then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.

SCH is not true in ZFC + large cardinals (Magidor), its failure is equiconsistent with $\exists \kappa o(\kappa) = \kappa^{++}$ (Gitik) but SCH holds if there are no large cardinals (Jensen),

Some well known properties of singular cardinals κ

Singular cardinal compactness (Shelah) : every almost-free structure on κ (e.g. group) is free.

Inner model theory: in the absence of large cardinals $\square_\kappa$ holds for all singular κ. Need a Woodin cardinal to kill $\square^*_\kappa$.

SCH: for every singular κ, if $2^{\mathrm{cf}(\kappa)} < \kappa$ then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.

SCH is not true in ZFC + large cardinals (Magidor), its failure is equiconsistent with $\exists\kappa o(\kappa) = \kappa^{++}$ (Gitik) but SCH holds if there are no large cardinals (Jensen), above a supercompact (Solovay)

Some well known properties of singular cardinals κ

Singular cardinal compactness (Shelah) : every almost-free structure on κ (e.g. group) is free.

Inner model theory: in the absence of large cardinals $\square_\kappa$ holds for all singular κ. Need a Woodin cardinal to kill $\square^*_\kappa$.

SCH: for every singular κ, if $2^{\mathrm{cf}(\kappa)} < \kappa$ then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.

SCH is not true in ZFC + large cardinals (Magidor), its failure is equiconsistent with $\exists\kappa o(\kappa) = \kappa^{++}$ (Gitik) but SCH holds if there are no large cardinals (Jensen), above a supercompact (Solovay) and in models of various forcing axioms, such as MM (Foreman, Magidor, Shelah) down to MRP (Viale).

Some well known properties of singular cardinals κ

Singular cardinal compactness (Shelah) : every almost-free structure on κ (e.g. group) is free.

Inner model theory: in the absence of large cardinals $\square_\kappa$ holds for all singular κ. Need a Woodin cardinal to kill $\square^*_\kappa$.

SCH: for every singular κ, if $2^{\mathrm{cf}(\kappa)} < \kappa$ then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.

SCH is not true in ZFC + large cardinals (Magidor), its failure is equiconsistent with $\exists\kappa o(\kappa) = \kappa^{++}$ (Gitik) but SCH holds if there are no large cardinals (Jensen), above a supercompact (Solovay) and in models of various forcing axioms, such as MM (Foreman, Magidor, Shelah) down to MRP (Viale). However:

Some well known properties of singular cardinals κ

Singular cardinal compactness (Shelah) : every almost-free structure on κ (e.g. group) is free.

Inner model theory: in the absence of large cardinals $\square_\kappa$ holds for all singular κ. Need a Woodin cardinal to kill $\square^*_\kappa$.

SCH: for every singular κ, if $2^{\mathrm{cf}(\kappa)} < \kappa$ then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.

SCH is not true in ZFC + large cardinals (Magidor), its failure is equiconsistent with $\exists\kappa o(\kappa) = \kappa^{++}$ (Gitik) but SCH holds if there are no large cardinals (Jensen), above a supercompact (Solovay) and in models of various forcing axioms, such as MM (Foreman, Magidor, Shelah) down to MRP (Viale). However:

(Shelah) $(\forall n < \omega) 2^{\aleph_n} < \aleph_\omega \implies 2^{\aleph_\omega} < \aleph_{\omega_4}$.

A universality problem

A universality problem

Consider now trees of size κ without un uncountable branch, under the notion of reduction $f : T \to T'$ which preserves strict order $s <_T t \implies f(s) <_T f(t)$.

A universality problem

Consider now trees of size κ without un uncountable branch, under the notion of reduction $f : T \to T'$ which preserves strict order $s <_T t \implies f(s) <_T f(t)$.

Basically, branches go into branches, but f is not 1-1.

A universality problem

Consider now trees of size κ without un uncountable branch, under the notion of reduction $f : T \to T'$ which preserves strict order $s <_T t \implies f(s) <_T f(t)$.

Basically, branches go into branches, but f is not 1-1. The (non-existing) κ-branch is universal.

A universality problem

Consider now trees of size κ without un uncountable branch, under the notion of reduction $f: T \to T'$ which preserves strict order $s <_T t \implies f(s) <_T f(t)$.

Basically, branches go into branches, but f is not 1-1. The (non-existing) κ-branch is universal.

These trees arise naturally in the study of EF games and various logics, and provide a connection between set theory and computer sciences (see the work of Väänänen).

A universality problem

Consider now trees of size κ without un uncountable branch, under the notion of reduction $f : T \to T'$ which preserves strict order $s <_T t \implies f(s) <_T f(t)$.

Basically, branches go into branches, but f is not 1-1. The (non-existing) κ-branch is universal.

These trees arise naturally in the study of EF games and various logics, and provide a connection between set theory and computer sciences (see the work of Väänänen). The resulting structure $\mathcal{T}_\kappa$ has been studied extensively, for example at $\aleph_1$ (see e.g. Todorčević-Väänänen).

A universality problem

Consider now trees of size κ without un uncountable branch, under the notion of reduction $f: T \to T'$ which preserves strict order $s <_T t \implies f(s) <_T f(t)$.

Basically, branches go into branches, but f is not 1-1. The (non-existing) κ-branch is universal.

These trees arise naturally in the study of EF games and various logics, and provide a connection between set theory and computer sciences (see the work of Väänänen). The resulting structure $\mathcal{T}_\kappa$ has been studied extensively, for example at $\aleph_1$ (see e.g. Todorčević-Väänänen). Interestingly, using Todorčević's σ-operator, we can see that the universality number at $\aleph_1$ under GCH is the maximal possible, $2^{\aleph_1} = \aleph_2$.

A universality problem

Consider now trees of size κ without un uncountable branch, under the notion of reduction $f : T \to T'$ which preserves strict order $s <_T t \implies f(s) <_T f(t)$.

Basically, branches go into branches, but f is not 1-1. The (non-existing) κ-branch is universal.

These trees arise naturally in the study of EF games and various logics, and provide a connection between set theory and computer sciences (see the work of Väänänen). The resulting structure $\mathcal{T}_\kappa$ has been studied extensively, for example at $\aleph_1$ (see e.g. Todorčević-Väänänen). Interestingly, using Todorčević's σ-operator, we can see that the universality number at $\aleph_1$ under GCH is the maximal possible, $2^{\aleph_1} = \aleph_2$. (GCH always gives universality number 1 if the structure is first order).

A universality problem

Consider now trees of size κ without un uncountable branch, under the notion of reduction $f : T \to T'$ which preserves strict order $s <_T t \implies f(s) <_T f(t)$.

Basically, branches go into branches, but f is not 1-1. The (non-existing) κ-branch is universal.

These trees arise naturally in the study of EF games and various logics, and provide a connection between set theory and computer sciences (see the work of Väänänen). The resulting structure $\mathcal{T}_\kappa$ has been studied extensively, for example at $\aleph_1$ (see e.g. Todorčević-Väänänen). Interestingly, using Todorčević's σ-operator, we can see that the universality number at $\aleph_1$ under GCH is the maximal possible, $2^{\aleph_1} = \aleph_2$. (GCH always gives universality number 1 if the structure is first order).

Open Question Is there a model of set theory where $u_{\mathcal{T}_{\aleph_1}} = 1$?

The singulars are easy!

The singulars are easy!

Theorem (Dž. + Väänänen, Journal of Mathematical Logic 2011) Let κ be a strong limit singular of cofinality ω.

The singulars are easy!

Theorem (Dž. + Väänänen, Journal of Mathematical Logic 2011) Let κ be a strong limit singular of cofinality ω. Then $u_{\mathcal{T}_\kappa} = \kappa^+$.

The singulars are easy!

Theorem (Dž. + Väänänen, Journal of Mathematical Logic 2011) Let κ be a strong limit singular of cofinality ω. Then $u_{\mathcal{T}_\kappa} = \kappa^+$.

Just in ZFC!

The singulars are easy!

Theorem (Dž. + Väänänen, Journal of Mathematical Logic 2011) Let κ be a strong limit singular of cofinality ω. Then $u_{\mathcal{T}_\kappa} = \kappa^+$.

Just in ZFC! This is the only ZFC universality result on the uncountable that I know.

The singulars are easy!

Theorem (Dž. + Väänänen, Journal of Mathematical Logic 2011) Let κ be a strong limit singular of cofinality ω. Then $u_{\mathcal{T}_\kappa} = \kappa^+$.

Just in ZFC! This is the only ZFC universality result on the uncountable that I know.

It is sort of a combinatorial version of SCH.

κ singular, $cf(\kappa)=\omega$, $\kappa=\sup_n\kappa_n$

- κ- *Tree*: a tree T of height and cardinality κ (levels may be large)
- T is *bounded* if it has no κ branch
- A *reduction* $f:T\rightarrow T'$ is a function preserving the strict order $x<_T y \Rightarrow f(x)<_{T'} f(y)$.
- We are interested in the structure of the class of κ-Trees under the reducibility relation

In the case of regular cardinals, e.g. $\kappa=\omega_1$ structure of the corresponding class is intensively studied (e.g. S. Todorcevic and J. Väänänen, *Trees and Ehrenfeucht-Fräissé games*, Annals of Pure and Applied Logic 100 (1999), pp. 69--97).

Motivation from infinitary logics determined by a game. Trees are used as clocks in these (or other) games and if there is a reductio T to T', then it is easier for the good player to win the game determined by T than the one determined by T'. These games generalise the ordinary Ehrenfeucht-Fräissé games where the clock is ω or a well-founded tree (the latter used by Karp and also by Barwise in 1970s for back and forth sequences).

There are many ZFC results about such trees but also many independence results. In particular, the *universality number* can have various values.

Here, the universality number is the smallest size of a family $\mathcal{F}$ of κ-Trees such that every κ-Tree has a reduction into a member of $\mathcal{F}$.

In the case of κ-Trees where κ is singular of countable cofinality, we obtain a ZFC calculation of the universality number. It is κ^+. This makes these κ-Trees really look like ordinals. In fact, we introduce a notion of rank and show that within each rank, the universality number is ω.

A ZFC calculation of the universality number cannot be obtained for κ-Trees where κ is singular of uncountable cofinality, because there are results that connect this number to the universality number of λ -Trees where $\lambda=cf(\kappa)$.

rank on κ-Trees

- $\rho(t) \geq \alpha$ if for every n and $\beta < \alpha$ there is $s \geq_T t$ of height $\geq \kappa_n$ with $\rho(s) \geq \beta$.
- $\rho(t) = \alpha$ if $\rho(t) \geq \alpha$ but not $\rho(t) \geq \alpha + 1$
- $\rho(T) = \rho(t^*)$, where t^* is the root of T

Note: (1) The value of the rank does not depend on the choice of $< \kappa_n : n<\omega >$.
(2) If $T \leq T'$ then $\rho(T) \leq \rho(T')$.

We can introduce a game which can be used to prove

Theorem 1 A κ-Tree T is bounded iff $\rho(T) < \kappa^+$.

Using the notion of the rank and a certain operation on κ-Trees we can directly prove the following

Theorem 2 The universality number of the class of bounded κ-Trees is κ^+.

However:

We can calculate the universality number within each rank, and this number is **not** 1, as the analogy with the ordinals would suggest, it is ω. The theorem we obtain implies Theorem 2.

Theorem 3 (1) The universality number of κ-Trees of rank α for $0< \alpha< \kappa^+$ is ω.
(2) For every rank α there is a tree T^α of rank $\alpha+1$ satisfying $T\leq T^\alpha$ for all κ-Trees of rank α.

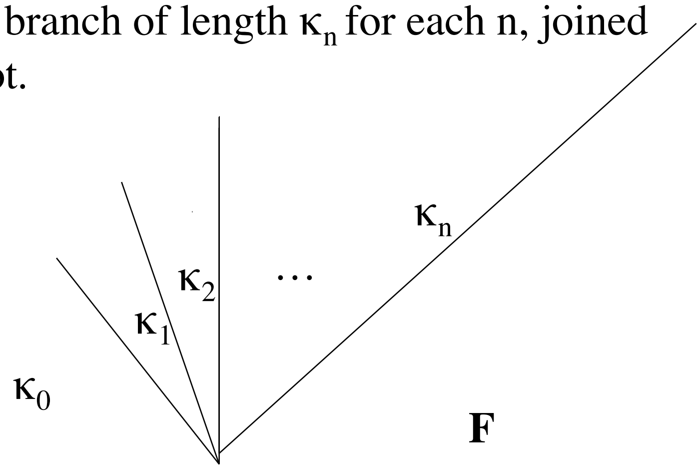
Corollary: Theorem 2.

Proof: The universal family is $\{T^{\alpha} : \alpha< \kappa^+ \}$ and this family is minimal because if $\alpha<\beta$ then T^β cannot embed into T^{α}.

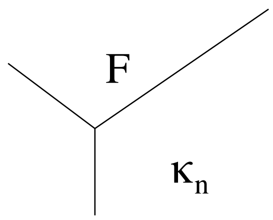
About Proof of Theorem 3

We illustrate the proof of Theorem 3 by concentrating on the case of rank=1. A typical tree of rank 1 is the **fan F:** κ_0

it consists of a branch of length κ_n for each n, joined by a common root.



For each n let F_n consist of a stem of length κ_n topped up with a copy of F. If n<m then there is no reduction from F_m to F_n, because the reduction would have to be to a branch of F_n.



If a tree T has rank 1, then there is n such that no point of height more than κ_n has rank 1, so we can map T into F_n.

Other cofinalities

Clearly, one cannot hope to get a rank with trees that have nothing to do with well-foundedness, for example for trees of singular cardinality κ whose cofinality is uncountable. We still may ask if it the analogue of Theorem 2 (the universality number for bounded κ-Trees is κ^+).

The answer is negative. Namely, suppose e.g. $cf(\kappa)=\omega_1$. Then we can to each bounded κ-Tree T associate its 'small twin' tw(T) so that tw(T) is an ω_1 -Tree with no uncountable branch, and if $T \leq T'$ then $tw(T) \leq tw(T')$. It is consistent that the universality number for bounded ω_1 -Trees is as large as desired (one can use a GMA).

More on ω_1

Mekler and Väänänen (1993) showed that it is consistent that the universality number of bounded ω_1-Trees is ω_2 while $2^{\omega_1} > \omega_2$. (Preliminary work with Katherine Thompson indicates that is also consistent to have one universal bounded ω_1-Tree.)

On the other hand, it is known that the universality number of bounded ω_1-Trees can be 2^{ω_1} which can be as large as desired.

Therefore, using the twinning operator, the universality number of κ-Trees for κ singular of cofinality ω_1 cannot have a ZFC value κ^+.

Boundedness theorems

The classical boundedness theorem in descriptive set theory is that, denoting by WO a complete Π^1_1 set, then for every Σ^1_1 subset A of WO we have that the sup of ordinals coded by is $< \omega_1$. Mekler and Väänänen (1993) gave a similar theorem for Σ^1_1-subsets of the topological space ${}^{\omega_1}\omega_1$, under CH. The topology here is generated by

$N(f,\xi)=\{g: g|\xi=f|\xi\}$, $f: \omega_1 \to \omega_1$, $\xi< \omega_1$

and the set WO is replaced by the set TO which is a universal Π^1_1 set of codes for ω_1-Trees with no uncountable branch.

Theorem (Mekler and Väänänen)(CH) If $A\subseteq TO$ is Σ^1_1 then there is T in TO with $t\leq T$ for all t in A.

(2) (Dz. and Väänänen) Analog for κ strong limit, $cf(\kappa)=\omega$.

Universality of graphs

Universality of graphs

Let κ be a cardinal $\geq \aleph_1$.

Universality of graphs

Let κ be a cardinal $\geq \aleph_1$. Consider the embeddings $f: G \to H$ between graphs on κ which preserve the edge and the non-edge relation and say that $G \leq H$ if there is such an embedding.

Universality of graphs

Let κ be a cardinal $\geq \aleph_1$. Consider the embeddings $f : G \to H$ between graphs on κ which preserve the edge and the non-edge relation and say that $G \leq H$ if there is such an embedding. We are interested in the smallest size of a dominating family in the resulting structure, call this u_κ.

Universality of graphs

Let κ be a cardinal $\geq \aleph_1$. Consider the embeddings $f: G \to H$ between graphs on κ which preserve the edge and the non-edge relation and say that $G \leq H$ if there is such an embedding. We are interested in the smallest size of a dominating family in the resulting structure, call this u_κ.

If GCH holds then $u_\kappa = 1$ for all κ.

Universality of graphs

Let κ be a cardinal $\geq \aleph_1$. Consider the embeddings $f : G \rightarrow H$ between graphs on κ which preserve the edge and the non-edge relation and say that $G \leq H$ if there is such an embedding. We are interested in the smallest size of a dominating family in the resulting structure, call this u_κ.

If GCH holds then $u_\kappa = 1$ for all κ.

For κ the successor of a regular Cohen forcing gives the consistency of $u_\kappa = 2^\kappa > \kappa^+$.

Universality of graphs

Let κ be a cardinal $\geq \aleph_1$. Consider the embeddings $f: G \to H$ between graphs on κ which preserve the edge and the non-edge relation and say that $G \leq H$ if there is such an embedding. We are interested in the smallest size of a dominating family in the resulting structure, call this u_κ.

If GCH holds then $u_\kappa = 1$ for all κ.

For κ the successor of a regular Cohen forcing gives the consistency of $u_\kappa = 2^\kappa > \kappa^+$.

For κ the successor of a regular it is consistent to have $u_\kappa < 2^\kappa > \kappa^+$ (Mekler, Shelah for $\aleph_1$, Dž. + Shelah in general).

Successors of singulars

Successors of singulars

The following was obtained by Dž. and Shelah for $\lambda = \aleph_0$ (2005)

Successors of singulars

The following was obtained by Dž. and Shelah for $\lambda = \aleph_0$ (2005) and by Cummings, Dž., Magidor, Morgan and Shelah (recent) in general:

Successors of singulars

The following was obtained by Dž. and Shelah for $\lambda = \aleph_0$ (2005) and by Cummings, Dž., Magidor, Morgan and Shelah (recent) in general:

Theorem If κ is a supercompact cardinal, $\lambda < \kappa$ is a regular cardinal and $\theta \geq \kappa^{+3}$ is a cardinal with $\mathrm{cf}(\theta) \geq \kappa^{++}$,

Successors of singulars

The following was obtained by Dž. and Shelah for $\lambda = \aleph_0$ (2005) and by Cummings, Dž., Magidor, Morgan and Shelah (recent) in general:

Theorem If κ is a supercompact cardinal, $\lambda < \kappa$ is a regular cardinal and $\theta \geq \kappa^{+3}$ is a cardinal with $\mathrm{cf}(\theta) \geq \kappa^{++}$, then there is a cardinal preserving forcing extension in which $\mathrm{cf}(\kappa) = \lambda$, $2^\kappa = 2^{\kappa^+} = \theta \geq \kappa^{+3}$ and $u_{\kappa^+} \leq \kappa^{+2}$.

Some ideas of the proof

Some ideas of the proof

Iterate a forcing which blows up the power of κ, builds the future universal graphs and controls the names in Radin forcing of graphs on κ^+.

Some ideas of the proof

Iterate a forcing which blows up the power of κ, builds the future universal graphs and controls the names in Radin forcing of graphs on κ^+. Radin forcing with respect to what, subsets of κ are being added all the time?

Some ideas of the proof

Iterate a forcing which blows up the power of κ, builds the future universal graphs and controls the names in Radin forcing of graphs on κ^+. Radin forcing with respect to what, subsets of κ are being added all the time? Well, a measure sequence is being constructed as we go.

Some ideas of the proof

Iterate a forcing which blows up the power of κ, builds the future universal graphs and controls the names in Radin forcing of graphs on κ^+. Radin forcing with respect to what, subsets of κ are being added all the time? Well, a measure sequence is being constructed as we go. The universal family is obtained using a cofinal sequence in λ.

For the rest of this section we fix an ultrafilter sequence w with $\mathrm{lh}(w) > 1$, and write κ for κ_w.

Definition 4.1. $M(w)$ is the forcing with conditions $p = (A, B)$, with:

- $A = \langle A_\rho \,|\, \rho < \rho^p \rangle$, where $\rho^p < \kappa$,
- $\forall \rho < \rho^p \,(A_\rho \subseteq \mathcal{U}_\kappa \;\&\; A_\rho \neq \emptyset \;\&\; \exists \kappa_\rho < \kappa \,\forall u \in A_\rho \; \kappa_u = \kappa_\rho)$,
- $\forall \rho < \rho^p \; \forall u \in A_\rho \, \forall \tau \in (0, \mathrm{lh}(u)) \; u \restriction \tau \in A_\rho$,
- $\langle \kappa_\rho \,|\, \rho < \rho^p \rangle$ is strictly increasing,
- $B \in \mathcal{F}(w)$,
- $\forall v \in B \, \forall \tau \in (0, \mathrm{lh}(v)) \; (v \restriction \tau \in B)$, and
- $\mathrm{ssup}(\{\kappa_\rho \,|\, \rho < \rho^p\}) \leq \min(\{\kappa_v \,|\, v \in B\})$.

Setting 4.4. Suppose $\mathcal{T}$ is a binary κ^+-tree with Υ many branches.

Definition 4.5. Let $\langle x_\alpha \,|\, \alpha < \Upsilon \rangle$ be an enumeration of a set of branches through $\mathcal{T}$. Let $\langle \dot{E}_\alpha \,|\, \alpha < \Upsilon \rangle$ be a list of canonical $\mathbb{R}_w$-names for binary relations on κ^+. We will use the sequences $\langle x_\alpha \,|\, \alpha < \Upsilon \rangle$ and $\langle \dot{E}_\alpha \,|\, \alpha < \Upsilon \rangle$ as parameters in the definition of the forcing $Q(w)$.

$Q^*(w)$ is the forcing with conditions $p = (A, B, t, f)$ satisfying the following four clauses.

(1) $(A, B) \in M(w)$ (see Definition (4.1)). We set $a = a^{(A,B)}$.

(2) $t \in [(a \cap \sup(a)) \times \Upsilon]^{<\kappa}$ and $f = \langle f^\eta_\alpha \,|\, (\eta, \alpha) \in t \rangle$. For $\eta \in a \cap \sup(a)$, set $t^\eta = \{\alpha \,|\, (\eta, \alpha) \in t\}$.

(3) $\forall \eta \in a \cap \sup(a)\ \forall \alpha \in t^\eta\ d^\eta_\alpha = \mathrm{dom}(f^\eta_\alpha) \in [\kappa^+]^{<\kappa}$.

(4) $\forall \eta \in a \cap \sup(a)\ \forall \alpha \in t^\eta\ \forall \zeta \in d^\eta_\alpha\ \exists \nu < \kappa\ \ f^\eta_\alpha(\zeta) = (x_\alpha \restriction \zeta, \nu)$.

If p, $q \in Q^*(w)$ then $q \leq p$ if $[A^q, B^q] \subseteq [A^p, B^p]$, $t^p \subseteq t^q$ and $\forall (\eta, \alpha) \in t^p\ (f^\eta_\alpha)^p \subseteq (f^\eta_\alpha)^q$; and $q \leq^* p$ if $q \leq p$ and $A^q = A^p$, $t^q = t^p$ and $f^q = f^p$. (If $q \leq^* p$ we say q is a *direct extension* of p.)

We write $Q(w)$ for the suborder of $Q^*(w)$ consisting of conditions which also satisfy:

(5) for all $\eta \in a \cap \sup(a)$, for all α, $\beta \in t^\eta$, for every lower part y for $\mathbb{R}_w$ harmonious with A past η, and for all ζ, $\zeta' \in d^\eta_\alpha \cap d^\eta_\beta$ we have:

$$f^\eta_\alpha(\zeta) = f^\eta_\beta(\zeta) \neq f^\eta_\alpha(\zeta') = f^\eta_\beta(\zeta') \implies$$
$$y^\frown(w, B) \Vdash_{\mathbb{R}_w} \text{“}\,\zeta \dot{E}_\alpha \zeta' \longleftrightarrow \zeta \dot{E}_\beta \zeta'\,\text{”}.$$

With Cummings and Morgan we have recent work where κ can be made to be $\aleph_\omega$.

With Cummings and Morgan we have recent work where κ can be made to be $\aleph_\omega$. Cummings and his student Jacob Davis are working on the case $\aleph_{\omega_1}$.

With Cummings and Morgan we have recent work where κ can be made to be $\aleph_\omega$. Cummings and his student Jacob Davis are working on the case $\aleph_{\omega_1}$.

The point is: *For all we know* u_{κ^+} might be κ^+ (so 1) in our model (and these other models)!

Question Can we have κ singular and $u_{\kappa^+} = 2^{\kappa^+} > \kappa^{++}$?

Question Can we have κ singular and $\mathfrak{u}_{\kappa^+} = 2^{\kappa^+} > \kappa^{++}$?

The good old proof with the Cohen reals does not seem to generalize in any sense!

Question Can we have κ singular and $u_{\kappa^+} = 2^{\kappa^+} > \kappa^{++}$?

The good old proof with the Cohen reals does not seem to generalize in any sense!

For all we know u_{κ^+} might be κ^+ (so 1) in *every* model, i.e. in ZFC!

Cardinal invariants at singulars and their successors

Cardinal invariants at singulars and their successors

The two examples presented show that the cardinal invariants at a singular and its successor are genuinely different than what we know and, I think, that they should be studied systematically.

Cardinal invariants at singulars and their successors

The two examples presented show that the cardinal invariants at a singular and its successor are genuinely different than what we know and, I think, that they should be studied systematically. Not everything can be a ZFC result, of course we know that SCH can fail (Magidor), and GCH can fail everywhere (Foreman and Woodin).

Cardinal invariants at singulars and their successors

The two examples presented show that the cardinal invariants at a singular and its successor are genuinely different than what we know and, I think, that they should be studied systematically. Not everything can be a ZFC result, of course we know that SCH can fail (Magidor), and GCH can fail everywhere (Foreman and Woodin). There are combinatorial results which show that some cardinal invariants can be made as high as possible,

Cardinal invariants at singulars and their successors

The two examples presented show that the cardinal invariants at a singular and its successor are genuinely different than what we know and, I think, that they should be studied systematically. Not everything can be a ZFC result, of course we know that SCH can fail (Magidor), and GCH can fail everywhere (Foreman and Woodin). There are combinatorial results which show that some cardinal invariants can be made as high as possible, for example Cummings and Shelah (1995) show that it is consistent modulo l.c. to have that every infinite Boolean algebra $\mathfrak{B}$ has $2^{|\mathfrak{B}|}$ subalgebras.

Cardinal invariants at singulars and their successors

The two examples presented show that the cardinal invariants at a singular and its successor are genuinely different than what we know and, I think, that they should be studied systematically. Not everything can be a ZFC result, of course we know that SCH can fail (Magidor), and GCH can fail everywhere (Foreman and Woodin). There are combinatorial results which show that some cardinal invariants can be made as high as possible, for example Cummings and Shelah (1995) show that it is consistent modulo l.c. to have that every infinite Boolean algebra $\mathfrak{B}$ has $2^{|\mathfrak{B}|}$ subalgebras.

We need a systematic study, involving also development - if possible of forcing axioms.

A Question suggested by Stevo

A Question suggested by Stevo

Consider the universality of graphs on κ with no complete subgraph of size κ,

A Question suggested by Stevo

Consider the universality of graphs on κ with no complete subgraph of size κ, as a generalization of the problem of tree reductions.

A Question suggested by Stevo

Consider the universality of graphs on κ with no complete subgraph of size κ, as a generalization of the problem of tree reductions.

This is a very general problem and has to do with the so called Rado conjecture (there one restricts to intersection graphs of linear orders).

Foundational and philosophical remarks

Foundational and philosophical remarks

The fact that ZFC determines to some extent the combinatorics at the singulars and their successors has a philosophical significance.

Foundational and philosophical remarks

The fact that ZFC determines to some extent the combinatorics at the singulars and their successors has a philosophical significance. If one is a platonist then the fact that we have independence in set theory and elsewhere speaks just of our inability to model the true universe by our methods.

Foundational and philosophical remarks

The fact that ZFC determines to some extent the combinatorics at the singulars and their successors has a philosophical significance. If one is a platonist then the fact that we have independence in set theory and elsewhere speaks just of our inability to model the true universe by our methods. The combinatorics at the singulars shows that to some extent we catch our tail at singular cardinals.

Foundational and philosophical remarks

The fact that ZFC determines to some extent the combinatorics at the singulars and their successors has a philosophical significance. If one is a platonist then the fact that we have independence in set theory and elsewhere speaks just of our inability to model the true universe by our methods. The combinatorics at the singulars shows that to some extent we catch our tail at singular cardinals. ZFC is capable of telling us the truth *asymptotically*.

Foundational and philosophical remarks

The fact that ZFC determines to some extent the combinatorics at the singulars and their successors has a philosophical significance. If one is a platonist then the fact that we have independence in set theory and elsewhere speaks just of our inability to model the true universe by our methods. The combinatorics at the singulars shows that to some extent we catch our tail at singular cardinals. ZFC is capable of telling us the truth *asymptotically*.

This is very pleasing and stands as a good answer, at least to me, to “what is the relevance of set theory in mathematics?

Foundational and philosophical remarks

The fact that ZFC determines to some extent the combinatorics at the singulars and their successors has a philosophical significance. If one is a platonist then the fact that we have independence in set theory and elsewhere speaks just of our inability to model the true universe by our methods. The combinatorics at the singulars shows that to some extent we catch our tail at singular cardinals. ZFC is capable of telling us the truth *asymptotically*.

This is very pleasing and stands as a good answer, at least to me, to "what is the relevance of set theory in mathematics? Why work in ZFC and not in some other system".