Large cardinals: the logic vs the set-theoretic perspective

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Large cardinals as reflection principles

... strong reflection principles, also known as large cardinal axioms [...] assert that certain properties of the universe V of all sets are shared by, or "reflect to", initial segments V_α of the cumulative hierarchy of sets.[1]

[1] Martin, D. A., and Steel, J. R., A proof of projective determinacy. Journal of the AMS 2 (1), 71 – 125 (1989).

Large cardinals as reflection principles

Examples

1. κ is strongly-inaccessible iff κ is regular and $V_\kappa \preceq_1 V$.
2. κ is strongly-inaccessible iff for every $A \subseteq V_\kappa$ there is $\alpha < \kappa$ such that
$$\langle V_\alpha, \in, A \cap V_\alpha \rangle \preceq \langle V_\kappa, \in, A \rangle.$$
3. κ is Mahlo iff for every $A \subseteq V_\kappa$ there is $\alpha < \kappa$ regular such that
$$\langle V_\alpha, \in, A \cap V_\alpha \rangle \preceq \langle V_\kappa, \in, A \rangle.$$
4. κ is weakly-compact iff for every $A \subseteq V_\kappa$ and every Π^1_1 sentence φ in the language of set theory with one additional predicate symbol, if $\langle V_\kappa, \in, A \rangle \models \varphi$, then there is $\alpha < \kappa$ such that $\langle V_\alpha, \in, A \cap V_\alpha \rangle \models \varphi$.

Structural Reflection

> *Structural reflection (SR): For every definable (in the first-order language of set theory, with parameters) class of structures $\mathcal{C}$ of the same type, there exists α that reflects $\mathcal{C}$, i.e., for every A in $\mathcal{C}$ there exists B in $\mathcal{C} \cap V_\alpha$ and an elementary embedding from B into A.*

The SR principle can be formulated in the first-order language of set theory as an axiom schema, to wit, for each natural number n let

> *Σ_n-Structural Reflection (Σ_n-SR): For every Σ_n-definable, with parameters, class $\mathcal{C}$ of structures of the same type, there exists an ordinal α that reflects $\mathcal{C}$.*

Π_n-SR is defined analogously.

Structural Reflection

The first observation is that Σ_1-SR is provable in ZFC.

Proposition

Σ_1-SR holds. In fact, every uncountable cardinal κ with $V_\kappa = H_\kappa$ and such that V_κ contains the parameters of some Σ_1 definition of a given class $\mathcal{C}$ of structures reflects $\mathcal{C}$.

But Π_1-SR is already very strong.

Structural Reflection

Theorem (Magidor, 1970)

If κ is the least cardinal that reflects the Π_1 definable proper class $\mathcal{C}$ of structures of the form $\langle V_\lambda, \in\rangle$, then κ is supercompact.

The converse is also true, in a strong sense. Namely, if κ is supercompact, then κ reflects $\mathcal{C}$. In fact, κ reflects all classes of structures of the same type that are Π_1 definable with parameters in V_κ.

Structural Reflection

More generally, we have the following equivalences:[2]

Theorem

The following are equivalent:

1. Π_1-*SR.*
2. *There exists a proper class of supercompact cardinals.*

A cardinal κ witnesses Π_1-SR if and only if either κ is supercompact or a limit of supercompacts.

[2] Bagaria, J., Casacuberta, C., Mathias, A. R. D., and Rosicky, J., *Definable orthogonality classes in accessible categories are small*. To appear in the Journal of the EMS.
Bagaria, J., $C^{(n)}$-*Cardinals.* Archive for Mathematical Logic (2012).

Structural Reflection

For the next level of complexity we have the following:

Theorem

The following are equivalent:

1. Π_2-*SR.*
2. *There exists a proper class of extendible cardinals.*

Structural Reflection

For the higher levels of complexity we need the notion of $C^{(n)}$-extendible cardinal:

Definition

κ is $C^{(n)}$-extendible if for every λ greater than κ there exists an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with $crit(j) = \kappa$, $j(\kappa) > \lambda$, and $V_{j(\kappa)}$ is a Σ_n-elementary substructure of V.

Notice that κ is extendible if and only if it is $C^{(1)}$-extendible.

Structural Reflection

Theorem

The following are equivalent for $n \geq 1$:

1. Π_{n+1}-*SR.*
2. *There exists a proper class of $C^{(n)}$-extendible cardinals.*

Corollary

The following are equivalent:

1. *SR, i.e., Π_n-SR for all n.*
2. *There exists a $C^{(n)}$-extendible cardinal, for every n.*
3. *Vopeňka's Principle.*

Reflecting the structure of the V_α's

For each n, let Π_n-SR$_0$ be like Π_n-SR, but restricted to Π_n definable classes of structures of the form $\langle V_\alpha, \in, A\rangle$, where A is a predicate.

Theorem

1. Π_1-*SR*$_0$ *is equivalent to the existence of a proper class of supercompact cardinals.*
2. Π_2-*SR*$_0$ *is equivalent to the existence of a proper class of extendible cardinals.*
3. Π_{n+1}-*SR*$_0$ *is equivalent to the existence of a proper class of $C^{(n)}$-extendible cardinals.*
4. Π_n-*SR*$_0$ *for all n is equivalent to Vopeňka's Principle.*

Main question

Question

Can large cardinals below supercompact (e.g., measurable, strong, Woodin, etc.) be characterized in terms of structural reflection, or some other natural from of reflection?

Structural reflection and small large cardinals

Consider the principle of structural reflection restricted to particular definable classes of structures.

> *Structural Reflection for $\mathcal{C}$ (SR($\mathcal{C}$)): There exists an ordinal α that reflects $\mathcal{C}$. i.e., for every A in $\mathcal{C}$ there exists B in $\mathcal{C} \cap V_\alpha$ and an elementary embedding $e : B \preceq A$.*

Structural reflection for L and $0^\sharp$

Let $\mathcal{C}$ be the class of structures of the form $\langle L_\beta, \in, \gamma \rangle$, where γ and β are cardinals (in V) and $\gamma < \beta$. The class $\mathcal{C}$ is Π_1 definable (without parameters).

Theorem

1. *$SR(\mathcal{C})$ if and only if $0^\sharp$ exists.*
2. *$0^\sharp$ exists implies $SR(\mathcal{D})$, for all classes $\mathcal{D}$ of structures of the same type that are definable in L.*

Given a set of ordinals X, let $\mathcal{C}_X$ be the class of structures of the form $\langle L_\beta[X], \in, \gamma\rangle$, where γ and β are cardinals and $sup(X) < \gamma < \beta$. Clearly, $\mathcal{C}$ is Π_1 definable with X as a parameter.

Theorem

1. *$SR(\mathcal{C}_X)$ if and only if $X^\sharp$ exists.*
2. *$X^\sharp$ exists implies $SR(\mathcal{D})$, for all classes $\mathcal{D}$ of structures of the same type that are definable in $L[X]$.*

For M an inner model, let $\Pi_1(M)$-SR be the same as the principle Π_1-SR, but restricted to Π_1 classes of structures $\mathcal{C} \subseteq M$.

Corollary

1. *$\Pi_1(L)$-SR if and only if $0^\sharp$ exists.*
2. *$\Pi_1(L(A))$-SR if and only if $A^\sharp$ exists, for every set A.*

$$\frac{\mathbf{0}^\sharp}{\mathbf{L}} = \frac{\mathbf{A}^\sharp}{\mathbf{L(A)}} = \frac{\textbf{Supercompact}}{\mathbf{V}}$$

The following is a joint work with **Jouko Väänänen.**

Let $\mathcal{R}$ be a set of Π_1 predicates or relations. A class $\mathcal{K}$ of models (in a fixed vocabulary) is $\Sigma_1(\mathcal{R})$ if it is definable by means of an existential formula of the first-order language of set theory, with additional predicates from $\mathcal{R}$.

Consider the following kind of principles, for $\mathcal{R}$ a set of Π_1 predicates or relations, and κ an infinite cardinal.

$(1)_{\mathcal{R}}$: If $\mathcal{K}$ is a $\Sigma_1(\mathcal{R})$ class of models of the same (countable) type, then for every $\mathcal{A} \in \mathcal{K}$, there exist $\mathcal{B} \in \mathcal{K}$ of cardinality less than κ and $e : \mathcal{B} \preceq \mathcal{A}$.

Note that if $(1)_{\mathcal{R}}$ holds for κ, then it also holds for any cardinal greater than κ. We write $(1)_{\mathcal{R}} = \kappa$ to indicate that κ is the least cardinal for which $(1)_{\mathcal{R}}$ holds.

We have that $(1)_\emptyset = \aleph_1$.

And, as we have already seen, if R is the Π_1 relation "x is an ordinal and $y = V_x$", then $(1)_R = \kappa$ if and only if κ is the first supercompact cardinal. Moreover, if κ is supercompact, then $(1)_\mathcal{R}$ holds for κ, for any set $\mathcal{R}$ of Π_1 predicates.

Let Cd be the Π_1 predicate "x is a cardinal".

Proposition

If κ satisfies $(1)_{Cd}$, then there exists a weakly inaccessible cardinal $\lambda \leq \kappa$.

Magidor-Väänänen[3] show that, starting form a supercompact cardinal, it is consistent that $(1)_{Cd}$ holds for the first weakly inaccessible cardinal. So, we cannot prove in ZFC that more large-cardinal properties hold for some cardinal $\leq \kappa$ just by assuming that $(1)_{Cd}$ holds at κ.

[3] Magidor, M. and Väänänen, J. A. (2011) On Löwenheim-Skolem-Tarski numbers for extensions of first order logic. *Journal of Mathematical Logic*, Vol. 11, No. 1, 87–113.

Let *Rg* be the predicate "*x* is a regular ordinal".

Proposition

If κ satisfies $(1)_{Rg}$, then there exists a weakly Mahlo cardinal $\lambda \leq \kappa$.

Magidor-Väänänen[4] show that we cannot hope to get from $(1)_{Rg}$ more than one weakly Mahlo cardinal $\leq \kappa$. Moreover, we cannot hope either to obtain from $(1)_{Rg}$ that κ is strongly inaccessible, for one can have $(1)_{Rg}$ for $\kappa = 2^{\aleph_0}$.

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[4] Magidor, M. and Väänänen, J. A. (2011) On Löwenheim-Skolem-Tarski numbers for extensions of first order logic. *Journal of Mathematical Logic*, Vol. 11, No. 1, 87–113.

There is a condition between $(1)_{Cd}$ and $(1)_{Rg}$, namely $(1)_{Cd,WI}$, where WI is the Π_1 predicate "x is weakly inaccessible".

Proposition

If κ satisfies $(1)_{Cd,WI}$, then there exists a 2-weakly inaccessible cardinal $\lambda \leq \kappa$.

Let $WC(x, \alpha)$ be the Π_1 relation "x is a partial ordering with no chain of length α".

Proposition

If κ satisfies $(1)_{Cd,WC}$, then there exists a weakly compact cardinal $\lambda \leq \kappa$.

Since the first weakly Mahlo cardinal can satisfy $(1)_{Rg}$ (Magidor-Väänänen), we cannot get a weakly compact cardinal $\leq \kappa$ just from $(1)_{Rg}$. Hence, $(1)_{Cd,WC}$ is stronger than $(1)_{Rg}$.

The model-theoretic approach

(Reformulating $(1)_{\mathcal{R}}$ in model-theoretic terms.)

Suppose $\mathcal{L}^*$ is a logic. E.g.,

- First-order logic ($\mathcal{L}_{\omega,\omega}$).
- Infinitary logic ($\mathcal{L}_{\kappa,\lambda}$).
- Higher-order logic ($\mathcal{L}^n$, $n \geq 2$).

possibly extended with generalized quantifiers. In all cases of logics under consideration, isomorphism of models implies $\mathcal{L}^*$-equivalence.

A model class $\mathcal{K}$ (i.e., a class of models in some fixed vocabulary) is said to be $\mathcal{L}^*$-definable if there is a sentence $\varphi \in \mathcal{L}^*$ such that $\mathcal{K} = Mod(\varphi)$, i.e., $\mathcal{K} = \{\mathcal{A} : \mathcal{A} \models \varphi\}$.

Sometimes, for some logic $\mathcal{L}^*$, a model class is a projection of an $\mathcal{L}^*$-definable model class, and at the same time the complement of the model class is also a projection of an $\mathcal{L}^*$-definable model class. Then we say that the model class is $\Delta(\mathcal{L}^*)$-definable.

A paradigm example

The model class $\mathcal{W}$ of structures $(M, <)$, where $<$ well-orders M is $\Delta(\mathcal{L}_{\omega,\omega}(I))$-definable, where I is the Härtig quantifier, given by

$$Ixy\varphi(x)\psi(y) \leftrightarrow |\varphi(\cdot)| = |\psi(\cdot)|.$$

Why?: First, consider the model class $\mathcal{K}_0$ of models $(M, <, X)$, where $<$ is a linear ordering and X is a subset of M that has no $<$-least element (a first-order property). The projection $\mathcal{K}_0 \restriction \{<\}$ is the class of non well-ordered structures.

Then, to represent the class of well-ordered structures as the projection of a model class that is $\mathcal{L}_{\omega,\omega}(I)$-definable, use that a linear order $(M, <)$ is a well-order if and only if there are sets A_a, for $a \in M$, such that $a <_M b$ if and only if $|A_a| < |A_b|$.

A paradigm example

So let $\mathcal{K}_1$ be the class of structures $(A, M, <, R)$ such that:

1. $M \subseteq A$
2. $(M, <)$ is a linear order,
3. $R \subseteq M \times A$
4. $a <_M b$ implies $|R(a, \cdot)| < |R(b, \cdot)|$

So, the class of well-ordered models is the projection $\mathcal{K}_1 \restriction \{<\}$. As a result, both the class $\mathcal{W}$ of well-ordered $(M, <)$ and the class of non well-ordered $(M, <)$ are projections of $\mathcal{L}_{\omega,\omega}(I)$-definable model classes, i.e., $\mathcal{W}$ is $\Delta(\mathcal{L}_{\omega,\omega}(I))$-definable.

Given a definable n-ary relation R, let

$$Q_R := \{\mathcal{A} : \mathcal{A} \cong (M, \in, \bar{a}_1, \ldots, \bar{a}_n), M \text{ transitive, and } R(\bar{a}_1, \ldots, \bar{a}_n)\}.$$

The class Q_R yields a generalized quantifier (in the sense of Mostowski-Lindström). Namely,

$$\mathcal{A} \models Q_R uvx_1 \ldots x_n(uEv)(x_1 = a_1) \ldots (x_n = a_n)$$

if and only if

$$\langle A, E^{\mathcal{A}}, a_1, \ldots, a_n \rangle \in Q_R.$$

Proposition

Suppose R is an n-ary relation. The following are equivalent for any logic $\mathcal{L}^*$ that contains the first-order language of set theory with an additional n-ary relation symbol and a constant symbol:

1. *Every $\Delta_1(R)$-definable model class that is closed under isomorphisms is $\Delta(\mathcal{L}^*)$-definable.*
2. *The model class Q_R is $\Delta(\mathcal{L}^*)$-definable.*

The following notion of symbiosis, between a logic and a predicate of set theory, is due to Väänänen[5]

Definition

A (finite set of) *n*-ary relation(s) $\mathcal{R}$ and a logic $\mathcal{L}^*$ are symbiotic if the following conditions are satisfied:

1. Every $\mathcal{L}^*$-definable model class is $\Delta_1(\mathcal{R})$-definable.
2. Every $\Delta_1(\mathcal{R})$-definable model class closed under isomorphisms is $\Delta(\mathcal{L}^*)$-definable.

By the proposition above, in the case $\mathcal{R} = \{R\}$, and for a suitable $\mathcal{L}^*$, (2) may be replaced by: The model class Q_R is $\Delta(\mathcal{L}^*)$-definable.

[5] J. Väänänen, Applications of set theory to generalized quantifiers, Ph. D. Thesis, University of Manchester, 1977.
J. Väänänen, Abstract Logic and Set Theory, I. Definability. In Logic Colloquium 78. M. Boffa, D. van Dalen, and K. McAloon (Eds.). North.Holland, 1979.

The following are examples of symbiosis:

Examples

1. R: Cd
   $\mathcal{L}^*$: $\mathcal{L}_{\omega,\omega}(I)$, where $Ixy\varphi(x)\psi(y) \leftrightarrow |\varphi| = |\psi|$ is the Härtig quantifier.

2. R: Cd
   $\mathcal{L}^*$: $\mathcal{L}_{\omega,\omega}(W^{Cd})$, where $W^{Cd}xy\varphi(x,y) \leftrightarrow \varphi(\cdot,\cdot)$ is a well-ordering of the order-type of a cardinal.

3. R: Cd, WI
   $\mathcal{L}^*$: $\mathcal{L}_{\omega,\omega}(I, W^{WI})$, where $W^{WI}x\varphi(x) \leftrightarrow |\varphi(\cdot)|$ is weakly-inaccessible.

Examples

1. R: Rg
   $\mathcal{L}^*$: $\mathcal{L}_{\omega,\omega}(I, W^{Rg})$, where $W^{Rg}xy\varphi(x,y) \leftrightarrow \varphi(\cdot,\cdot)$ has the order-type of a regular cardinal.
2. R: Cd, WC
   $\mathcal{L}^*$: $\mathcal{L}_{\omega,\omega}(I, Q_{Br})$, where $Q_{Br}xy\varphi(x,y) \leftrightarrow \varphi(\cdot,\cdot)$ is a tree order of height some α and no branch of length α.
3. R: Cd, WC
   $\mathcal{L}^*$: $\mathcal{L}_{\omega,\omega}(I, \bar{Q}_{Br})$, where $\bar{Q}_{Br}xyuv\varphi(x,y)\psi(u,v) \leftrightarrow \varphi(\cdot,\cdot)$ is a partial order with a chain of order-type $\psi(\cdot,\cdot)$.

The Löwenheim-Skolem-Tarski property

The Löwenheim-Skolem-Tarski property (of cardinals), for a logic $\mathcal{L}^*$, denoted by $LST(\mathcal{L}^*)$, is defined as follows.

Definition

A cardinal κ has the *$LST(\mathcal{L}^*)$ property* if for any $\mathcal{L}^*$-definable model class $\mathcal{K}$ and any $\mathcal{A} \in \mathcal{K}$, there is $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \in \mathcal{K}$ and $|\mathcal{B}| < \kappa$.

Notice that if κ has the $LST(\mathcal{L}^*)$ property, then any larger cardinal also has it. We call the least cardinal κ that has the $LST(\mathcal{L}^*)$ property the *$LST(\mathcal{L}^*)$-number*, and we write $LST(\mathcal{L}^*) = \kappa$ to indicate this.

Examples

- $LST(\mathcal{L}_{\omega,\omega}) = LST(\mathcal{L}_{\omega_1,\omega}) = \aleph_1$.
- $LST(\mathcal{L}_{\omega,\omega}(MM^n_{\aleph_1})) = \aleph_2$, where $MM^n_{\aleph_1}$ is the Magidor-Malitz quantifier. Namely,

  $$MM^n_{\aleph_1} x_1, \ldots, x_n \varphi(x_1, \ldots, x_n, \vec{y})$$

  if and only if there exists X such that $|X| \geq \aleph_1$ and $\varphi(a_1, \ldots, a_n, \vec{y})$ holds for all $a_1, \ldots, a_n \in X$.
- $LST(\mathcal{L}_{\omega,\omega}(W)) = \aleph_1$, where

  $$Wxy\varphi(x, y, \vec{z}) \text{ iff } \varphi(\cdot, \cdot) \text{ is a well-ordering.}$$

Theorem

Suppose $\mathcal{L}^*$ and R are symbiotic. Then the following are equivalent:

(i) $(1)_R$ *holds for* κ.

(ii) $LST(\mathcal{L}^*)$ *holds for* κ.

It follows that if $\mathcal{L}^*$ and R are symbiotic, then $LST(\mathcal{L}^*) = \kappa$ if and only if $(1)_R = \kappa$. Thus, writing $\equiv$ to indicate that the corresponding cardinals are the same, we have:

1. $(1)_{Cd} \equiv LST(\mathcal{L}_{\omega,\omega}(I))$.
2. $(1)_{Rg} \equiv LST(\mathcal{L}_{\omega,\omega}(W^{Rg}))$.
3. $(1)_{Cd,WI} \equiv LST(\mathcal{L}_{\omega,\omega}(I, W^{WI}))$.
4. $(1)_{Cd,WC} \equiv LST(\mathcal{L}_{\omega,\omega}(I, Q_{Br}))$.

Thus, the $LST(\mathcal{L}^*)$-number yields a hierarchy of logics, and in the case of symbiotic R and $\mathcal{L}^*$ it also yields a hierarchy of $(1)_R$ principles.

The case of second-order logic

Let *PowerSet* be the Π_1 relation $\{(x, y) : y = \mathcal{P}(x)\}$. Let $\mathcal{L}^2$ be second-order logic. Then we have the following.

Lemma

The PowerSet relation and $\mathcal{L}^2$ are symbiotic.

Theorem

$\kappa = LST(\mathcal{L}^2)$ *iff* $\kappa = (1)_{PowerSet}$ *iff* κ *is the first supercompact cardinal.*

A weakening of $(1)_{\mathcal{R}}$ is the following.

$(1)^{-}_{\mathcal{R}}$: If $\mathcal{K}$ is a $\Sigma_1(\mathcal{R})$ class of models and $\mathcal{A} \in \mathcal{K}$ has cardinality κ, then there exists $\mathcal{B} \in \mathcal{K}$ of cardinality less than κ and an elementary embedding $e : \mathcal{B} \preceq \mathcal{A}$.

The following theorem is a strengthening of Proposition 1.

Theorem

If $(1)^{-}_{Cd}$ holds for κ, then there exists a weakly inaccessible cardinal $\lambda \leq \kappa$.

The Strict Löwenheim-Skolem-Tarski property

Definition

We say that the strict Löwenheim-Skolem-Tarski property for $\mathcal{L}^*$, written $SLST(\mathcal{L}^*)$, holds at κ if whenever $\mathcal{A}$ is a model and $\varphi \in \mathcal{L}^*$ is such that $\mathcal{A} \models \varphi$, and $|A| = \kappa$, then there is $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \models \varphi$ and $|B| < \kappa$.

The results above show that if the property $SLST(\mathcal{L}_{\omega,\omega}(I))$ holds at κ, then there exists a weakly inaccessible cardinal less than or equal to κ.

Theorem

If κ is weakly inaccessible, then $SLST(\mathcal{L}_{\omega,\omega}(I))$ holds at κ.

Corollary

$SLST(\mathcal{L}_{\omega,\omega}(I)) = \kappa$ if and only if κ is the first weakly inaccessible cardinal.

Theorem

If κ is weakly Mahlo, then $SLST(\mathcal{L}_{\omega,\omega}(I, W^{Rg}))$ holds for κ.

Corollary

$SLST(\mathcal{L}_{\omega,\omega}(I, W^{Rg})) = \kappa$ if and only if κ is the first weakly Mahlo cardinal.

Theorem

If κ is weakly compact, then $SLST(\mathcal{L}_{\omega,\omega}(I, Q_{Br}))$ holds for κ.

Corollary

$SLST(\mathcal{L}_{\omega,\omega}(I, Q_{Br})) = \kappa$ if and only if κ is the first weakly compact cardinal.