MTH-3D10 : Arithmetic

1. Introduction: The course concentrates upon algebraic and geometric properties of integers (as distinct from analytic properties). Even though we concentrate on equations which are mainly quadratic or cubic, we find that the study of their (integral or rational) solutions is very rich. The pre-requisites are **Algebra I** and **II**.

2. Timetable Hours, Credits, Assessments: A 20 UCU course of 33 lectures, supported by seminars to be arranged informally. The overall mark comes from coursework (20%) and one three-hour examination (80%).

3. Overview: The first part of the course is based upon a study of the consequences of unique factorization. In Z for example, this leads to a characterization of all Pythagorean triples. When we study the same phenomenon over the Gaussian integers, it leads to the solution of the problem about which (positive) integers are the sum of 2 squares of integers. By mimicking this technique in the ring of quaternions, we are led to a proof of Lagrange's famous theorem that every positive integer is the sum of 4 squares of integers.

We will show how this theory is complemented by the theory of quadratic reciprocity. In particular, it is possible to characterize which primes can be represented by binary quadratic forms in integers.

The second part of the course is based upon a study of elliptic curves. As a bridge between the two parts, we will study the integral theory and show how unique factorization can be used to show the finiteness of the integral solutions.

During the century, the most influential results have been concerned with the rational points on elliptic curves. We will discuss Modell's famous theorem that the group of rational points is finitely generated. Overall, the study of elliptic curves represents a remarkable confluence of algebra, geometry and complex analysis. Finally, we discuss how elliptic curves can be used to shed light on problems such as the Congruent Number Problem and Fermat's Last Theorem.

4. Recommended Reading:

I Niven, Zuckermann "An Introduction to the Theory of Numbers"

& H Montgomery

Students will be encouraged to make use of the www as a background source.

5. Lecture Contents:

Fundamental Theorem of Arithmetic, Pythagorean Triples. **(2 lectures)**

Euclidean algorithm unique factorization in other rings, 2-square theorem. **(4 lectures)**

Lagrange's 4-square theorem, quaternions. **(4 lectures)**

Quadratic reciprocity and representation of primes by binary forms, proof of QRL. **(5 lectures)**

Some cubic equations, finitely many integral solutions. **(2 lectures)**

Elliptic curves, geometric group law, complex curve. **(2 lectures)**

Elliptic functions including Weierstrass P-function, differential equation satisfied by P. **(3 lectures)**

Rational elliptic curves, points of finite order. **(1 lecture)**

The congruent number problem, interpretation using elliptic curves. **(3 lectures)**

Modell's Theorem, heights, functoriality of height. **(3 lectures)**

Elliptic curves over finite fields, Lenstra's factorizating methods. **(3 lectures)**

Revision, problem sessions. **(2 lectures)**

MTH-3D13 : Functional Analysis

1. Introduction: This course extends methods of linear algebra and analysis to spaces of functions, in which the interaction between the algebra and the analysis allows powerful methods to be developed. The course will be mathematically sophisticated and uses ideas both from linear algebra and from analysis. The ideas developed will be applied to various problems from Fourier Analysis to Differential Equations. Familiarity with differential and integral equations is NOT essential, but a good background in linear algebra and analysis is essential. The pre-requisite is **Algebra II**.

2. Hours, Credits and Assessment: There will be 33 one hour lectures and four or five office/tutorial hours. There will be 5 or 6 problem sheets which will make up the coursework component of the unit. Sketch solutions to these will be distributed or gone over in lectures, and a consulting hour arranged (every 2 weeks). The assessment will be coursework 20% (from homework); 3 hour examination 80%.

3. Overview: Functional analysis is the language developed by mathematicians for talking about functions, in the same way that analysis (as taught in 1A11, 1A22 and 2A11) deals with numbers. In analysis a basic object of study is a function F:R -> R that sends a number to a number. The Intermediate Value Theorem (for example) gives conditions under which an equation like F(x) = x must have a solution.

A basic object in Functional analysis is something like F:V -> V where V is a *space of functions* , and F is a *functional* (something that sends a function to a function). For example, if V is the vector space of differentiable functions on the reals, then F might be a *differential operator* like F(u) = v, where v(x) = du/dx + cos(xu(x)). If one can develop general methods for showing that equations like F(u) = u must have solutions, then you can show that the differential equation du/dx + cos(xu) = u must have a solution.

Developing these ideas involves understanding spaces like V above: in all interesting settings, the space on which our functionals act will be an infinite dimensional vector space, or a *normed linear space.* Finite-dimensional linear spaces can be studied using algebra (matrices, eigenvalues, eigenvectors), but infinite-dimensional spaces need to be studied using both algebra and analysis. It is this basic dual nature that gives functional analysis its flavour.

For infinite-dimensional spaces, completeness is an issue, so much of the time will be spent on Banach and Hilbert spaces. These are types of linear spaces with very good properties.

Along the way we will see many examples and applications: existence and uniqueness results for differential equations and Fredholm and Volterra integral equations; a continuous periodic function whose Fourier series does not converge; the theory of Fourier analysis itself.

4. Recommended literature and references: The following can all be found in the UEA library.

[1] *Functional Analysis*, W. Rudin. McGraw-Hill (1973). (This book is thorough, sophisticated, and demanding.)

[2] *Functional Analysis,* F. Riesz and B. Sz.-Nagy. Dover (1990). (A classic text, much more advanced than the course.)

[3] *Foundations of Modern Analysis* , A. Friedman. Dover (1982). (Cheap and cheerful, includes excellent sections on background.)

[4] *Essential results of Functional Analysis* , R. Zimmer. University of Chicago Press (1990). (Many good problems and a useful chapter on background.)

[5] *Functional Analysis Lecture Notes*, T. Ward (unpublished).

[6] *Functional Analysis in Modern Applied Mathematics,* R.F. Curtain and A.J. Pritchard. Academic Press (1977). (This book is fairly close to the course and includes a lot of interesting other material on control theory.)

It is not necessary to buy any of these books, but if you do [3] or [6] is probably the best one to get.

5. Contents:

1. Normed Linear Spaces:

Review of linear (vector) spaces, subspaces, independence. **(3 lectures)**

Norms on linear spaces, maps between linear spaces, sequences and completeness of linear spaces. **(4 lectures)**

Topological language, quotient spaces. **(4 lectures)**

2. Banach Spaces:

Complete normed spaces, and completions of normed spaces. **(2 lectures)**

The Contraction Mapping Theorem. **(2 lectures)**

Applications to differential equations. **(2 lectures)**

Applications to integral equations. **(2 lectures)**

3. Linear Transformations:

The space of linear operators and the uniform boundedness principle. Application to Fourier analysis. **(3 lectures)**

Open mapping and Hahn-Banach theorems. **(3 lectures)**

4. Integration:

Lebesgue measure and L^p spaces as Banach spaces (review). **(3 lectures)**

5. Hilbert Spaces:

Projections and self-adjoint operators. Orthonormal sets and Gram-Schmidt. **(3 lectures)**

6. Special Topic:

As time allows, a special topic chosen from spectral theory, K_0 of AF algebras, applications of the current algebra to the moment problem, convergence results in Fourier analysis. **(3 lectures)**

MTH-3D15 : Theory of Finite Groups

1. Introduction: This third year course is a thorough introduction to Finite Groups with **Algebra I** and **II** as prerequisites. Group theory is a very large field which interconnects with many branches of pure and applied mathematics. The unit is a good accompaniment to other pure mathematics units such as **Representation Theory**, **Ring Theory** and **Algebraic Number Theory**. (It is good advice where practicable to do these units concurrently or after the group theory course.)

2. Hours, Credits and Assessment: The course is a 20 UCU unit of 33 lectures and 3 to 5 additional hours for group discussions. Assessment is by course work (20%) through assessed homework and examination (80%).

3. Overview: Historically Group Theory has two main roots, one in geometry where groups of geometrical transformations were studied, the other in algebra and the theory of equations where groups of substitutions of variables (i.e. permutations) in polynomial functions were analysed. The revolutionary work of Galois (1820's) about the solvability of polynomial equations, for instance, made it necessary to study such groups of substitutions. Finite group theory evolved to a great extent from this second root. Abstract groups began to emerge with Jordan's seminal *Traité des substitutions et des equations algébriques* (1870) while the definition of abstract groups in general appears to be due to Weber (1882).

The course starts with a review of elementary facts such as the correspondence and isomorphism theorems, composition series and composition factors. The theorem of Jordan and Höelder will be proved which shows that a finite group is constructed in some fashion from simple groups.

The idea that a group acts on a set is fundamental. Therefore *group actions* will be studied in great detail. For finite groups the orbit stabilizer theorem, a relatively easy result on group actions, plays a central role and a great many results are a consequence of it. One instance is the theorem which determines the number of orbits of a permutation group which is used for pattern counting more generally. Other applications include the Class Equation which leads to an elementary introduction to groups whose order is a power of a prime.

Sylow's Theorem is an early high point of finite group theory. It says that a finite group of order p n.x with p a prime not dividing x has a subgroup of order p n . It also says that any two such subgroups are conjugate and that their number is congruent to 1 modulo p . Several proofs of this theorem will be given, various strengthenings will be proved and a variety of numerical non-simplicity will be derived.

The course concludes with a short exposition of the general linear group GL(V) . When V is a finite dimensional vector space over finite field $GF(p_n)$ then GL(V) itself is finite and has a very rich structure. Its Sylow-p-subgroup is easy to describe explicitly and from this one can obtain a further independent proof of Sylow's theorem. Also investigated are the finite subgroups of the orthogonal group on V when instead V is a real 2- or 3-dimensional vector space. This leads to the classification of the regular Platonic solids.

4. Recommended literature and references: While there is no single text book for the course there are a great number of books on group theory. The following list begins with some more basic texts which are useful to look up elementary facts about groups:

(1) Herstein: Topics in Algebra.

(Useful for the beginner, does not cover all you need)

(2) PM Cohn: Algebra.

(Useful for the beginner, does not cover all you need)

(3) J Rotman: Introduction to Group Theory, Springer Verlag

(Contains all of what you need and much more).

(4) J Rose: A Course on Group Theory, CUP

(Contains a lot of exercises and examples).

(5) M Hall: The Theory of Groups, Macmillan.

(A thorough treatment, not in print anymore)

(6) Burnside: Group Theory

(A classic text, not always easy to read).

Not all of these books are still in print but all should be in the library.

5. Contents:

1 . Factor groups, the correspondence and isomorphism theorems. Composition series, composition factors and the Jordan-Hölder theorem. Solvable and simple groups. **(6 lectures)**

2 . Permutation groups, the sign function and definition of the alternating group. Review of group actions and applications: action on cosets, conjugation. Simplicity of alternating groups. Orbit stabilizer theorem, orbit counting theorem. Examples: Symmetry groups of combinatorial structures such as Petersen graph. The rotation groups of the cube and dodecahedron. The Class equation for a finite group. **(7 lectures)**

3 . The proof of the three Sylow theorems with various extensions. Counting Sylow subgroups with applications to non-existence of simple groups of certain orders. A5 is the only non-abelian simple group of order less than 100. **(8 lectures)**

4. Introduction to p-groups. The centre intersect any normal subgroup non-trivially, normalizers of subgroups, the Frattini argument. Nilpotency and the Frattini subgroup. Generators for a p-group. **(6 lectures)**

5. Linear groups. The general linear and special linear groups, their action on affine and projective spaces. GL(n,q) , its order and some of its subgroups, Sylow's theorem revisited. O(n,R) for small n , classification of finite subgroups and Platonic bodies. **(6 lectures)**

MTH-3D23 : Mathematical Logic

1. Introduction: The course in concerned with foundational issues of modern pure mathematics. It is a rigorous introduction to first-order logic. Proofs will be given for most of the results discussed. Some degree of mathematical sophistication is called for and familiarity with (and a taste for) mathematical proofs, such as would be seen in a rigorous first-year analysis or algebra course, will be assumed. The prerequisite is **Algebra I** and there are connections with Discrete Mathematics II.

2. Timetable Hours, Credits, Assessments: 33 one hour lectures; 20 UCU. Assessment: Coursework 20% via assessed homework; 3 hour examination 80%. There will be 4 problem sheets which will make up the coursework component of the unit. Sketch solutions will be distributed and consulting hours arranged.

3. Overview: Mathematical logic analyses symbolically the way in which we reason formally, particularly about mathematical structures. The subject was developed the 20th century with Alfred Tarski and Kurt Gödel as its major figures. Although it is highly abstract, its ideas are fundamental in theoretical computer science and artificial intelligence and its results have applications in other parts of Mathematics (via an area known as Model Theory).

The first level of the subject is the propositional calculus. We look at the way simple statements (propositions) can be built into more complicated ones using connectives (`or,' `and,' `not,' `implies') and make precise how the truth or falsity of the component statements influences the truth or falsity of the compound statement. This is done using truth tables and can be useful for testing the validity of various forms of reasoning. It provides a way of analysing deductions of the form `if the following statements are true: ..... ; then so is .... .' We then move on to a completely symbolic process of deduction and describe the formal deduction system for propositional calculus. The statements we consider (propositional formulas) are regarded as strings of symbols and we give rules for deducing a new formula from a given collection of formulas. We want these deduction rules to have the property that anything that could be deduced using truth tables (so by considering truth or falsity of the various statements), can be deduced in this formal way, and vice versa. This is the soundness and completeness of our formal system.

The next level of the subject is the predicate calculus. This is what is needed to analyse `real' mathematics and the extra ingredient is the use of quantifiers (`for all' and `there exists'). We introduce the notion of a first-order structure, which is general enough to include many of the algebraic objects you come across in mathematics (groups, rings, vector spaces). We then have to be precise about the expressions (formulas) which make statements about these structures, and give a precise definition of what it means for a particular formula to be true in a structure. This is quite intricate, and the clever part is in getting the definitions right (- this is due to Tarski), but it corresponds to ordinary mathematical usage. Once this is done, we set up a formal deduction system for predicate calculus. This parallels what we did for propositional calculus, but is much harder. Nevertheless, the end result is the same: the formulas which are produced by our formal deduction system (the `theorems') are precisely the formulas which are true in all first-order structures. This is Gödel's Completeness Theorem.

The final section of the unit is concerned with model theory: the study and classification of mathematical structures in terms of what can be said about them in 1st order languages.

4. Recommended Reading: The following can be found in the UEA library. An asterisk (*)

indicates that a copy of the book has been placed in the Restricted Loan collection and can only be taken out for short periods of time. There may be another copy available and other suitable books. Check the Library Catalogue.

1.* A G Hamilton "Logic for Mathematicians", (Cambridge University Press, 1988)

2. Elliott Mendelson "Introduction to Mathematical Logic" (2nd edition), (van Nostrand, 1979)

3.* Wilfrid Hodges "A Shorter Model Theory", (Cambridge University Press)

5. Lecture Contents:

Propositional calculus: Truth tables; propositional formulas; adequacy of sets of connectives; disjunctive normal form. **(4 lectures)**

The formal system L for propositional calculus. Proofs and deduction in L. Soundness; the Deduction Theorem. Proof of the Completeness Theorem (Adequacy) for propositional calculus. **(5 lectures)**

Predicate calculus: First-order structures. Construction of first-order languages and formulas. Interpretations. Satisfaction and truth of formulas. **(6 lectures)**

The formal system for predicate calculus. Soundness; the Deduction Theorem. Gödel’s completeness theorem for countable first-order languages (Henkin’s proof). The compactness theorem. Normal models and applications of the compactness theorem. **(9 lectures)**

Set theory: Countability and a brief discussion of cardinality and the Axiom of Choice. The General version of the completeness theorem. **(4 lectures)**

Model theory: Substructures and embeddings. elementary equivalence. The Löwenheim-Skolem theorems, categoricity and the Los-Vaught test. Applications: dense linear orders, the random graph, and the zero-one law for graphs. **(5 lectures)**

MTH-3D32 : Partial Differential Equations

1. Introduction: This is a 3rd year course running in conjunction with MTH-4D32. The two courses are offered in two levels of difficulty. MTH-3D32 is geared towards a third year student rather than a MMath student. With additional topics offered as a reading element of the course, MTH-4D32 is geared toward a more advanced student.

This unit has as prerequisites **Advanced Calculus I&II.** The solution of partial differential equations pervades applied mathematics. However the subject is rich in the concepts and techniques deployed and can be studied detached from particular applications; indeed that is the reason for its generality. The material is presented in this mathematical form and should be of interest to pure and applied mathematicians.

2. Hours, Credits and Assessment: 33 lectures, 20 UCU. Assessment is 20% coursework and 80% examination. Exercises are provided (with outline solutions following) and a subset are used for frequent coursework assignments.

3. Overview: The primary aims of this unit are to develop an appreciation of the relationship between partial differential equations and data, an ability to recognise properly posed problems and an awareness of circumstances in which a solution can fail. The unit involves classification of partial differential equations, analysis of linked systems of partial differential equations and exploring the implications of variable coefficients and non-linearity. It deploys techniques from preceding units: vector fields, linear algebra, differential equations; it uses geometrical ideas to motivate procedures. In illustrating the theory it covers a range of methods of analytical solution but do not expect a collection of recipes. Finite difference and spectral methods will be introduced.

The history of the subject is not readily traced as methods were developed to solve problems in diverse fields. In the 18th century there were contributions from D'Alembert, D. Bernoulli, Euler and Clairaut. In the early 19th century Green introduced integral representations, Fourier established separation of variables and Cauchy used characteristic supports; Monge had provided geometrical insight. Laplace and Poisson derived particular solutions. Later Riemann developed characteristic theory for a system. Weierstrass pioneered ideas of continuation and Sonia Kowaleski developed local solutions for a broad class of higher order equations. Kirchoff and Helmholtz were active with particular equations. Hadamard addressed the question of what constitutes a properly posed problem. Ideas from functional analysis began to make an impact in the early 20th century. Finite difference methods were used by Courant et al as a theoretical tool for existence in the limit as mesh size approached zero. The advent of electronic computers prompted a re-appraisal of the significance of past contributions. The importance of the function space in which solutions were sought was realised and the effects of non-linearity could be explored.

The availability of computer programmes allowing numerical solution for practical geometries has vastly enhanced applicability and commercial packages incorporate a wealth of experience and expertise. Currently partial differential equations find application in Finance, Economics, Biology, Medicine, Chemistry & Environmental Sciences, as well as the traditional fields of Astronomy, Engineering and Physics. However the limitations of the methods in packages are frequently not adequately documented or flagged and there is need for genuine understanding both in formulating problems and in obtaining solutions.

4. Recommended literature and references:

E Zauderer "Partial Differential Equations of Applied Mathematics" (J Wiley)

W E Williams "Partial Differential Equation" (OUP)

C R Chester "Techniques in Partial Differential Equations" (McGraw-Hill)

Courant & Hilbert "Methods in Mathematical Physics II" (Academic)

5. Contents:

Preliminaries: Generating of surfaces by curves. Envelopes. Elimination of arbitrary functions. The 'generality' of a solution. General solution. Complete integral. Systems of ode's. Solving dx/P = dy/Q = dz/R and Pdx + Qdy + Rdz = 0 (integrability condition). **(3 lectures)**

First Order PDE's: Solution of 1st order linear and semi-linear PDE's. Characteristic supports. Quasi-linear PDE's: characteristic generators, Initial value problem, dependence on data, non-linear breakdown, examples. Characteristic supports obtained by considering jumps in partial derivatives. **(3½ lectures)**

General non-linear PDE's. Special methods for complete integrals; envelopes to solve initial value problems. **(1 lecture)**

Non-linear PDE's - characteristic strip theory. Monge cones. Initial Value Problem, non-uniqueness via data, for 2 independent variables. N > 2 independent variables: inner derivatives, characteristic hypersurfaces, strip equations. Examples. **(3½ lectures)**

Systems of 1st order PDE's: Relation between systems and higher order equations. Quasi-linear systems. Classification. Characteristics and canonical form for hyperbolic case. Riemann invariants: 1-D flow example. Systems N > 2 independent variables. Wavefronts, bicharacteristics and rays. **(5 lectures)**

Higher order PDE's: Characteristic surface for a single higher order equation. Second order equations. Classification and reduction to canonical form. Examples for 2 and 3 independent variables. **(3 lectures)**

Boundary and initial conditions: Common types of data for second order PDE's; relation to classification by considering examples. **(1 lecture)**

Diffusion & parabolic type PDE's. Solution by separation of variables, superposition, singular solutions and integral representation. Extension to 2 & 3 space variables. **(2 lectures)**

Laplace, Poisson & elliptic type PDE's. Green's formulae, integral representations, mean-value property, min-max principle, continuous dependence on data. Green's functions for 2&3 space variables, symmetry in arguments. Helmholtz & related equations-uniqueness not settled by classification. **(2 lectures)**

Wave & hyperbolic PDE's (1D). D'Alembert solution - Domains of dependence & influence. Boundary conditions for a bounded interval, uniqueness via energy integral. Timelike and spacelike boundary curves. Extension to more space dimensions. **(2 lectures)**

Uniqueness, continuous dependence on data. **(2 lectures)**

Systems of first order constant coefficient PDE's, initial-boundary value problems and conditions for uniqueness. **(1½ lectures)**

Finite difference and spectral methods. **($2\frac{1}{2}$ lectures)**

MTH-3D37 : Advanced Mathematical Techniques

1. Introduction: This second semester course at level 3 follows on from the techniques stream courses of level 2 **Advanced Calculus I and II**. It is designed to be equally suitable for students taking pure or applied units in their final year.

2. Timetable Hours, Credits, Assessments: The unit is of 20 UCU and is taught in the Spring Semester by means of 33 hours of lectures, three a week. No formal support teaching is timetabled, but office hours are advertised and students are encouraged to seek help from the lecturer when required. *The Assessment is by coursework set three times through the semester (20%) and a three-hour examination (80%).

* A laboratory class using the computer software package MAPLE will also be timetabled.

3. Overview: Calculus of Variations includes techniques for maximising integrals subject to constraints. A typical problem addressed is the curve described by a heavy chain hanging under the effect of gravity.

The Sturm-Liouville eigenvalue problem gives rise to the construction of sequences of mutually orthogonal functions. Functions can then be expressed as a sum of such orthogonal functions. In particular Legendre & Chebyshev polynomials will be studied.

Analysis of eigenfunction expansions leads naturally to Fourier series and hence to the concept of Fourier transforms. Indeed, the inversion of Fourier integrals can be investigated by analogy with convergence of Fourier series. Applications of Fourier and Laplace transforms include the solution of differential equations.

Techniques are available for the evaluation of integrals involving large or small parameters, a topic known as asymptotic analysis. Such techniques are particularly suitable for the inversion integrals arising from transform theory. In addition techniques for studying differential equations involving small parameters will be studied.

This unit will include illustration of concepts using numerical investigation with MAPLE.

4. Recommended Reading: No one book covers all the course. References are provided at the end of each section. Books which prove useful to several parts of the course include Arfken and Hinch.

Arfken 'Mathematical Methods for Physicists'

Lighthill 'An introduction to Fourier analysis & generalised functions'

Kreyszig 'Advanced Engineering Mathematics'

Hinch 'Perturbation Methods'

5. Lecture Contents:

Calculus of Variations

Variations subject to constraints. Eulers equation and applications. Lagrangian multipliers. **(4 lectures)**

Sturm-Liouville Theory

Adjoint operators and orthogonality of eigenfunctions. Orthogonal polynomials. Examples including Legendre & Chebyshev polynomials. **(5 lectures)**

Fourier transforms

Including inversion theorems and convolution. Relevance of theory developed for Fourier series. Examples of applications to differential equations. **(5 lectures)**

Laplace transforms

Emphasis on inversions by contour deformation. Examples of application (including integral equations). Inversions involving branch cuts. **(5 lectures)**

Asymptotic series

Definitions and notation. Asymptotic solution of differential equations. Method of Multiple Scales. **(8 lectures)**

Asymptotic Evaluation of Integrals

Method of Stationary Phase. Contour deformation. Method of Steepest Descent. **(6 lectures)**

MTH-3D42 : Fluid Dynamics

1. Introduction: This course is a natural successor to the MTH units **Hydrodynamics I and II**. It is available to students with that background, or other appropriate background as provided, for example, for students in the School of Environmental Sciences.

2. Timetable Hours, Credits, Assessments: This unit is of 20 UCU and is taught in the Autumn Semester by means of 33 lectures at a rate of three lectures per week. The Assessment is by set regular coursework (20%) and a three-hour examination (80%).

3. Overview: Modern fluid mechanics has its roots set firmly in the 19th Century. But the rapid developments in this century, with their impact on flight, ocean engineering and the climate (among others), demonstrate that fluid mechanics has contributed to the shaping and understanding of our world on a par with the other great advances of 20th century physics. One key to this was Prandtl's revolutionary discovery of the boundary layer in 1904 which had the same transforming effect on fluid mechanics as Einstein's 1905 discoveries had on other parts of physics. The purpose of this unit is to introduce you to advanced topics from inviscid flow theory and to elementary viscous flow. The Navier-Stokes equations of motion are solved for a few special cases but to make realistic progress we must make approximations, suitable for boundary layers, thin films, jets and wakes.

4. Recommended Reading: The notes taken in lectures are intended to be complete and self-contained in themselves. Recommended books in order of increasing technical detail are:

M J Lighthill An Informal Introduction to Theoretical Fluid Mechanics (Oxford UP)

D J Acheson Elementary Fluid Dynamics, (Oxford UP)

D J Tritton Physical Fluid Dynamics, (Oxford UP)

G K Batchelor An Introduction to Fluid Dynamics, (Cambridge UP)

5. Lecture Contents:

(1) Introduction:

(2) Inviscid flow. Revision. Euler's equations, Bernoulli's equation, stream function, (complex) velocity potential. Point source, line source, vortex. Flow past circular cylinder. D'Alembert's paradox. Spherical bubble oscillation. **(3 lectures)**

(3) Slender Wing theory. (i) symmetric flow past slender body - source distribution; (ii) thin cambered aerofoil - vortex distribution. Lift, and lift coefficient. **(4 lectures)**

(4) Navier-Stokes Equations:

(4.1) Introduction to the stress tensor for incompressible Newtonian fluid only. Coefficients of viscosity μ and ν for several fluids.

(4.2) Boundary conditions at fixed surface for inviscid/viscous flow; for moving body; for free surface; on interface.

(4.3) Reynolds number Re from dimensionless Navier-Stokes. Froude Number. Examples. Re and drag. Vorticity equations. Energy dissipation. **(2 lectures)**

(5) Exact solutions of Navier-Stokes Equations:

(5.1) 2D flow between 2 parallel plates, one sliding wall in steady motion. Poiseuille flow. One plate suddenly moved from rest, and when 2nd plate absent: Similarity Solution. Harmonic Solution - oscillating plate.

(5.2) Exact solutions in cylindrical polars. Pipe flow, concentric cylinders, Couette flow, radial and transverse flow between concentric cylinders. Unsteady line vortex, Burgers vortex. **(4 lectures)**

(6) The mathematics of boundary layers:

(6.1) Regular and singular perturbation expansions for algebraic equations.

(6.2) Boundary layer for an o.d.e. **(2 lectures)**

(7) Boundary-Layer Theory:

(7.1) Derivation of the boundary-layer (b.l.) equations.

(7.2) B.L. on a flat plate, with zero pressure gradient, similarity solution and thickening b.l.

(7.3) B.L. thicknesses.

(7.4) As (7.2) but with non-uniform flow in mainstream: separation induced by adverse pressure gradient. **(4 lectures)**

(8) Jets and Wakes:

(8.1) B.L. equations applied to thin 2D, steady, submerged jet.

(8.2) Wake: Slender wake at $Re \gg 1$. **(2 lectures)**

(9) Very Viscous Flow. Derivation of the thin film equations. Flow in a layer between $z = 0$ and $z = h$ $\ll$ horizontal length scale. Integration when h = constant. Squeezing film flow when $h = h(t)$. Lubrication theory: general theory. Plane slider bearing. **(6 lectures)**

(10) Flow stability or other topics as time permits. **(6 lectures)**

MTH-3D45 : Nonlinear Elasticity

1. Introduction: The prerequisite is **Hydrodynamics II**, replaced by **Mathematics for Geophysical Science II** for ENV students, which ensures a little prior knowledge of linear elasticity. Continuum mechanics underpins all theoretical work in both solid and fluid mechanics. It concerns the deformation and motion of a continuous body, the balance laws and the formulation of constitutive equations, which distinguish one sort of material from another. These general ideas are applied in particular to nonlinear elasticity.

2. Timetable Hours, Credits, Assessments: 33 lectures; 20 UCU; assessment is 20% coursework and 80% examination. Example sheets are handed out containing many examples to support and illustrate the lecture material. Some are set as coursework and detailed solutions to most exercises are handed out.

3. Overview: Continuum mechanics is the mechanics of continuous materials, i.e. materials which occupy every point of a continuous region of space. Real materials are not like this, they consist of atoms and molecules and are mostly empty space! Continuum mechanics approximates real materials by smearing out the atoms and molecules uniformly in space. The justification is that it works! Those theoretical and experimental phenomena of engineering and physics which take place over a length scale greater than the interatomic spacing in solids or the mean free path in gases are found to be well described by continuum mechanics.

It took a hundred years to get from Newton's laws for particles and rigid bodies to the equivalent integral balance law formulations of Euler and Cauchy for continuous materials, leading to the definition of a stress tensor and the partial differential equations governing the motion of a body. It is the precise nature of the dependence of the stress upon the motion of the body, termed the constitutive equation, that distinguishes one sort of material from another.

The resurgence of modern continuum mechanics, and of nonlinear elasticity in particular, began c. 50 years ago with Rivlin's discovery of nonlinear exact solutions which are valid for all incompressible isotropic elastic materials (e.g. rubber). In this course we examine the general foundations of continuum mechanics, setting up the basic equations for both fluid and solid mechanics. In the second half we consider various aspects of nonlinear isotropic elasticity, including Rivlin's work.

4. Recommended Reading:

(1) G A Holzapfel "Nonlinear Solid Mechanics" (Wiley)

(2) R W Ogden "Non-Linear Elastic Deformations" (Dover)

(3) P Chadwick "Continuum Mechanics" (Dover)

(4) R J Atkin & N Fox "An Introduction to the Theory of Elasticity" (Longman)

(5) A J M Spencer "Continuum Mechanics" (Longman)

5. Lecture contents:

CONTINUUM MECHANICS

Introduction: The nature and structure of continuum mechanics

1. Kinematics of a continuum

Basic description of motion: two illustrations, velocity and acceleration. The referential and spatial descriptions: material time derivative. Mass: referential and spatial equations of continuity. Notes on tensors. Reynold's transport formula. **(3 lectures)**

2. Dynamics of a continuum

Momentum, angular momentum, force and torque: Euler's laws of motion. Cauchy's equations of motion: equilibrium of a body. Principal stresses and principal axes of stress. **(3 lectures)**

3. Other stress measures, elastic strain energy and constraints

Introduction: basic constitutive assumptions and principle of objectivity, material symmetry. First and second Piola-Kirchhoff stresses. Energy balance. The strain energy function and constitutive forms. Constraints, reaction stress and constitutive forms. Incompressibility. **(3 lectures)**

ELASTICITY LINEAR ISOTROPIC ELASTICITY

4. Linear isotropic elasticity

The infinitesimal stress and strain tensors. The generalized Hooke's law, equations of motion and equilibrium. Homogeneous deformations: simple dilatation, simple extension, simple shear (bulk, Young's and shear moduli, Poisson's ratio). Elastic constants and physically reasonable response. **(2 lectures)**

NON-LINEAR ISOTROPIC ELASTICITY

Compressible isotropic elasticity:

5. Constitutive equation

Equations of motion, equilibrium equations. Interpretation of left and right Cauchy-Green strain tensors. **(1 lecture)**

6. Homogeneous deformations

Dilatation: pressure a monotonic decreasing function of stretch (example, foam rubber), incremental bulk modulus. Simple extension: axial tension a monotonic increasing function of axial stretch (example, Hadamard material), incremental Young's modulus and Poisson's ratio. Simple shear: generalized shear modulus is positive, Kelvin effect, universal relations, Poynting effect, stresses on the inclined faces. **(4 lectures)**

INCOMPRESSIBLE ISOTROPIC ELASTICITY:

7. Constitutive equation (1 lecture)

8. Homogeneous deformations

Dilatation (impossible). Simple extension, incremental Young's modulus and Poisson's ratio. Stretching of a plane sheet. Simple shear. Examples of constitutive equations (neo-Hookean, Mooney-Rivlin). Principal forces, Rivlin's cube. **(2 lectures)**

9. Non-homogeneous deformations

The five families. Family 1: bending and extension of a rectangular block. Family 2: straightening and extension of an annular wedge. Family 3: extension and torsion of a circular cylinder.

Appendix: strain, stress and equilibrium equations in orthogonal curvilinear coordinate systems. **(5 lectures)**

BELL MATERIALS

10. Bell-constrained isotropic elasticity

Bell's constraint: geometry, invariant triangle. Generalised shear with normal stretch. The reaction stress, constitutive equations. Inequalities among the response functions. Uniaxial loading. The general universal relation. Simple tension produces simple stretch. Square rooting the left Cauchy-Green strain tensor to find the left stretch tensor. The Bell stress. Shear deformation with shear stress! **(9 lectures)**

MTH-3D49 : Geophysical Fluid Dynamics

1. Introduction: This level 3 course in semester I covers various aspects of modelling the circulation of the oceans and atmosphere. It requires prior completion of second level courses, either **Hydrodynamics I** or **Mathematics for Geophysical Science II** or the ENV course **Mathematics II**.

2. Timetabled Hours, Credits, Assessment: This unit is of 20 UCU and is taught in Semester I by 33 lectures. It is supported by occasional problem classes, as required. Assessment is by set regular homework (20%) and an examination (80%).

3. Overview: The mathematical modelling of the oceans and atmosphere in this course demonstrates how the techniques developed in second year units on fluid dynamics and differential equations can be used to explain some interesting phenomena in the real physical world. The course begins with a discussion of the effects of rotation in fluid flows, then continues with models of the ocean circulation and waves, and ends with simplified models of the atmosphere.

4. Recommended texts: The recommended reference books are:

A Gill "Atmosphere-Ocean Dynamics", (Academic Press)

J Pedlosky "Ocean Circulation Theory", (Springer)

J Holton "An Introduction to Dynamic Meteorology", (Academic Press)

5. Lecture Contents:

A. Introduction:

What is GFD? Equations of motion. Rotation. Geostrophic flow. Ekman boundary layer. Ekman transport and suction. Rossby and Ekman numbers. **(3 lectures)**

B. Ocean dynamics:

(i) Wind-driven ocean circulation: Geostrophic and hydrostatic balance. Time scales. Steady flow away from lateral boundaries. Coastal boundary layers. Munk layer and nonlinear solutions. Coastal and equatorial upwelling. **(5 lectures)**

(ii) Topography: Ertel's theorem for conservation of potential vorticity. Applications to mid-Atlantic ridge and continental shelves. **(4 lectures)**

(iii) Oceanic coastal upwelling: Theories of coastal and equatorial upwelling. Coastal currents. Steady and unsteady flows. Cool upwelling filaments. **(5 lectures)**

C. Atmospheric dynamics:

(i) Basic forces in the atmosphere: Gravity, Coriolis force, hydrostatic approximation. Pressure gradient force, geostrophy. Using pressure or vertical coordinate. Equations of motion. **(3 lectures)**

(ii) Thermodynamics of the dry atmosphere**:** Equation of state, potential temperature, lapse rates and vertical stability. **(2 lectures)**

(iii) Simple consequences of the equations of motion: Scaling for typical wind systems. Equations of motion in pressure coordinates. Geostrophy, gradient wind, inertial flow. **(3 lectures)**

(iv) Circulation and vorticity: Circulation on a rotating body. Kelvin's theorem; Bjerknes theorem. Planetary and relative vorticity. Conservation of potential vorticity. Ertel's theorem. **(4 lectures)**

(v) Planetary boundary layers: Scaling of equations. Perturbation techniques and turbulence. Well-mixed and stratified boundary layers. Mixing lengths and the Ekman spiral. Logarithmic surface layer. **(4 lectures)**

MTH-3S50 : Advanced Statistics

1. Introduction: This is a course in Applied Statistics. Some use of statistical packages is required.

2. Timetable Hours, Credits, Assessments: The course is a 20 UCU unit of 36 lectures. Assessment is by coursework exercises (30%) and examination (70%).

3. Overview: The course develops the ideas of linear models. These are widely used in data modelling. Perhaps the most important development in statistics over the last 30 years has been the extension of linear models. These generalised models are of real practical value.

Not all data series are uncorrelated and we spend some time looking at time series techniques.

4. Recommended Reading:

A J Dobson "An Introduction to Generalized Linear Models" (Chapman & Hall)

G Janacek & A Swift "Time Series" (Ellis Horwood)

G Janacek "Practical Time Series" (Arnold)

5. Lecture Contents:

1. Introduction to Regression.

2. General Linear Model.

3. Extension to generalised Linear Models.

4. Time Series.

(36 lectures)