MTH-3D79 : Free Surface Flows with Advanced Topics

1. Introduction: This course provides an introduction to the theory of free surface flows, in particular of water waves, liquid jets and liquid sheets. It requires some knowledge of hydrodynamics and multivariable calculus. The unit is suitable for those with an interest in Applied Mathematics.

2. Timetable Hours, Credits, Assessments: 20 credits(UCU) assessed by a three-hour examination (80%) and by coursework (20%). There will be 33+4 lectures.

3. Overview: Free surface problems occur in many aspects of science and everyday life. Examples of free surface problems are waves on a beach, bubbles rising in a glass of champagne and a liquid jet flowing from a tap. In these examples the free surface is the surface of the sea, the interface between the gas and the champagne and the boundary of the falling jet. We will study aspects of linear and nonlinear water waves using analytical techniques. In the second part we will investigate the dynamics and breakup of liquid jets or sheets using linear stability analysis.

4. Recommended literature and references (The notes taken in lectures are intended to be complete and self-contained):

J. Billingham and A.C. King, *Wave Motion,* Cambridge Univ. Press, 2000.
G.B. Whitham, *Linear and nonlinear waves,* Wiley, 1999.
J.-M. Vanden-Broeck*, Gravity-Capillary Free-Surface Flows,* C.U.P., 2010.
P. Drazin, *Introduction to hydrodynamic stability,* C.U.P., 2002.
S.P.Lin, *Breakup of Liquid Sheets and Jets,* C.U.P., 2003.
James Lighthill, *Waves in Fluids*, C.U.P., 2001.
R. S. Johnson - *A Modern Introduction to the Mathematical Theory of Water Waves,* C.U.P., 1997.

In addition, material may be taken from recent journal articles and appropriate references will be given during the lectures.

5. Lecture Contents:

Introduction and overview: Equations and boundary conditions, the concept of surface tension.
(3 lectures)

A. Theory of water waves.
- Linear aspects: dispersion relation and group velocity, wave energy, wave shoaling, the wavemaker problem, initial value problems, gravity-capillary waves.

(10 lectures)

- Nonlinear waves: Stokes periodic waves, solitary waves, shallow-water approximations, model equations (Korteweg-de Vries equation, nonlinear Schrödinger equation).

(10 lectures)

B. Theory of liquid jets and sheets: Linear stability analysis. One-dimensional approximations. Liquid sheets

(10 lectures)

6. Advanced Topic: Water waves as a dynamical system, or internal waves.

MTH-ME29: Galois Theory

1. Introduction: This module is an introduction to Galois Theory, which beautifully brings together the notions of a group and of a field, from 2C3Y Algebra. In particular, the ideas developed will be applied to looking at the question of solving polynomial equations.

2. Hours, Credits and Assessment: A 20 UCU module of 37 lectures, supported by office hours. Assessment is based on coursework (20%) and exam (80%).

3. Overview: Galois theory is one of the most spectacular mathematical theories. It gives a beautiful connection between the theory of polynomial equations and group theory. In fact, many fundamental notions of group theory originated in the work of Galois. For example, why are some groups called "solvable"? Because they correspond to the equations which can be solved (by some formula based on the coefficients and involving algebraic operations and extracting roots of various degrees). Galois theory explains why we can solve quadratic, cubic and quartic equations, but no similar formulae exist for equations of degree greater than 4. In modern exposition, Galois theory deals with "field extensions", and the central topic is the "Galois correspondence" between extensions and groups.

4. Recommended literature and references: Suitable books include

Stewart, I., Galois Theory, Chapman and Hall. [QA214STE]

Artin, E., Galois Theory, Dover.

Cohn, P.M., Algebra Vol. 1, Wiley. [QA154COH]

Herstein, I.N., Topics in Algebra, Wiley. [QA154HER]

Snaith, V.P., Groups, rings and Galois theory, World Scientific. [QA171SNA]

5. Lecture Contents:
Fields and polynomial rings; irreducibility of polynomials and irreducibility criteria for polynomials over Q; maximal ideals and construction of algebraic field extensions; degree; tower law; splitting fields. (9 lectures)

Artin’s Extension Theorem, separability, inseparability, Primitive Element Theorem. (5 lectures)

Normal and Galois extensions, the Fundamental Theorem of Galois Theory, examples of the explicit computation of Galois groups. (7 lectures)

Radical extensions, solvable groups, proof that a polynomial can be solved using radicals if and only if the associated Galois group is a solvable group, radical solution to general quadratic, cubic and quartic equations, explicit examples of polynomials which are not solvable by radicals. (7 lectures)

Finite fields: basic structure of finite fields, all extensions of finite degree are Galois with cyclic Galois group. (5 lectures)

Advanced topics: Some aspects of the inverse Galois problem of constructing a field extension with a given Galois group. (4 lectures)

MTH-ME23: Graph Theory with Advanced Topics

1. Introduction: This module is a thorough introduction on modern Graph Theory. This subject plays an important role in many branches of mathematics and the sciences generally. Graph Theory therefore has applications almost everywhere. There are no formal prerequisites from MTH Year 2 modules.

2. Hours, Credits and Assessment: The module is a 20 UCU module of 33 lectures and additional time for group discussions on demand. Assessment is based on a three-hour examination and course work.

3. Overview: Graphs are among the most basic structures in mathematics. A graph consists of a set of points, the *vertices* of the graph, and a set of *edges* which link certain pairs of vertices. By their very simplicity it is therefore not surprising that graphs are important in many part of mathematics, computing and the sciences.

This module is designed as an introduction to the theory of graphs. Thus we first develop the basic notions of connectivity and matchings. Graphs appear also in topology and these aspects will be investigated in a section on planarity. This addresses the question if a graph can be drawn in the plane in such a fashion that edges do not intersect except at a vertex. We shall aim to prove a famous theorem due to Kuratowski which provides the exact conditions for the planarity of a graph.

An important area of interest are graph colourings. These are assignments of 'colours' to the edges of the graphs in such a way that two different edges of the same colour never meet at a vertex. There are also vertex colourings where vertices of the same colour may not be joined by an edge. Such questions lead to many beautiful results and interesting applications. One of the best known theorems in graph theory is the Four-Colour-Theorem. While this result is not within our reach we shall aim to prove the Five-Colour-Theorem.

This part of the module is based on Diestel's book below, and this text will be the designated course text.

In the **Advanced Topic** we will study Strongly Regular Graphs**.** This is a particular class of graphs that have many regularity and symmetry properties that are of great importance in combinatorics and group theory. This material will be presented by a few lecture and reading material.

4. Recommended literature and references: The literature on the subject is extensive. We will be using the first book as the main course text. (Booksellers can be found at http://www.AbeBooks.co.uk)

R Diestel "Graph Theory", Springer Graduate Texts in Mathematics 173, Electronic searchable versions are available.

NL Biggs "Discrete Mathematics", OUP, (a very useful introductory text)

JA Bondy, USR Murty "Graph Theory” Springer GTM 244 (a comprehensive text)

NL Biggs "Algebraic Graph Theory", CUP

5. Lecture Contents:

Basics: Definitions, subgraphs, trees, Euler circuits, cycle. **(9 lectures)**

Matchings: The theorems of König and Hall. **(5 lectures)**

Connectivity : k-connected graphs, Menger's theorem. **(5 lectures)**

Planarity: Basic topological ideas, introduction to Kuratowski’s theorem. **(6 lectures)**

Colourings: vertex colourings, edge colourings, the 5-colour theorem. **(8 lectures)**

6. Advanced Topic:

Strongly regular graphs**.** **(4 lectures and reading material)**

MTH-ME46 : Dynamical Oceanography with Advanced Topics

1. Introduction: This masters level module covers modelling the large scale ocean circulation and structure, internal waves and coastal flows. It requires prior completion of second level modules, either **MTH-2C2Y** or **ENV-2A22** and level 3 study of some aspect of fluid dynamics**.**

2. Timetabled Hours, Credits, Assessment: This is a 20 credit module of 37 lectures. Assessment is based on a three-hour examination (80%) and two pieces of coursework (20%).

3. Overview: The mathematical modelling of the oceans in this module provides a demonstration of how the techniques developed in second year modules on fluid dynamics and differential equations can be used to explain some interesting phenomena in the real physical world. The module begins with a discussion of the effects of rotation in fluid flows. The dynamics of large scale ocean circulation is discussed including the development of ocean gyres and strong western boundary currents. The thermal structure associated with these flows is examined. These large scale currents are responsible for the variation in climate between land on the eastern and western side of major ocean basins. The dynamics of equatorial waves are examined. Such waves are intimately linked with the El Niño phenomena which affects the climate throughout the globe.

4. Recommended texts: There are no set textbooks for the module. However there are a number of useful books for background reading and reference in library (as both hard copy and e-books). The following is particularly recommended:

G Vallis "Atmospheric and Oceanic Fluid Dynamics", (Cambridge)

5. Lecture Contents:

Introduction: Equations of motion. Rotation. Geostrophic flow. Rossby and Ekman numbers. Geostrophic and hydrostatic balances. Stratification and its impact on flows. The Sverdrup relation. **(5 lectures)**

Abyssal Circulation. Stommel Arons Theory and circlulation of deep and bottom waters. **(3 lectures)**

Gyre models: Wind-driven ocean circulation. Surface Ekman layer. Ekman transport and suction. Sverdrup interior circulation. Stommel boundary layer. **(7 lectures)**

Southern Ocean Dynamics: Antarctic Circumpolar Current. Steady circulation with topography. Unsteady circulation without topography. Ertel's theorem. **(6 lectures)**

Internal waves: Coastal and equatorial Kelvin Waves. Rossby Waves. **(6 lectures)**

Equatorial dynamics: Kelvin and Rossby waves. **(6 lectures)**

6. Advanced Topic, Flow instability
Instability of parallel shear flow. Conditions for instability. Baroclinic Instability. **(4 lectures)**

MTH-ME58: Semigroup Theory with Advanced Topics

1. Introduction: This module is an introduction to Semigroup Theory. Semigroups are algebraic objects which generalize groups (seen in Algebra 2C3Y). They are of interests because they arise naturally in many parts of mathematics: whenever we are composing functions, multiplying matrices, or considering homomorphisms between objects, there are semigroups underlying our mathematics.

2. Hours, Credits and Assessment: A 20 UCU module of 37 lectures, supported by office hours. Assessment is based on coursework (20%) and examination (80%).

3. Overview: This module is concerned with the study of a class of algebraic objects called semigroups. A semigroup is an algebraic structure consisting of a set together with an associative binary operation. For example, every group is a semigroup, but the converse is far from being true. This course will cover the fundamentals of semigroup theory, with a focus on studying the structure of semigroups by looking at their ideals, and at the groups that embed into them.

4. Recommended literature and references: The main reference is

J. M. Howie. *Fundamentals of semigroup theory*, Academic Press [Harcourt Brace Jovanovich Publishers], London, 1995. L.M.S. Monographs, No. 7.

Other useful books include:

A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups. Vol. I & II, Mathematical Surveys*, No. 7. American Mathematical Society, Providence, R.I. 1961.

G. Lallement, *Semigroups and Combinatorial Applications*, John Wiley & Sons, New York, 1979.

M. V. Lawson, *Inverse Semigroups: The Theory of Partial Symmetries*, World Scientific, 1998.

J. Rhodes & B. Steinberg. *The q-theory of finite semigroups*. Springer Monographs in Mathematics. Springer, New York, 2009.

5. Lecture Contents: Basic definitions and examples of semigroups and monoids: transformation semigroups, partial transformations, semigroup of binary relations, symmetric inverse semigroup, full linear monoid, bands, rectangular bands, semilattices. Subsemigroups, homomorphisms and congruences; First Isomorphism Theorem; Cayley's Theorem; direct products of semigroups. **(7 lectures)**

Ideals and Rees congruences; principal left and right ideals; Green's relations; Green's Lemma; maximal subgroups of semigroups; von Neumann regularity; semigroup inverses; D-class structure and egg-box diagrams; partially ordered sets of R, L and J-classes. **(8 lectures)**

Simple and 0-simple semigroups; principal factors; completely simple and completely 0-simple semigroups; the Rees matrix construction; the Rees-Sushkevich Theorem. **(7 lectures)**

Inverse semigroups; semilattices; Brandt semigroups; bicyclic monoid; symmetric inverse semigroup and Vagner-Preston Theorem, Clifford monoids, Bruck-Reilly extensions. **(6 lectures)**

Free semigroups; presenting semigroups with generators and relations; free groups; free inverse semigroups and Munn trees. **(5 lectures)**

Advanced Topic: Finiteness conditions of infinite semigroups: finite generation/presentability, local finiteness, residual finiteness. **(4 lectures)**

MTH-ME60: INTRODUCTION TO NUMERICAL ANALYSIS WITH ADVANCED TOPICS

1. Introduction: This is an introductory course in numerical analysis that will cover approximating a function and it's derivative numerically. Further topics will include the numerical solution to boundary and initial value problems, numerical integration and nonlinear equations. Advanced topics will include an introduction to the numerical solution to hyperbolic partial differential equations.

2. Timetable Hours, Credits, Assessments: 20 credits(UCU) assessed by a three-hour examination (80%) and by coursework (20%). This module is mainly lecture based with two (required) scheduled lab sessions.

3. Overview: The modelling of many physical phenomena involve equations that may not have closed form solutions. For example, blood flow through the heart or turbulence in the wake of an aircraft. For these examples we resort to numerical simulations. This module is designed to introduce the basics of numerical simulation used in more complex modelling.

4. Recommended literature and references (The notes taken in lectures are intended to be complete and self-contained):

R. Burden and J.D. Faires, *Numerical Analysis,* Brooks/Cole, 2010.
A. Quarteroni, F. Saleri and P. Gervasio, *Scientific Computing with MATLAB and Octave,* Springer, 2010.
E. Isaacson and H.B. Keller*, Analysis of Numerical Methods,* Dover, 2003.

5. Lecture Contents:

Introduction and overview: Motivation and review of important theorems.

(2 lectures)

A. Solving to Nonlinear equations: Bisection method, Fixed point iteration and the Newton-Raphson scheme. Comparison of the effectiveness of these methods and the expected convergence.

(4 lectures)

B. Approximating functions and derivatives using Taylor series and Interpolation and their expected convergence rates.

(4 lectures)

C. Numerical integration: Midpoint, integral and trapezoidal rule. Obtaining higher accuracy through Gauss quadrature. Convergence

(4 lectures)

D. Numerical ODEs and Finite Difference schemes: Single-step schemes (Euler, Modified Euler, Trapezoidal). Local truncation and round-off error. Consistency and stability. Aspects of Linear Algebra that are important in computing

(~10 lectures)

E. Further Advanced topics: Issues important in solving hyperbolic partial differential equations.

MTH-ME77 : Financial Mathematics with Advanced Topics

1. Introduction: This module is primarily concerned with the valuation of certain financial instruments known as derivatives. It has great application to finance and banking. The module assumes some knowledge of differential equations and MTH-2C4Y Differential Equations and Algorithms, or equivalent, is a prerequisite. Some of the mathematical modelling is stochastic but no prior knowledge of probability or statistics is assumed. Neither is any previous background in finance theory necessary.

2. Timetable Hours, Credits, Assessments: This module is of 20 UCU and is taught in the Autumn Semester by 33 lectures. Regular question sheets will be handed out, and assessment is 20% on course work and 80% by a three-hour examination.

3. Overview: The Mathematical Modelling of Finance is a relatively new area of application of mathematics yet it is expanding rapidly and has great importance for world financial markets. The module is concerned with the valuation of financial instruments known as derivatives. Introduction to options, futures and the no-arbitrage principle. Mathematical models for various types of options are discussed. We consider also Brownian motion, stochastic processes, stochastic calculus and Ito's lemma. The Black-Scholes partial differential equation is derived and its connection with diffusion brought out. It is applied and solved in various circumstances.

4. Recommended Reading:

Any of the following should prove useful, especially the first two:

1) Paul Wilmott introduces quantitative finance, 2nd ed.
   P. Wilmott. Wiley, 2007.

   This is also available as an e-book in the library.

2) The mathematics of financial derivatives
   P. Wilmott, S. Howison, J. Dewynne. CUP, 1995.

3) An elementary introduction to mathematical finance, 2nd ed.
   S.M. Ross. CUP, 2003.

4) A course in financial calculus
   A. Etheridge. CUP, 2002.

5) The concepts and practice of mathematical finance
   M. Joshi. CUP, 2003.

5. Lecture Contents:

Introduction to options, futures and the no-arbitrage principle - using this to calculate fair delivery prices for futures. **(6 lectures)**

Basic probability theory. **(2 lectures)**

The Binomial Model for random asset prices and option valuation. **(5 lectures)**

Basics of stochastic calculus and Ito's lemma. Brownian motion and geometric Brownian motion. Stochastic and deterministic processes. **(4 lectures)**

The Black-Scholes analysis. Derivation of the Black-Scholes partial differential equation and the assumptions behind it (i.e. Hedging and no-arbitrage). Formulating the mathematical problem, determining boundary conditions for option pricing problems. **(4 lectures)**

Solving the Black-Scholes equation. Connection with heat conduction equation, solution of the heat conduction equation - similarity solutions and the Dirac delta function. Derivation of the price of European options. European options with continuous dividend yield. **(5 lectures)**

The Greeks. Barrier options. More than one asset. American options. **(7 lectures)**

6. Advanced topic

The advanced topic will be stochastic interest rate models.

MTH-ME86: Slow Viscous Flow

1 Introduction

This module considers fluid flows in the limiting case of the flow being very viscous and/or slow. In this regime the viscosity of the fluid dominates its inertia. The full Navier–Stokes equations simplify somewhat, making solutions more straightforward. We will consider the general properties of such flows, various solution techniques, and a number of applications.

Prerequisite modules are MTH-1C24: Multivariable Calculus, MTH-2C2Y: Fluids and Solids, and MTH-2C4Y: Differential Equations and Algorithms. (The third-year Fluid Dynamics module MTH-3D41 is helpful, but not required.)

This course was previously given in 2011/12 and 2012/13 with the module code MTH-ME84. The syllabus for the current course will be essentially the same as that from 2012/13, but without the section on flow in porous media. There will also be an added coursework component. The content covered in 2012 was a slightly different again.

2 Hours, Credits and Assessment

This is a 20-credit module, comprising 22 lectures in the Spring Semester. A number of formative problem sheets will be provided. Some of the lecture hours will be used for seminars/tutorials on the problem sheets, and there will also be some optional interactive demonstrations. Assessment is by coursework (20%) and examination (80%).

3 Overview

The dimensionless parameter Re, known as the Reynolds number, gives the ratio of inertial to viscous effects in a fluid flow. It is usually defined by

$$Re = \frac{UL}{\nu},$$

where U is a typical velocity in the flow, L is a typical length-scale, and ν is the kinematic viscosity of the fluid. When $Re \ll 1$, inertial effects are negligible and the $\mathrm{D}\boldsymbol{u}/\mathrm{D}t$ term in the Navier–Stokes equations may be neglected. This simplifies the equations, making them linear (no $\boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{u}$ term) and instantaneous (no $\partial\boldsymbol{u}/\partial t$ term). These simplifications make solving low-Reynolds-number flow problems much easier than problems involving high-Reynolds-number flow.

This module will consider the circumstances under which the Reynolds number is small, examine the basic properties of low-Reynolds-number flows, present a number of solution techniques, and show how they can be applied to simple problems.

For example, we will see how solutions in simple geometries can be constructed using Papkovich–Neuber potentials (e.g. a flow around a sedimenting rigid sphere); examine flow in a thin layer using lubrication theory (e.g. modelling the spreading of a lava flow); and look at flow around a slender body (e.g. to model a swimming micro-organism).

4 Recommended Reading

The lectures will be self-contained and cover all of the examinable material. But for students who wish to read around the subject, or see an alternative treatment of certain topics, the following books may be useful.

Basic texts

- Acheson, D. J., *Elementary Fluid Dynamics*, OUP (1990). [Chapter 7 in particular.]
- Ockendon & Ockendon, *Viscous Flow*, CUP (1995). [Chapters 3 and 4 in particular.]

More advanced texts (some of which go well beyond this course)

- Batchelor, G. K., *An Introduction to Fluid Dynamics*, CUP (1967). [Chapter 4 in particular.]
- Happel & Brenner, *Low Reynolds Number Hydrodynamics*, Springer (1983).
- Pozrikidis, C., *Boundary Integral and Singularity Methods for Linearised Viscous Flow*, CUP (1992).
- Kim & Karrila, *Microhydrodynamics*, Dover (1991).

5 Lecture Content

- **Introduction**

  Introduction to low-Reynolds-number flow, the Stokes equations and boundary conditions. Basic properties, uniqueness theorem, reciprocal theorem, minimum dissipation theorem. Oscillating Couette flow and Poiseuille flow. [8 lectures]

- **Fundamental solutions and potentials**

  Solution using potentials. Papkovich–Neuber potentials, flow past a rigid sphere. Boundary integrals and the multi-pole expansion. [5 lectures]

- **Slender-body theory**

  Basic derivation. Applications to sedimenting slender objects and swimming microorganisms [4 lectures]

- **Lubrication theory**

  Basic derivation. Application to squeeze-film flows and gravity currents. Similarity solutions. [5 lectures]

MTH-ME90: Model Theory 2014/15

1. Introduction: Model Theory is a branch of mathematical logic that studies the connection between mathematical structures and their formal descriptions, or theories. It has deep and surprising applications in many branches of mathematics including number theory, algebra, and analytic geometry. One of the central questions which this course will address is when a theory has exactly one model. In most cases a theory is ambiguous in that it has many different models, but this apparent defect is in fact a rich source of ideas in Model Theory.

2. Hours, Credits and Assessment: The module is a 20 UCU unit of 22 lectures in the Spring Semester. Assessment is by coursework (20%) and examination (80%).

3. Overview: Model theory is a branch of mathematical logic which seeks to study mathematics itself using the ideas of logic, specifically formal (first-order) languages. The idea is that by fixing a specific formal language, although one can say less about any particular mathematical object, one can understand completely what can be said in that language. The set of all things we can say about a structure M is called the theory of M. Conversely, if T is some theory, then a structure which satisfies T is called a model of T. It turns out that, in most cases, there are many models of a given theory T, and the study of the class of models and the relationships between them is very rich. We can even learn a lot about the structure we first started with by looking at the other models of the same theory.

The main technique in model theory is the compactness theorem, which says roughly that a first-order language cannot tell the difference between arbitrarily large finite sets and infinite sets. We will show how this simple theorem has a myriad of consequences.

4. Recommended literature and references:

Katrin Tent and Martin Ziegler: **A Course in Model Theory**, CUP 2012
David Marker: **Model Theory: An Introduction**, Springer
Wilfrid Hodges: **A Shorter Model Theory**, CUP

Tent and Ziegler's new book has quickly replaced David Marker's book as the standard graduate textbook on the subject, although they cover slightly different material. Hodges has a lot more explanations, and explains the ideas behind the details in a way that the other books do not.

5. Lecture Contents:

Languages and Structures: terms, formulas, structures, satisfaction	**(2 lectures)**
Automorphisms and definable sets	**(2 lectures)**
Theories and compactness: examples including groups, vector spaces, Peano Arithmetic	**(4 lectures)**
Axiomatizable classes: finite and infinite models, examples including fields	**(2 lectures)**
Elementary embeddings: Lowenheim-Skolem theorems.	**(3 lectures)**
Categoricity and completeness: back-and-forth method	**(3 lectures)**
Types and their realisations: examples including prime factorisation, infinitesimals	**(4 lectures)**
Countable models and the Ryll-Nardzewski theorem	**(2 lectures)**
Algebraic applications	**(2 lectures)**