# MTHA5001Y: Analysis

**1. Introduction**: This module continues the study of the analysis of functions started in the first year. The first part gives a unified treatment of the basic properties of real and complex functions, and discusses the impact of the geometry of the complex plane on complex analytic functions. The second part introduces complex integration and exposes the remarkable rigidity that the property of differentiability imposes on a complex function. The material is of central importance to both pure and applied mathematicians.

**2. Hours, Credits and Assessment**: The module is a compulsory (MTH students) 20 UCU unit of 40 lectures, half in the Autumn Semester and half in the Spring Semester. Support teaching is via seminars. Assessment is by coursework (20%) and an examination (80%).

**3. Overview**: The study of Analysis is of central importance to Mathematics, in particular underpinning all of calculus. Complex Analysis, especially the concept of a path integral, was primarily developed by Cauchy in the early 19th century (although under restrictive assumptions), and further contributions were made by Liouville, Laurent and Riemann. There followed work to relax assumptions and widen the applicability of results, which had profound implications in understanding the geometry of sets of reals (Bolzano, Weierstrass, Cantor), as well as work on function theory. Weierstrass, in particular, developed a systematic theory of complex functions, though the use of complex variable techniques was already widespread among 19th century mathematicians, physicists, and engineers (and many scientists in this era combined these roles).

This module continues the study from the first year in two ways: first, functions of a complex variable are treated alongside real functions; second, the notion of convergence is strengthened to uniform convergence, and this allows us to include functions defined by power series, such as the exponential and trigonometric functions, in our setting. The module then goes on to look at how the topology of the complex plane puts an extra rigidity on the complex theory, and sees how the theory of integration has some surprising implications and applications to problems in real analysis which are not solvable using only real analysis.

**4. Recommended literature and references**:

Books for Real Analysis:
Kopp, P.E., Analysis, Arnold. [QA303 KOP in short loan collection]
Protter, M.H., & Morrey, C.B., A first course in real analysis, Springer-Verlag. [QA300 PRO]
Stirling, D.S.G., Mathematical analysis and proof, Albion. [QA300 STI]

Books for Complex Analysis:
Priestley, H.A., Introduction to Complex Analysis, Oxford University Press. [QA331 PRI].
Spiegel, M.R., Schaum's Outline of Theory and Problems of Complex Variables, Mc-Graw Hill. [QA331 SPI]
Stewart, I., & Tall, D., Complex Analysis, Cambridge University Press. [QA331 STE]

It is strongly recommended that students consult books in addition to consulting their lecture notes and course material.

**5. Lecture Contents**:

AUTUMN SEMESTER
The topics will include:

The complex plane; the modulus as a measure of distance. (2 lectures)

Review of continuity for real and complex functions. (2 lectures)

Sequences of functions. Uniform and pointwise convergence for sequences of functions. (4 lectures)

Power series representing real and complex functions. Radius of convergence. (4 lectures)

Differentiability of real and complex functions. Pathological behaviour of real differentiable functions; Cauchy-Riemann equations. (4 lectures)

Differentiation of power series representing complex functions. (4 lectures)

SPRING SEMESTER
The topics will include:

Topology of the complex plane: open, closed and compact sets. (2 lectures)

Review of differentiablility of complex functions, holomorphic functions, the Cauchy-Riemann equations, and elementary functions defined by power series. (2 lectures)

Paths, contours, connectedness, and simple-connectedness. (2 lectures)

Integration along a path, the Estimation Theorem, integration of power series, the Fundamental Theorem of Calculus. (3 lectures)

Cauchy's Theorem, Cauchy's Integral Formulae, Taylor's Theorem, Liouville's Theorem, the Identity Theorem, Laurent expansions. (5 lectures)

Singularities, residues and Cauchy's residue theorem, techniques for finding residues, summation of series, and other applications. (6 lectures)

## MTHA5002Y: Fluid Dynamics, Theory and Computation

**1. Introduction:** This module is at year long module and is an introduction to the dynamics of fluid motion, including the motion of both liquids and gases. The module follows on naturally from Multivariable Calculus (MTH-1C24), which is a prerequisite.

**2. Timetable Hours, Credits, Assessments**: This is a 20 credit module spread across both the Autumn and Spring semesters. Teaching in the autumn semester is done via lectures and seminars. Teaching in the spring semester is done via lectures, seminars, and computer lab classes. Assessment is by set coursework across the two semesters (20%) and by an end-of-year examination covering the material across both semesters (80%).

**3. Overview**:
Autumn: Hydrodynamics has been used as the topic to introduce mathematical modelling for many years in UK. It describes a variety of situations that can be described mathematically and for which the solutions are realistic. We work up to solving the partial differential equations needed to describe frictionless flows in pipes and channels, water-flow over weirs in rivers. The theory ends with applications to the theory of water waves on the surface of lakes and oceans.

Spring: The first ten lectures cover computational fluid dynamics, and the second ten lectures cover aerodynamics. Numerical computation has emerged over the last 50 years or so as an invaluable tool for studying fluid motion, from computing simple particle trajectories to studying highly complex turbulent motions. In many cases, progress cannot be made analytically and computation is a necessity. The second ten lectures cover analytical methods in hydrodynamics and their applications to vortex motions and aerofoils.

**4. Recommended Reading**:

Autumn:
The two recommended text books are D.J. Acheson 'Elementary Fluid Dynamics' (Oxford UP) and A. Paterson 'A First Course in Fluid Dynamics', (Cambridge UP). A useful book of pictures of fluid flows, is edited by M. Samimy 'A Gallery of Fluid Motion' (Cambridge UP). Further physical background is described by D. J. Tritton in his 'Physical Fluid Dynamics' (Oxford UP).
Spring:
Pozrikidis, C. Fluid Dynamics: Theory, Computation, and Numerical Simulation. Springer.

Acheson, D.J. Elementary Fluid Dynamics. Oxford UP.

**5. Lecture Contents**:

Autumn:
(1) Introduction: Examples of flows in science and nature. Mathematical background -- revision of vector calculus. The Physical background -- liquids, gases and fluids. Pressure, density, hydrostatics and Archimedes Principle with full or partial immersion. (3 lectures)

(2) Kinematics: Velocity fields, particle paths, streamlines, streaklines. Material derivative. Mass continuity: Compressibility and incompressibility. (4 lectures)

(3) One-Dimensional Flows: Pipes. Open channel flows: flow over a weir, under a sluice gate, and in a hydraulic jump (e.g. Severn Bore) including energy loss. (3 lectures)

(4) Dynamics: Euler's equations and boundary conditions. Bernoulli's equation for unsteady flow. Vorticity. Velocity potentials for irrotational flows. (3 lectures)

(5) Two-dimensional Flows: Stream function -- streamlines. Line-vortices and line-sinks. (2 lectures)

(6) Vorticity and Circulation: Kelvin's Circulation Theorem; Helmholtz's Vortex Theorems. The tornado. Motion of vortices. (2 lectures)

(7) Water Waves: Linearised equations for waves on water of constant depth. Simple solutions periodic in space and time. Wave speed as a function of wavelength in deep or shallow water. Group velocity and wave groups. (3 lectures)

Spring:
Lectures 1-10: Computational Fluid Dynamics

Revision of basic ideas, including steady/unsteady flow, particle paths, and streamlines. One dimensional flows: Euler’s method (algorithm and error analysis); Runge-Kutta methods; the trapezium rule method. Flow in two and three dimensions. Consistency, convergence and stability. Practical computation of particle paths using interpolation methods.

Lectures 11-20: Irrotational and incompressible flows

Applications of Hydrodynamics. Complex velocity potential: sources, sinks and dipoles. Line vortices: circulation, motion of vortices. Schwartz Reflection Principle. Conformal mappings. Joukowski aerofoils. Blasius’s formula.

Computer Lab Classes (Spring)

Class 1: Introduction to Matlab
Class 2: Numerical Integration of ODEs and visualization
Class 3: Project work
Class 4: Project work

# MTHA5003Y: Algebra

**Hours, Credits and Assessment:** This module has 20 UCU and runs over two semesters. The Autumn Semester part is on Group Theory while the Spring Semester part is on Ring Theory. There are seminars and support classes. Assessment is by examination (80%) and coursework (20%) via assessed homework.

**Overview:** This module introduces the theory of groups and rings, which together with vector spaces are the most important algebraic structures. They appear in many branches of mathematics and are fundamental examples of axiomatic systems. The development of the theory is methodical and self-contained, and encourages working formally in an axiomatic environment.

At the heart of group theory in the first semester is the study of geometric transformations and symmetry. The module remains close to such fundamental mathematical whilst also introducing a theory with enough generality to be applied elsewhere. This includes the notion of factor groups and basic structural results such as Lagrange's theorem and the isomorphism theorems. We shall also revisit vector spaces which are of course also groups.

In the second semester we introduce rings, using the Integers as a model. The subsequent theory is developed with a variety of examples, giving new insights into familiar concepts such as substitution and factorisation. In contrast to the first section of the module, a commutative setting is soon adopted. Important examples of commutative rings are fields and domains. New constructions of fields are introduced using quotients of rings modulo maximal ideals. The concepts of divisibility and factorisation are tackled in domains.

**Recommended literature and references:** The following should be in the Library, which also has numerous other texts covering the course material. Much can be found on the internet, and second hand copies of books are available cheaply online.

Joseph J. Rotman, "A first course in abstract algebra"

R.B.J.T. Allenby, "Rings, Fields and Groups: Introduction to Abstract Algebra"

J.B. Fraleigh, "A first course in abstract algebra"

Group Theory wikipedia: http://groupprops.subwiki.org/wiki/Main_Page

**Lecture Contents:**

**Groups (20 Lectures)**

i) Groups, subgroups, orders of elements. Examples, such as symmetric groups, general linear groups, and dihedral groups

ii) Cosets, Lagrange's Theorem, consequences

iii) Homomorphisms, normal subgroups, with example

iv) Quotient groups and the first isomorphism theorem.

v) Vector spaces, homomorphisms and quotient spaces

vi) Group actions, time permitting

**Rings (20 Lectures)**

i) Basic properties of Rings and Integral Domains

ii) Polynomials

iii) Euclidean Domains

iv) Unique Factorization Domains

v) Ideals and Principal Ideal Domains

vi) Quotient Rings, Ring Homomorphisms and the First Isomorphism Theorem

vii) Polynomials and Matrices (time permitting)

## MTHA5004Y: Differential Equations and Applied Methods

**1. Introduction:** This year long unit is in the mainstream of methods teaching for degree programmes in mathematics. It studies important methods and differential equations that have many applications.

**2. Hours, Credits and Assessment:** This unit is of 20 UCU and is taught throughout the year means of 40 lectures, supported by 6 seminars and 6 Problem Classes. Assessment is by set regular coursework and a final examination.

**3. Overview:**

**Autumn:** Ordinary differential equations with variable coefficients are solved by means of infinite series. The Legendre and Bessel differential equations are of particular importance especially for Laplace's partial differential equation. Fourier Series and Separation of Variables together provide a basic technique used in level 2 & 3 applied mathematics units, for the solution of Linear Partial Differential Equations.

**Spring:** Fourier transforms are a continuous analogue of Fourer series for use with non-periodic functions. The course will study their basic properties and show how these can be used to find inverse transforms. Students will also learn how Fourier transforms can be used to solve certain classes of intergral equations, ODEs and PDEs. The method of characteristics is a technique that can be used to solve certain types of partial differential equation. Students will learn how to classify PDEs, and how to apply the method of characteristics to solve first-order semi-linear and quasi-linear equations. Second-order hyperbolic PDEs will also be considered briefly. Dynamical Systems: equilibrium points and their stability; the phase plane; theory and applications.

**4. Recommended literature and references:**
**Autumn:** A full coverage of the unit and much further reading are offered by "Advanced Engineering Mathematics" by E. Kreyszig and "Elementary Differential Equations and Boundary Value Problems" by W.E. Boyce and R.C. Diprima (both published by Wiley).
**Spring:**

- *Mathematical Methods for Physics and Engineering (3rd edition). K. F. Riley, M. P. Hobson, S. J. Bence, Cambridge University Press (2006). ISBN 0-521-67971-0*
- *Applied Partial Differential Equations*. J. R. Ockendon, S. D. Howison, A. A. Lacey and A. B. Movchan, Oxford University Press (2003). ISBN 0-19-853243-1.
- *Nonlinear Dynamics and Chaos*. S H Strogatz (Addison-Wesley)

**5. Lecture Contents:**

**Autumn:**

**A. Differential Equations, Special Functions**

Linear ordinary differential equations on the real line. Linear dependence; the Wronskian, Liouville/Abel formula. The complete solution, method of variation of parameters. **(4 Lectures)**

Second-order linear equations on the real line. Series solutions. Singular points and the method of Frobenius. The indicial equation; equal roots and roots differing by an integer. Legendre polynomials and Bessel functions. **(6 Lectures)**

**B. Fourier Series:**

Fourier series for $2\pi$ periodic functions. Orthogonality integrals & the selection of Fourier coefficients. Odd and even functions. Fourier's Theorem stated. Extension to Fourier series of functions of arbitrary period. Functions defined on a finite interval. **(4 lectures)**

**C. Elementary partial differential equations**

**Separation of variables:** Derivation of the Heat Equation and Wave Equation. Boundary and initial conditions. Separation of Variables Method in Cartesian coordinates.

**Separation for non-rectangular regions:** Solutions to Laplace's equation in cylindrical & spherical polar co-ordinates. Separation in the axisymmetric case and the emergence of polynomial coefficient ordinary differential equations i.e. Bessel and Legendre equations.

**(6 lectures)**

**Spring:**

**D. Fourier Transforms**

Definition and basic properties of the Fourier transform
Inversion and convolution theorems
Applications to integral equations, PDEs, and ODEs (**5 lectures**)

**E. Method of Characteristics**

Classification of first-order PDEs in two variables (linear, semi-linear, quasi-linear), chain rule for differentiation along a parametric curve in the plane.
Solution of first-order scalar PDEs using the Method of Characteristics
Consideration of 2nd-order PDEs, and classification into elliptic, parabolic, and hyperbolic types. (**5 lectures**)

**F. Dynamical Systems**

1D discrete and continuous dynamical systems. Fixed point, stability, phase portrait. Bifurcations: saddle-node, transcritical, pitchfork. (**3 lectures**)
Planar continuous systems. General first order systems in two variables. Classification of equilibrium points. Conservative systems. Periodic orbits. Limit cycles. Gradient systems, Liapunov functions. Hopf bifurcations. (**4 lectures**)
Chaos. Lorenz equation (attractor, strange attractor). Logistic map (periodic-doubling, transition to chaos). (**3 lectures**)

## MTHA5005Y : Mathematics project

**1. Introduction:** This module gives the students the opportunity to work on a project of their own choosing.

**2. Timetable, Hours, Credits, Assessments:** The module is a 20 UCU year long module with 75% of the marks given for the written report and 25% given for the poster session.

**3. Overview:** The students each pick a topic from a list of subjects offered by members of faculty. Each project will give a suggested reading list, but students are encouraged to find new sources for themselves. The students will each write a project report which is up to 15 pages long and which will be submitted towards the end of the semester. They will also design an A1 poster on the project. These posters will be displayed at a poster session where the students will have the opportunity to discuss their projects and posters with members of faculty. There will be lectures on how to write the project and design the poster, and the students will have four meetings with their supervisor. The mathematics in the project will not be taught.

# MTHF5TopicA (2014/15): Elementary number theory

**Overview:** For a long time mathematicans have been trying to understand the behaviour of the integers, and many areas of pure mathematics have been born out of this quest. Elementary number theory may roughly be described as the study of the most basic properties of the integers, using minimal prerequisites from other areas of maths. In addition to its central, motivational role in pure mathematics, concepts and techniques from elementary number theory are constantly being utilised around us in our increasingly technology-dependent society. The course will provide you with a good understanding of the basic concepts of elementary number theory, and will make you aware of its hugely important real world applications, as well as its interaction with other areas of mathematics.

**Assessment:** 20% coursework (deadline Thursday of week 7), 80% written exam.

There are no prerequisites for this course, and there will be 15 lectures, divided into three chapters:

**1: Prime numbers**

The Euclidian algorithm will be used to prove the fundamental theorem of arithmetic, which describes how numbers factorise into primes. We will then briefly look at the distribution of primes amongst the integers, and give the statement of the prime number theorem.

**2: Congruence modulo $n$**

The concept of modular arithmetic will be introduced. We will prove the Chinese remainder theorem, look at Euler's totient function, and then study the multiplicative group modulo $n$. Finally we will apply what we have just learned by looking at one of elementary number theory's most spectacular 'real-world' applications, public-key cryptograpy.

**3: Quadratic reciprocity**

We will prove Gauss's law of quadratic reciprocity, which describes a highly non-trivial relationship between primes. Finally we will apply quadratic reciprocity to solve some Diophantine problems, including when a prime may be expressed as a sum of two squares.

The course is self contained and lecture notes will be provided, but for a second opinion you could try:

- William Stein's (free) online lecture notes: http://wstein.org/ent/ent.pdf
- *'Elementary number theory'* by Burton.

## MTHF5TopicB (2014/15): Quantum Mechanics

**Overview:** In classical mechanics, a physical system is described in terms of particles moving with a particular linear momentum. Other phenomena such as the transmission of light are described in terms of the propagation of electromagnetic waves. In the $20^{th}$ century it became clear that some physical observations can not be explained in such terms – for example the formation of fringe patterns due to the scattering of light through two slits. The concept of a photon having both particle and wave-like properties is at the heart of Quantum Mechanics. In this unit the emphasis is on detailed mathematical study of simplified model systems rather than broad descriptions of quantum phenomena. The main mathematical topics from Year One mathematics modules that this module builds on are differential equations and vector calculus (definitions of grad etc).

**Recommended Reading:** The module is self-contained.
Two recommended books for additional reading are:

"Quantum Mechanics" by Alistair Rae (Taylor & Francis)
"Introduction to Quantum Mechanics" by A.C. Phillips (Wiley).

Both these books go beyond the scope of the unit. Lecture notes will be provided online after each section has been completed.

**Lecture Contents:**

**Introduction:** Photons, wave-particle duality, uncertainty principle, basic wave theory **(2 lectures)**

**One-dimensional Schrodinger Equation:** Motivation, position and momentum probabilities.
Energy levels. Hamiltonian operators. Orthogonality of eigensolutions. **(6 lectures)**

**Square wells and barriers:** Eigensolutions for potential well.
Transmission and reflection coefficients at barrier. Tunnelling phenomena. **(4 lectures)**

**Harmonic Oscillator:** Eigensolutions and Hermite polynomials.
Application to physical systems. **(2 lectures)**

**Three-dimensional Schrodinger Equation:** Application to hydrogen atoms. **(1 lectures)**

**(NB** Numbers of lectures for each topic are approximate.**)**

## MTHF5TopicC (2014/15): Combinatorics

**Overview:** Combinatorics is one of the most applicable and accessible part of mathematics, yet it is also full of challenging problems. We shall cover many basic combinatorial concepts, such as binomial coefficients, counting arguments, and Ramsey theory.

In the first half of the course, we will discuss some enumerative combinatorics, that is, ways of counting objects. We will look at binomial coefficients, Stirling's Formula, the inclusion-exclusion formula and properties of partitions. For example, how many ways are there of dividing a set of n objects into k non-empty subsets? If 20 students each hand in a piece of work and the work is then randomly shuffled and distributed back to the students, what is the probability that no student receives his or her own work?

The second half of the course will look at colourings, in particular Ramsey Theory. For example, for a group of people, draw a blue line between a pair of people if they know each other and a red line if not. We shall prove that in any group of 6 people there are either 3 people who all know each other or there are 3 people, none of whom know each other (or both) - that is, we get a red triangle or a blue triangle. How many people would we have to choose in order to guarantee that there are 4 people who all know each other or 4 people, none of whom know each other (or both)? What if we replace 4 by an arbitrary integer n?

At the end of the course we will look at Ramsey theory for infinite sets.

**Recommended Reading:** The module is self-contained. Lecture notes will be provided. The following books will be useful for supplementary reading.


1. Miklos Bona, A Walk Through Combinatorics, 3rd ed. World Scientific, 2011

2. Peter J. Cameron, Combinatorics: Topics, Techniques, Algorithms, CUP 1996.

3. J.H. van Lint and R.M.Wilson, A Course in Combinatorics, CUP 1992.

3. Bela Bollobas, Combinatorics, CUP 1986.

4. Alan Tucker, Applied Combinatorics, Second Ed., Wiley 1984.

## MTHF5TopicD (2014/15) : Mathematical Modelling

**Overview:** Mathematical modelling is concerned with how to convert real problems, arising in industry or other sciences, into mathematical equations, and then solving them and using the results to better understand, or make predictions about, the original problem.

The course will start with a brief overview of the topic and look at the technique of non-dimensionalising, a method for simplifying any model and identifying what features are important.

We will then consider models from various sources, focusing how the model is developed and what the solutions tell us about the problem. The solutions will be found using techniques you see in other modules, such as solutions of differential equations.

Topics considered will include traffic flow, looking at both individual car based models and continuum models, and population dynamics, examining how numbers of an individual species grow and how different species interact. Further examples may be drawn from industry, biology or other sciences.

**Pre-requisites:** Ideally you should be studying MTH-2C6Y Differential Equations and Applied Methods (everybody in MTH!) which underpins the solution techniques used during the course. If not, we will consider nonlinear systems of ODEs and method of characteristics for first-order PDEs in particular, which you could pick-up during the course with a little further reading.

### Recommended Reading:

No one book covers the entire scope of the course but useful books include:

**R Haberman** "Mathematical Models" (which looks at some population models and an introduction to traffic flow)

**S Howison** "Practical Applied Mathematics" (An excellent and approachable book on mathematical modeling, that touches on traffic flow among many other applications)

**A Fowler** "Mathematical Models in the Applied Sciences" (not directly relevant to the module but a very good introduction to a wide variety of mathematical models)

### Main Contents:

**Introduction:** General approaches. Non-dimensionalisation.

**Traffic Flow:** Discrete models, modelling driver behaviour. Continuum models, velocity-density relationships. Traffic lights, junctions. (Considers differential equations and non-linear first-order PDEs/method of characteristic)

**Population Models:** Single species models. Competing species models, predator-prey systems. (Solving systems of linear and non-linear ODEs)

**Other:** As time allows we will consider a further one or two models, probably based on cooking, or dialysis machines (or something else...).