Total positivity:
A concept at the interface between algebra, analysis and combinatorics

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Key references:

1. Flajolet, Combinatorial aspects of continued fractions, Discrete Math. **32**, 125–161 (1980).

2. Viennot, *Une théorie combinatoire des polynômes orthogonaux généraux* (UQAM, 1983).

Positive semidefiniteness vs. total positivity

Compare the following two properties for matrices $A \in \mathbb{R}^{m\times n}$:

- A is called *positive semidefinite* if it is square ($m = n$), symmetric, and all its *principal* minors are nonnegative (i.e. $\det A_{II} \geq 0$ for all $I \subseteq [n]$).
- A is called *totally positive* if *all* its minors are nonnegative (i.e. $\det A_{IJ} \geq 0$ for all $I \subseteq [m]$ and $J \subseteq [n]$).

From the point of view of general linear algebra:

- Positive semidefiniteness is *natural*: it is equivalent to the positive semidefiniteness of a quadratic form on a vector space, and hence is basis-independent.
- Total positivity is *unnatural*: it is grossly basis-dependent.

This talk is about the "unnatural" property of total positivity.

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What total positivity is *really* about:
Functions $F\colon\ S \times T \to R$ where

- S and T are *totally ordered* sets, and
- R is a *partially ordered commutative ring* (traditionally $R = \mathbb{R}$, but we will generalize this)

Some references on total positivity

The classics:

1. Gantmakher and Krein, Sur les matrices complètement non négatives et oscillatoires, Compositio Math. **4**, 445–476 (1937).
2. Gantmakher and Krein, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems* (2nd Russian edition, 1950; English translation by AMS, 2002).
3. Karlin, *Total Positivity* (Stanford UP, 1968).
4. Ando, Totally positive matrices, Lin. Alg. Appl. **90**, 165–219 (1987).

Two recent books:

1. Pinkus, *Totally Positive Matrices* (Cambridge UP, 2010).
2. Fallat and Johnson, *Totally Nonnegative Matrices* (Princeton UP, 2011).

Applications to combinatorics:

1. Brenti, Unimodal, log-concave and Pólya frequency sequences in combinatorics, Memoirs AMS **81**, no. 413 (1989).
2. Brenti, The applications of total positivity to combinatorics, and conversely. In: *Total Positivity and its Applications* (1996).
3. Skandera, Introductory notes on total positivity (2003).

Log-concavity and log-convexity in combinatorics

A sequence $(a_i)_{i \in I}$ of nonnegative real numbers (indexed by an interval $I \subset \mathbb{Z}$) is called

- *log-concave* if $a_{n-1}a_{n+1} \le a_n^2$ for all n
- *log-convex* if $a_{n-1}a_{n+1} \ge a_n^2$ for all n

Many important combinatorial sequences are log-concave (cf. Stanley 1989 review article) or log-convex.

For a triangular array $T_{n,k}$ $(0 \le k \le n)$, typically:

- "Horizontal sequences" (n fixed, k varying) are log-concave.
- "Vertical" sequence of row sums is log-convex.

Examples: Binomial coefficients, Stirling numbers of both kinds, Eulerian numbers, ...

Proofs can be combinatorial or analytic.

Strengthenings of log-concavity and log-convexity: Toeplitz- and Hankel-total positivity

To each two-sided-infinite sequence $\boldsymbol{a} = (a_k)_{k\in\mathbb{Z}}$ we associate the *Toeplitz matrix*

$$T_\infty(\boldsymbol{a}) \;=\; (a_{j-i})_{i,j\geq 0} \;=\; \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_{-1} & a_0 & a_1 & \cdots \\ a_{-2} & a_{-1} & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

If $\boldsymbol{a}$ is one-sided infinite $(a_0, a_1, \ldots)$ or finite $(a_0, a_1, \ldots, a_n)$, set all "missing" entries to zero.

- We say that the sequence $\boldsymbol{a}$ is *Toeplitz-totally positive* if the Toeplitz matrix $T_\infty(\boldsymbol{a})$ is totally positive. [Also called "Pólya frequency sequence".]
- This implies that the sequence is *log-concave*, but is much stronger.

To each one-sided-infinite sequence $\boldsymbol{a} = (a_k)_{k\geq 0}$ we associate the *Hankel matrix*

$$H_\infty(\boldsymbol{a}) \;=\; (a_{i+j})_{i,j\geq 0} \;=\; \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- We say that the sequence $\boldsymbol{a}$ is *Hankel-totally positive* if the Hankel matrix $H_\infty(\boldsymbol{a})$ is totally positive.
- This implies that the sequence is *log-convex*, but is much stronger.

Characterization of Toeplitz-total positivity

Aissen–Schoenberg–Whitney–Edrei theorem (1952–53):

1. Finite sequence $(a_0, a_1, \ldots, a_n)$ is Toeplitz-TP iff the polynomial $P(z) = \sum\limits_{k=0}^{n} a_k z^k$ has all its zeros in $(-\infty, 0]$.
2. One-sided infinite sequence $(a_0, a_1, \ldots)$ is Toeplitz-TP iff

$$\sum_{k=0}^{\infty} a_k z^k \;=\; C e^{\gamma z} \frac{\prod\limits_{i=1}^{\infty} (1 + \alpha_i z)}{\prod\limits_{i=1}^{\infty} (1 - \beta_i z)}$$

   in some neighborhood of $z = 0$, with $C, \gamma, \alpha_i, \beta_i \geq 0$ and $\sum\limits_i \alpha_i, \sum\limits_i \beta_i < \infty$.
3. Similar but more complicated representation for two-sided-infinite sequences.

Proofs of #2 and #3 rely on Nevanlinna theory of meromorphic functions.

Open problem: Find a more elementary proof.

See Brenti for many combinatorial applications of Toeplitz-total positivity.

Characterization of Hankel-total positivity

For a sequence $\boldsymbol{a} = (a_k)_{k\geq 0}$, define also the m-shifted Hankel matrix

$$H_\infty^{(m)}(\boldsymbol{a}) \;=\; (a_{i+j+m})_{i,j\geq 0} \;=\; \begin{pmatrix} a_m & a_{m+1} & a_{m+2} & \cdots & \\ a_{m+1} & a_{m+2} & a_{m+3} & \cdots & \\ a_{m+2} & a_{m+3} & a_{m+4} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Recall that the sequence $\boldsymbol{a}$ is *Hankel-totally positive* in case the Hankel matrix $H_\infty^{(0)}(\boldsymbol{a})$ is totally positive.

Fundamental result (Stieltjes 1894, Gantmakher–Krein 1937, ...):

For a sequence $\boldsymbol{a} = (a_k)_{k=0}^\infty$ of real numbers, the following are equivalent:

(a) $H_\infty^{(0)}(\boldsymbol{a})$ is totally positive.

(b) Both $H_\infty^{(0)}(\boldsymbol{a})$ and $H_\infty^{(1)}(\boldsymbol{a})$ are positive-semidefinite.

(c) There exists a positive measure μ on $[0,\infty)$ such that $a_k = \int x^k \, d\mu(x)$ for all $k \geq 0$.

[That is, $(a_k)_{k\geq 0}$ is a Stieltjes moment sequence.]

(d) There exist numbers $\alpha_0, \alpha_1, \ldots \geq 0$ such that

$$\sum_{k=0}^{\infty} a_k t^k \;=\; \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}$$

in the sense of formal power series.

[Steltjes-type continued fraction with nonnegative coefficients]

From numbers to polynomials
[or, From counting to counting-with-weights]

Some simple examples:

1. Counting subsets of $[n]$: $a_n = 2^n$

   Counting subsets of $[n]$ by cardinality: $P_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$

2. Counting partitions of $[n]$: $a_n = B_n$ (Bell number)

   Counting partitions of $[n]$ by number of blocks:
   $P_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$ (Bell polynomial)

3. Counting non-crossing partitions of $[n]$: $a_n = C_n$ (Catalan number)

   Counting non-crossing partitions of $[n]$ by number of blocks:
   $P_n(x) = \sum_{k=0}^{n} N(n,k)\, x^k$ (Narayana polynomial)

4. Counting permutations of $[n]$: $a_n = n!$

   Counting permutations of $[n]$ by number of cycles:
   $P_n(x) = \sum_{k=0}^{n} \left[{n \atop k} \right] x^k$

   Counting permutations of $[n]$ by number of descents:
   $P_n(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k$ (Eulerian polynomial)

An industry in combinatorics: q-Narayana polynomials, p,q-Bell polynomials, ...

Sequences and matrices of polynomials

- Consider sequences and matrices whose entries are *polynomials* with real coefficients in one or more indeterminates $\mathbf{x}$.
- $P \succeq 0$ means that P has nonnegative coefficients. ("coefficientwise partial order on the ring $\mathbb{R}[\mathbf{x}]$")
- More generally, consider sequences and matrices with entries in a *partially ordered commutative ring* R.

We say that a sequence $(a_i)_{i \in I}$ of nonnegative elements of R is

- *log-concave* if $a_{n-1}a_{n+1} - a_n^2 \le 0$ for all n
- *strongly log-concave* if $a_{k-1}a_{l+1} - a_k a_l \le 0$ for all $k \le l$
- *log-convex* if $a_{n-1}a_{n+1} - a_n^2 \ge 0$ for all n
- *strongly log-convex* if $a_{k-1}a_{l+1} - a_k a_l \ge 0$ for all $k \le l$

For sequences of *real* numbers,

- Strongly log-concave $\iff$ log-concave with no internal zeros.
- Strongly log-convex $\iff$ log-convex.

But on $\mathbb{R}[x]$ this is not so:

Example: The sequence (a_0, a_1, a_2, a_3) with

$$\begin{aligned} a_0 &= a_3 = 2 + x + 3x^2 \\ a_1 &= a_2 = 1 + 2x + 2x^2 \end{aligned}$$

is log-convex but not strongly log-convex.

We say that a matrix with entries in R is *totally positive* if every minor is nonnegative (in R).

Toeplitz (resp. Hankel) total positivity implies the *strong* log-concavity (resp. *strong* log-convexity) — but of course goes far beyond it.

Coefficientwise Hankel-total positivity for sequences of polynomials

Many interesting sequences of polynomials $(P_n(x))_{n\geq 0}$ have been proven in recent years to be coefficientwise (strongly) log-convex:

- Binomials $\sum\limits_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$ [trivial]
- Bell polynomials $B_n(x) = \sum\limits_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$
  (Liu–Wang 2007, Chen–Wang–Yang 2011)
- Narayana polynomials $N_n(x) = \sum\limits_{k=0}^{n} N(n,k)\, x^k$
  (Chen–Wang–Yang 2010)
- Narayana polynomials of type B: $W_n(x) = \sum\limits_{k=0}^{n} \binom{n}{k}^2 x^k$
  (Chen–Tang–Wang–Yang 2010)
- Eulerian polynomials $A_n(x) = \sum\limits_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k$
  (Liu–Wang 2007, Zhu 2013)

Might these sequences actually be coefficientwise Hankel-totally positive?

- In many cases I can prove that the answer is **yes**, by using the Flajolet–Viennot method of *continued fractions*.
- In several other cases I have strong **empirical evidence** that the answer is **yes**, but no proof.
- The continued-fraction approach gives a *sufficient but not necessary* condition for coefficientwise Hankel-total positivity.

The combinatorics of continued fractions (Flajolet 1980)

Let $\boldsymbol{a} = (a_n)_{n \geq 0}$ be a sequence of elements in a commutative ring R. We associate to $\boldsymbol{a}$ the formal power series

$$f(t) \;=\; \sum_{n=0}^{\infty} a_n t^n \;\in\; R[[t]]$$

We now consider two types of continued fractions:

- Continued fractions of Stieltjes type (S-type):

$$f(t) \;=\; \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \cdots}}}} \;,$$

  which we denote by $S(t; \boldsymbol{\alpha})$ where $\boldsymbol{\alpha} = (\alpha_n)_{n \geq 1}$.

- Continued fractions of Jacobi type (J-type):

$$f(t) \;=\; \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \gamma_2 t - \cfrac{\beta_3 t^2}{1 - \gamma_3 t - \cdots}}}} \;,$$

  which we denote by $J(t; \boldsymbol{\beta}, \boldsymbol{\gamma})$ where $\boldsymbol{\beta} = (\beta_n)_{n \geq 1}$ and $\boldsymbol{\gamma} = (\gamma_n)_{n \geq 0}$.

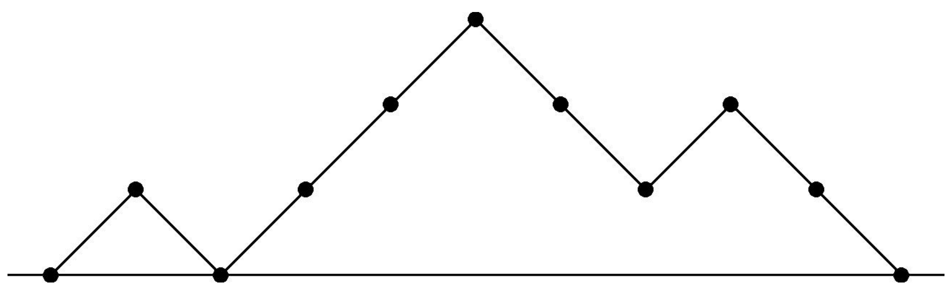
The combinatorics of Stieltjes-type continued fractions

Theorem (Flajolet 1980): As an identity in $\mathbb{Z}[\boldsymbol{\alpha}][[t]]$, we have

$$\cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}} \;=\; \sum_{n=0}^{\infty} S_n(\alpha_1, \ldots, \alpha_n)\, t^n$$

where $S_n(\alpha_1, \ldots, \alpha_n)$ is the generating polynomial for Dyck paths of length $2n$ in which each fall starting at height i gets weight α_i.

A **Dyck path** of length $2n$ is a path in the right quadrant $\mathbb{N} \times \mathbb{N}$ from $(0, 0)$ to $(2n, 0)$ using steps $(1, 1)$ ["rise"] and $(1, -1)$ ["fall"]:



A Dyck path of length $2n = 10$, which will get weight $\alpha_1^2 \alpha_2^2 \alpha_3$ in $S_5(\boldsymbol{\alpha})$.

$S_n(\boldsymbol{\alpha})$ is called the **Stieltjes–Rogers polynomial** of order n.

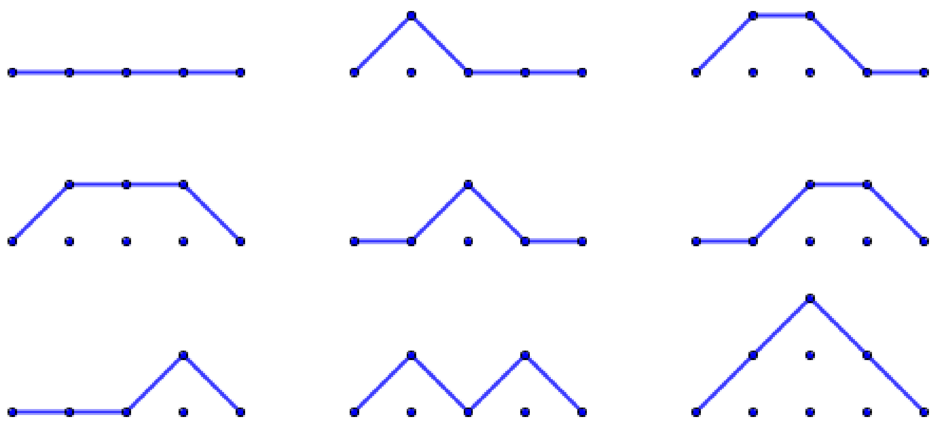
The combinatorics of Jacobi-type continued fractions

Theorem (Flajolet 1980): As an identity in $\mathbb{Z}[\boldsymbol{\beta}, \boldsymbol{\gamma}][[t]]$, we have

$$\cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \gamma_2 t - \cdots}}} \;=\; \sum_{n=0}^{\infty} J_n(\boldsymbol{\beta}, \boldsymbol{\gamma})\, t^n$$

where $J_n(\boldsymbol{\beta}, \boldsymbol{\gamma})$ is the generating polynomial for Motzkin paths of length n in which each level step at height i gets weight γ_i and each fall starting at height i gets weight β_i.

A **Motzkin path** of length n is a path in the right quadrant $\mathbb{N} \times \mathbb{N}$ from $(0, 0)$ to $(n, 0)$ using steps $(1, 1)$ ["rise"], $(1, -1)$ ["fall"] and $(1, 0)$ ["level"]:



All the Motzkin paths of length $n = 4$.

$$J_4(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \gamma_0^4 + 3\gamma_0^2\beta_1 + 2\gamma_0\gamma_1\beta_1 + \gamma_1^2\beta_1 + \beta_1^2 + \beta_1\beta_2$$

$J_n(\boldsymbol{\beta}, \boldsymbol{\gamma})$ is called the **Jacobi–Rogers polynomial** of order n.

Hankel matrix of Stieltjes–Rogers polynomials

Now form the infinite Hankel matrix corresponding to the sequence $\boldsymbol{S} = (S_n(\boldsymbol{\alpha}))_{n \ge 0}$ of Stieltjes–Rogers polynomials:

$$H_\infty(\boldsymbol{S}) \;=\; \bigl(S_{i+j}(\boldsymbol{\alpha})\bigr)_{i,j \ge 0}$$

And consider any minor of $H_\infty(\boldsymbol{S})$:

$$\Delta_{IJ}(\boldsymbol{S}) \;=\; \det H_{IJ}(\boldsymbol{S})$$

where $I = \{i_1, i_2, \ldots, i_k\}$ with $0 \le i_1 < i_2 < \ldots < i_k$
and $J = \{j_1, j_2, \ldots, j_k\}$ with $0 \le j_1 < j_2 < \ldots < j_k$

Theorem (Viennot 1983): The minor $\Delta_{IJ}(\boldsymbol{S})$ is the generating polynomial for families of *disjoint* Dyck paths $P_1, \ldots, P_k$ where path P_r starts at $(-2i_r, 0)$ and ends at $(2j_r, 0)$, in which each fall starting at height i gets weight α_i.

The proof uses the Karlin–McGregor–Lindström–Gessel–Viennot lemma on families of nonintersecting paths.

Corollary: The sequence $\boldsymbol{S} = (S_n(\boldsymbol{\alpha}))_{n \ge 0}$ is a Hankel-totally positive sequence in the polynomial ring $\mathbb{Z}[\boldsymbol{\alpha}]$ equipped with the coefficientwise partial order.

Now specialize $\boldsymbol{\alpha}$ to nonnegative elements in any partially ordered commutative ring:

Corollary: Let $\boldsymbol{\alpha} = (\alpha_n)_{n \ge 1}$ be a sequence of nonnegative elements in a partially ordered commutative ring R. Then $(S_n(\boldsymbol{\alpha}))_{n \ge 0}$ is a Hankel-totally positive sequence in R.

Hankel matrix of Stieltjes–Rogers polynomials (continued)

Can also get explicit formulae for the Hankel determinants $\Delta_n^{(m)}(\boldsymbol{S}) = \det H_n^{(m)}(\boldsymbol{S})$ for small m:

Theorem:

$$\Delta_n^{(0)}(\boldsymbol{S}) \;=\; (\alpha_1\alpha_2)^{n-1}(\alpha_3\alpha_4)^{n-2} \,\cdots\, (\alpha_{2n-3}\alpha_{2n-2})$$

$$\Delta_n^{(1)}(\boldsymbol{S}) \;=\; \alpha_1^n(\alpha_2\alpha_3)^{n-1}(\alpha_4\alpha_5)^{n-2} \,\cdots\, (\alpha_{2n-2}\alpha_{2n-1})$$

These formulae are classical in the theory of continued fractions, but Viennot 1983 gives a beautiful combinatorial interpretation.

See also Ishikawa–Tagawa–Zeng 2009 for extensions to $m = 2, 3$.

Hankel matrix of Jacobi–Rogers polynomials

What about J-type continued fractions?

As before, we form the Hankel matrix

$$H_\infty(\boldsymbol{J}) \;=\; \bigl(J_{i+j}(\boldsymbol{\beta},\boldsymbol{\gamma})\bigr)_{i,j\ge 0}$$

But the story is more complicated than for S-type fractions, because:

- The matrix $H_\infty(\boldsymbol{J})$ is *not* totally positive in $\mathbb{Z}[\boldsymbol{\beta},\boldsymbol{\gamma}]$.
- It is not even totally positive in $\mathbb{R}$ for all $\boldsymbol{\beta},\boldsymbol{\gamma} \ge 0$.
- Rather, the total positivity of $H_\infty(\boldsymbol{J})$ holds only when $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ satisfy suitable *inequalities*.

Form the infinite tridiagonal matrix

$$M_\infty(\boldsymbol{\beta},\boldsymbol{\gamma}) \;=\; \begin{pmatrix} \gamma_0 & 1 & 0 & 0 & \cdots \\ \beta_1 & \gamma_1 & 1 & 0 & \cdots \\ 0 & \beta_2 & \gamma_2 & 1 & \cdots \\ 0 & 0 & \beta_3 & \gamma_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Theorem: If $M_\infty(\boldsymbol{\beta},\boldsymbol{\gamma})$ is totally positive, then so is $H_\infty(\boldsymbol{J})$.

So we will need to test the tridiagonal matrix for total positivity.

Luckily, there is a simple criterion:

> A tridiagonal matrix is totally positive if and only if all its *off-diagonal elements* and all its *contiguous principal minors* are nonnegative.

This is classical for real-valued matrices; but the proof extends easily to matrices with values in a partially ordered commutative ring.

Finding Hankel-totally positive sequences of polynomials

A general strategy:

1. Start from a sequence $(c_n)_{n\ge 0}$ of positive real numbers that is a Stieltjes moment sequence, i.e. is Hankel-totally positive.

   [This property is easy to test empirically: just expand the generating series $\sum_{n=0}^{\infty} c_n t^n$ as an S-type continued fraction and test whether all coefficients α_i are ≥ 0.]

2. Refine this sequence somehow to a row-finite array $(c_{n,k})_{0\le k\le k_{\max}(n)}$ satisfying $\sum\limits_{k=0}^{k_{\max}(n)} c_{n,k} = c_n$;

   then define the polynomials $P_n(x) = \sum\limits_{k=0}^{k_{\max}(n)} c_{n,k}\, x^k$.

3. By construction, the sequence $(P_n(1))_{n\ge 0}$ is Hankel-totally positive; and if we are lucky, we will find that two successively stronger properties of Hankel-total positivity also hold:

   (a) For each real number $x \ge 0$, the sequence $(P_n(x))_{n\ge 0}$ of real numbers is Hankel-totally positive (i.e. is a Stieltjes moment sequence).

   (b) The sequence $(P_n(x))_{n\ge 0}$ of polynomials is coefficientwise Hankel-totally positive.

- Usually $(c_n)_{n\ge 0}$ will usually be a sequence of *positive integers* having some combinatorial interpretation, i.e. as the cardinality of some "naturally occurring" set $\mathcal{S}_n$.

- Then the $c_{n,k}$ will arise from the partition of $\mathcal{S}_n$ into disjoint subsets $\mathcal{S}_{n,k}$ according to some "natural" statistic $\kappa\colon \mathcal{S}_n \to \mathbb{N}$.

Some examples of combinatorial Stieltjes moment sequences

	$n=0$	1	2	3	4	5	6	Continued fraction α_{2k-1}	α_{2k}
Catalan numbers C_n	1	1	2	5	14	42	132	1	1
Central binomials $\binom{2n}{n}$	1	2	6	20	70	252	924	$\alpha_1 = 2$, all others 1	1
Bell numbers B_n	1	1	2	5	15	52	203	1	k
Irreducible Bell numbers IB_{n+1}	1	1	2	6	22	92	426	k	1
Factorials $n!$	1	1	2	6	24	120	720	k	k
Ordered Bell numbers OB_n	1	1	3	13	75	541	4683	k	$2k$
Odd semifactorials $(2n-1)!!$	1	1	3	15	105	945	10395	$2k-1$	$2k$
Even semifactorials $(2n)!!$	1	2	8	48	384	3840	46080	$2k$	$2k$
Genocchi medians H_{2n+1}	1	1	2	8	56	608	9440	k^2	k^2
Genocchi numbers G_{2n+2}	1	1	3	17	155	2073	38227	k^2	$k(k+1)$
Secant numbers E_{2n}	1	1	5	61	1385	50521	2702765	$(2k-1)^2$	$(2k)^2$
Tangent numbers E_{2n+1}	1	2	16	272	7936	353792	22368256	$(2k-1)(2k)$	$(2k)(2k+1)$

So our polynomial examples will divide naturally into "families": the Catalan family, the Bell family, the factorial family, etc.

Can also pursue this strategy in reverse:

- Find the S-type continued fraction for the generating series $\sum_{n=0}^{\infty} c_n t^n$.
- Generalize it by inserting one or more indeterminates $\mathbf{x}$.
- Try to compute the corresponding polynomials $P_n(\mathbf{x})$ and/or find a combinatorial interpretation for them.

Caveat:

- There also exist important combinatorial Stieltjes moment sequences that do *not* seem to have nice continued fractions.
- Some of them have polynomial refinements that are **empirically** Hankel-totally positive; but new methods will be needed to prove it!

Example 1: Narayana polynomials

- Narayana numbers $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ for $n \ge k \ge 1$ with convention $N(0,k) = \delta_{k0}$
- They refine Catalan numbers: $\sum_{k=0}^{n} N(n,k) = C_n$
- They count numerous objects of combinatorial interest:
  - Dyck paths of length $2n$ with k peaks
  - Non-crossing partitions of $[n]$ with k blocks
  - Non-nesting partitions of $[n]$ with k blocks
- Define Narayana polynomials $N_n(x) = \sum_{k=0}^{n} N(n,k)\, x^k$
- Define ordinary generating function $\mathcal{N}(t,x) = \sum_{n=0}^{\infty} t^n\, N_n(x)$
- Elementary "renewal" argument on Dyck paths implies

$$\mathcal{N} \;=\; \frac{1}{1 - tx - t(\mathcal{N} - 1)}$$

which can be rewritten as

$$\mathcal{N} \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - t\mathcal{N}}}$$

- Leads immediately to S-type continued fraction

$$\sum_{n=0}^{\infty} t^n\, N_n(x) \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{t}{1 - \cfrac{xt}{1 - \cfrac{t}{1 - \cdots}}}}}$$

with coefficients $\alpha_{2k-1} = x$, $\alpha_{2k} = 1$.

Narayana polynomials (continued)

Conclusions:

1. The sequence $\boldsymbol{N} = (N_n(x))_{n \geq 0}$ of Narayana polynomials is coefficientwise Hankel-totally positive. The minor $\Delta_{IJ}(\boldsymbol{N})$ counts families of disjoint Dyck paths as specified by Viennot 1983, with weights $\alpha_{2k-1} = x, \ \alpha_{2k} = 1$.

2. The first Hankel determinants $\Delta_n^{(m)}(\boldsymbol{N})$ are

$$\Delta_n^{(0)}(\boldsymbol{N}) = x^{n(n-1)/2}$$

$$\Delta_n^{(1)}(\boldsymbol{N}) = x^{n(n+1)/2}$$

Remarks:

1. The strong log-convexity was known previously (Chen–Wang–Yang 2010), but with a much more difficult proof.

2. The formula for $\Delta_n^{(0)}(\boldsymbol{N})$ was also known (Sivasubramanian 2010), by an explicit bijective argument.

Example 2: Bell polynomials

- Stirling number $\left\{ {n \atop k} \right\}$ = # of partitions of $[n]$ with k blocks
- Convention $\left\{ {0 \atop k} \right\} = \delta_{k0}$
- They refine Bell numbers: $\sum\limits_{k=0}^{n} \left\{ {n \atop k} \right\} = B_n$
- Define Bell polynomials $B_n(x) = \sum\limits_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$
- Define ordinary generating function $\mathcal{B}(t,x) = \sum\limits_{n=0}^{\infty} t^n \, B_n(x)$
- Flajolet (1980) expressed $\mathcal{B}(t,x)$ as a J-type continued fraction
- Can be transformed to an S-type continued fraction

$$\sum_{n=0}^{\infty} t^n \, B_n(x) \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{1t}{1 - \cfrac{xt}{1 - \cfrac{2t}{1 - \cdots}}}}}$$

with coefficients $\alpha_{2k-1} = x$, $\alpha_{2k} = k$.

Bell polynomials (continued)

Conclusions:

1. The sequence $\boldsymbol{B} = (B_n(x))_{n\geq 0}$ of Bell polynomials is coefficientwise Hankel-totally positive. The minor $\Delta_{IJ}(\boldsymbol{B})$ counts families of disjoint Dyck paths as specified by Viennot 1983, with weights $\alpha_{2k-1} = x,\ \ \alpha_{2k} = k$.

2. The first Hankel determinants $\Delta_n^{(m)}(\boldsymbol{B})$ are

$$\Delta_n^{(0)}(\boldsymbol{B}) \;=\; x^{n(n-1)/2} \prod_{i=1}^{n-1} i!$$

$$\Delta_n^{(1)}(\boldsymbol{B}) \;=\; x^{n(n+1)/2} \prod_{i=1}^{n-1} i!$$

Remarks:

1. The strong log-convexity was known previously (Chen–Wang–Yang 2011).
2. The formula for $\Delta_n^{(0)}(\boldsymbol{B})$ has also been known for a long time (Radoux 1979, Ehrenborg 2000).
3. For each real number $x \geq 0$, the sequence $(B_n(x))_{n=0}^{\infty}$ is the moment sequence for the Poisson distribution of expected value x:

$$B_n(x) \;=\; \sum_{k=0}^{\infty} k^n \left(e^{-x} \frac{x^k}{k!} \right)$$

Hence $(B_n(x))_{n=0}^{\infty}$ is a Hankel-totally positive sequence of real numbers. But the weights $e^{-x}x^k/k!$ here are not nonnegative elements of $\mathbb{R}[x]$ or $\mathbb{R}[[x]]$, so this approach cannot be used to prove the *coefficientwise* total positivity.

Example 3: Interpolating between Narayana and Bell

- Let $\pi = \{B_1, B_2, \ldots, B_k\}$ be a partition of $[n]$
- Associate to π a graph $\mathcal{G}_\pi$ with vertex set $[n]$ such that i, j are joined by an edge iff they are *consecutive* elements within the same block
- Always write an edge e of $\mathcal{G}_\pi$ as a pair (i, j) with $i < j$
- We say that edges $e_1 = (i_1, j_1)$ and $e_2 = (i_2, j_2)$ of $\mathcal{G}_\pi$ form
  - a *crossing* if $i_1 < i_2 < j_1 < j_2$
  - a *nesting* if $i_1 < i_2 < j_2 < j_1$
- We define $\mathrm{cr}(\pi)$ [resp. $\mathrm{ne}(\pi)$] to be number of crossings (resp. nestings) in π
- Write $|\pi| = k$ for the number of blocks in π
- Now define the three-variable polynomial

$$B_n(x, p, q) \;=\; \sum_{\pi \in \Pi_n} x^{|\pi|} p^{\mathrm{cr}(\pi)} q^{\mathrm{ne}(\pi)}$$

  with the convention $B_0(x, p, q) = 1$
- $B_n(x, 0, 1) = B_n(x, 1, 0) = N_n(x)$ and $B_n(x, 1, 1) = B_n(x)$, so this polynomial generalizes the Narayana and Bell polynomials.
- Kasraoui and Zeng (2006) have constructed an involution on Π_n that preserves the number of blocks (as well as some other properties) and exchanges the numbers of crossings and nestings; thus $B_n(x, p, q) = B_n(x, q, p)$.
- Define ordinary generating function $\mathcal{B}(t, x, p, q) = \sum_{n=0}^{\infty} t^n B_n(x, p, q)$

Interpolating between Narayana and Bell (continued)

- Kasraoui and Zeng (2006) have expressed $\mathcal{B}(t, x, p, q)$ as a J-type continued fraction
- Can be transformed to an S-type continued fraction

$$\sum_{n=0}^{\infty} t^n B_n(x,p,q) = \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{[1]_{p,q}t}{1 - \cfrac{xt}{1 - \cfrac{[2]_{p,q}t}{1 - \cdots}}}}}$$

with coefficients $\alpha_{2k-1} = x$, $\alpha_{2k} = [k]_{p,q}$, where

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}$$

Conclusions:

1. The sequence $\boldsymbol{B} = (B_n(x,p,q))_{n \geq 0}$ of three-variable polynomials is coefficientwise Hankel-totally positive. The minor $\Delta_{IJ}(\boldsymbol{B})$ counts families of disjoint Dyck paths as specified by Viennot 1983, with weights $\alpha_{2k-1} = x,\ \alpha_{2k} = [k]_{p,q}$.

2. The first Hankel determinants $\Delta_n^{(m)}(\boldsymbol{B})$ are

$$\Delta_n^{(0)}(\boldsymbol{B}) = x^{n(n-1)/2} \prod_{i=1}^{n-1} [i]_{p,q}!$$

$$\Delta_n^{(1)}(\boldsymbol{B}) = x^{n(n+1)/2} \prod_{i=1}^{n-1} [i]_{p,q}!$$

where

$$[n]_{p,q}! = \prod_{j=1}^{n} [j]_{p,q} \qquad (0.1)$$

Example 4: Eulerian polynomials

- Eulerian number $\left\langle {n \atop k} \right\rangle$ = # of permutations of $[n]$ with k descents
- Convention $\left\langle {0 \atop k} \right\rangle = \delta_{k0}$
- They obviously refine factorials: $\sum\limits_{k=0}^{n} \left\langle {n \atop k} \right\rangle = n!$
- Define Eulerian polynomials $A_n(x) = \sum\limits_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k$
- Define ordinary generating function $\mathcal{A}(t,x) = \sum\limits_{n=0}^{\infty} t^n A_n(x)$
- Flajolet (1980) expressed $\mathcal{A}(t,x)$ as a J-type continued fraction
- Can be transformed to an S-type continued fraction

$$\sum_{n=0}^{\infty} t^n A_n(x) = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{xt}{1 - \cfrac{2t}{1 - \cfrac{2xt}{1 - \cdots}}}}}$$

with coefficients $\alpha_{2k-1} = k$, $\alpha_{2k} = kx$.

Eulerian polynomials (continued)

Conclusions:

1. The sequence $\boldsymbol{A} = (A_n(x))_{n \ge 0}$ of Eulerian polynomials is coefficientwise Hankel-totally positive. The minor $\Delta_{IJ}(\boldsymbol{A})$ counts families of disjoint Dyck paths as specified by Viennot 1983, with weights $\alpha_{2k-1} = k,\ \ \alpha_{2k} = kx$.

2. The first Hankel determinants $\Delta_n^{(m)}(\boldsymbol{A})$ are

$$\Delta_n^{(0)}(\boldsymbol{B}) \;=\; x^{n(n-1)/2} \prod_{i=1}^{n-1} i!^2$$

$$\Delta_n^{(1)}(\boldsymbol{B}) \;=\; x^{n(n+1)/2} \prod_{i=1}^{n-1} i!^2$$

Remarks:

1. The (strong) log-convexity was known previously (Liu–Wang 2007, Zhu 2013).

2. The formula for $\Delta_n^{(0)}(\boldsymbol{A})$ was also known (Sivasubramanian 2010), by an explicit bijective argument.

3. Shin and Zeng (2012) have a p,q-generalization of this S-type continued fraction $\Longrightarrow$ their polynomials $A_n(x,p,q)$ form a coefficientwise (in x,p,q) Hankel-totally positive sequence.

Example 5: Narayana polynomials of type B

The polynomials

$$W_n(x) \;=\; \sum_{k=0}^{n} \binom{n}{k}^2 x^k$$

arise as

- Coordinator polynomial of the classical root lattice A_n
- Rank generating function of the lattice of noncrossing partitions of type B on $[n]$

I follow Chen–Tang–Wang–Yang 2010 in calling them the *Narayana polynomials of type B*.

- There is no S-type continued fraction *in the ring of polynomials*: we have

$$\alpha_1, \alpha_2, \ldots \;=\; 1{+}x,\; \frac{2x}{1+x},\; \frac{1+x^2}{1+x},\; \frac{x+x^2}{1+x^2},\; \frac{1+x^3}{1+x^2},\; \frac{x+x^3}{1+x^3},\; \frac{1+x^4}{1+x^3},\; \ldots$$

- However, there *is* a nice *J-type* continued fraction:

$$\sum_{n=0}^{\infty} t^n\, W_n(x) \;=\; \cfrac{1}{1-(1{+}x)t-\cfrac{2xt^2}{1-(1{+}x)t-\cfrac{xt^2}{1-(1{+}x)t-\cfrac{xt^2}{1-\cdots}}}}$$

  with coefficients $\gamma_n = 1+x$, $\beta_1 = 2x$, $\beta_n = x$ for $n \geq 2$.

- The corresponding tridiagonal matrix is totally positive.
- **Conclusion:** The sequence $(W_n(x))_{n\geq 0}$ is coefficientwise Hankel-totally positive.

Some cases I am *unable* (as yet) to prove ...

Finally, there are some cases where I find **empirically** that a sequence $(P_n(x))_{n\ge0}$ is coefficientwise Hankel-totally positive, but I am unable to prove it because there is neither an S-type nor a J-type continued fraction in the ring of polynomials:

- Domb polynomials
- Apéry polynomials
- Boros–Moll polynomials
- Inversion enumerators for trees
- Reduced binomial discriminant polynomials

  ⋮

The last two examples are closely related to the problem of *counting connected graphs* ...

Generating polynomials of connected graphs

- Let $c_{n,m} = \#$ of connected simple graphs on vertex set $[n]$ having m edges
- Define the *generating polynomial of connected graphs*

$$\begin{aligned} C_n(v) &= \sum_{m=n-1}^{\binom{n}{2}} c_{n,m}\, v^m \\ &= n^{n-2} v^{n-1} + \ldots + v^{\binom{n}{2}} \end{aligned}$$

- No useful explicit formula for the polynomials $C_n(v)$ or their coefficients is known.
- But they have the well-known exponential generating function

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}\, C_n(v) \;=\; \log \sum_{n=0}^{\infty} \frac{x^n}{n!}\, (1+v)^{n(n-1)/2}$$

- In particular we have

$$C_n(-1) \;=\; (-1)^{n-1}(n-1)!$$

- Of course we also have

$$C_n(0) \;=\; 0 \qquad \text{for } n \ge 2$$

since $C_n(v)$ has an $(n-1)$-fold zero at $v = 0$.

- Make change of variables $y = 1+v$ and define $\overline{C}_n(y) = C_n(y-1)$:

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}\, \overline{C}_n(y) \;=\; \log \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2}$$

- These formulae can be considered either as identities for formal power series or as analytic statements valid when $|1+v| \le 1$ (resp. $|y| \le 1$).

Inversion enumerator for trees

- Let T be a tree with vertex set $[n]$, rooted at the vertex 1.
- An *inversion* of T is an ordered pair (j,k) of vertices such that $j > k > 1$ and the path from 1 to k passes through j.
- Let $i_{n,\ell}$ denote the number of trees on $[n]$ having ℓ inversions.
- Define the *inversion enumerator for trees*

$$I_n(y) = \sum_{\ell=0}^{\binom{n-1}{2}} i_{n,\ell}\, y^\ell$$

$$= (n-1)! + \ldots + y^{\binom{n-1}{2}}$$

- The polynomial $I_n(y)$ turns out to be related to $C_n(v)$ by the beautiful formula

$$C_n(v) = v^{n-1}\, I_n(1+v)$$

or equivalently

$$\overline{C}_n(y) = (y-1)^{n-1}\, I_n(y)$$

- This shows in particular that $I_n(0) = (n-1)!$ and $I_n(1) = n^{n-2}$.
- It is useful to define the normalized polynomials

$$I_n^\star(y) = \frac{I_n(y)}{(n-1)!}$$

which have nonnegative rational coefficients and constant term 1.

Inversion enumerator for trees (continued)

Fact 1. $I_n(y)$ has strictly positive coefficients.

- Nonnegativity is obvious; strict positivity takes a bit of work.

Fact 2. $I_n(y)$ has log-concave coefficients.

- Special case of a deep result of Huh, arXiv:1201.2915, on the log-concavity of the h-vector of the independent-set complex for matroids representable over a field of characteristic 0: apply it to $M^*(K_n)$.
- **Open problem:** Find an elementary direct proof.

Now form the sequence $\boldsymbol{I} = (I_{n+1}(y))_{n \ge 0}$.

Conjecture 1. The sequence $\boldsymbol{I}$ is coefficientwise Hankel-totally positive.

- I have checked this through the 10×10 Hankel matrix.
- Even the log-convexity $I_{n-1} I_{n+1} \succeq I_n^2$ seems to be an open problem!

Conjecture 2. The 2×2 minors $I_{m-1} I_{n+1} - I_m I_n \;\; (1 \le m \le n)$ have coefficients that are log-concave.

- I have checked this through $n = 165$.
- It is false for minors of size 3×3 and higher.

Inversion enumerator for trees (continued)

Now look at the normalized polynomials $\boldsymbol{I}^\star = (I^\star_{n+1}(y))_{n\ge 0}$.

Conjecture 3. The sequence $\boldsymbol{I}^\star$ is coefficientwise Hankel-totally positive.

- I have checked this through the 10×10 Hankel matrix.
- The analogous result for *fixed real* $y \in [0,1]$ can be *proven* by using a result of Laguerre on the real-rootedness of the "deformed exponential function"

$$F(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2}$$

  This is what led me to conjecture the *coefficientwise* Hankel-total positivity.
- *At fixed real* y, the result for $\boldsymbol{I}^\star$ implies the one for $\boldsymbol{I}$, by virtue of a general fact about Hadamard products. But this argument does *not* work in $\mathbb{R}[y]$!

Conjecture 4. *All* the Hankel minors of $\boldsymbol{I}^\star$ have coefficients that are log-concave.

- I have checked this through the 10×10 Hankel matrix.
- For the 2×2 minors, I have checked it for $1 \le m \le n \le 165$.

Binomial discriminant polynomials

- Define $F_n(x,y) \;=\; \sum_{k=0}^{n} \binom{n}{k} x^k\, y^{k(k-1)/2}$
- Can be considered as a "y-deformation" of the binomial $(1+x)^n$. It is also the Jensen polynomial of the deformed exponential function.
- Now define the *binomial discriminant polynomial*

$$\overline{D}_n(y) \;=\; \mathrm{disc}_x\ F_n(x,y)$$

- $\overline{D}_n(y)$ is a polynomial with integer coefficients
- It has degree $n(n-1)^2/2$ and has first and last terms

$$\overline{D}_n(y) \;=\; b_n^2\, y^{n(n-1)(n-2)/3} \;+\; \ldots \;+\; (-1)^{n(n-1)/2} n^n y^{n(n-1)^2/2}$$

where

$$b_n \;=\; \prod_{k=1}^{n-1} \binom{n}{k} \;=\; \prod_{k=1}^{n} k^{2k-1-n} \;=\; \frac{\prod_{k=1}^{n} k^k}{\prod_{k=1}^{n} k!}$$

(does this sequence have any standard name?)

- The first few $\overline{D}_n(y)$ are:

$$\begin{aligned}
\overline{D}_0(y) &= 1 \\
\overline{D}_1(y) &= 1 \\
\overline{D}_2(y) &= 4-4y \\
\overline{D}_3(y) &= 81y^2 - 216y^3 + 162y^4 + 0y^5 - 27y^6 \\
\overline{D}_4(y) &= 9216y^8 - 44032y^9 + 76032y^{10} - 46080y^{11} - 15360y^{12} \\
&\quad +27648y^{13} - 4608y^{14} - 3072y^{15} + 0y^{16} + 0y^{17} + 256y^{18} \\
&\ \ \vdots
\end{aligned}$$

Reduced binomial discriminant polynomials

- $\overline{D}_n(y)$ has a factor $y^{n(n-1)(n-2)/3}$ and also a factor $(1-y)^{n(n-1)/2}$ [coming from the fact that the n roots of $F_n(x,y)$ all coalesce as $y \to 1$].
- So define the *reduced binomial discriminant polynomial*

$$J_n(y) \;=\; \frac{\overline{D}_n(y)}{y^{n(n-1)(n-2)/3}\,(1-y)^{n(n-1)/2}}$$

- $J_n(y)$ is a polynomial with integer coefficients
- It has degree $\binom{n}{3}$ and has first and last terms

$$J_n(y) \;=\; b_n^2 \;+\; \ldots \;+\; n^n y^{\binom{n}{3}}$$

- $J_n(1) = \prod_{k=1}^{n} k^k$ (hyperfactorials)
- The first few $J_n(y)$ are:

$$\begin{aligned}
J_0(y) &= 1 \\
J_1(y) &= 1 \\
J_2(y) &= 4 \\
J_3(y) &= 81 + 27y \\
J_4(y) &= 9216 + 11264y + 5376y^2 + 1536y^3 + 256y^4 \\
&\vdots
\end{aligned}$$

Conjecture 1. The coefficients of $J_n(y)$ are nonnegative (in fact, strictly positive).

Conjecture 2. The coefficients of $J_n(y)$ are log-concave (in fact, strictly log-concave).

- I have checked these conjectures for $n \leq 44$.
- What are the coefficients of $J_n(y)$ counting?

Reduced binomial discriminant polynomials (continued)

Now form the sequence $\boldsymbol{J} = (J_n(y))_{n \ge 0}$.

Conjecture 3. The sequence $\boldsymbol{J}$ is coefficientwise Hankel-totally positive.

- In fact, all the Hankel minors of $\boldsymbol{J}$ seem to have coefficients that are *strictly positive*.
- I have checked this through the 9×9 Hankel matrix.

Conjecture 4. All the Hankel minors of $\boldsymbol{J}$ have coefficients that are log-concave (in fact, strictly log-concave).

- I have checked this through the 9×9 Hankel matrix.
- For the 2×2 minors, I have checked it for $1 \le m \le n \le 42$.

Now look at the normalized polynomials $\boldsymbol{J}^\star = (J_n^\star(y))_{n \ge 0}$.

Conjecture 5. The sequence $\boldsymbol{J}^\star$ is coefficientwise *strongly log-convex*: that is, all the 2×2 minors $J_{m-1}^\star J_{n+1}^\star - J_m^\star J_n^\star$ have nonnegative coefficients.

- I have checked this for $1 \le m \le n \le 42$.
- The 3×3 and higher minors do *not* have nonnegative coefficients.

Conjecture 6. All the 2×2 minors $J_{m-1}^\star J_{n+1}^\star - J_m^\star J_n^\star$ have coefficients that are log-concave (in fact, strictly log-concave except when $m = n = 1$).

- I have checked this for $1 \le m \le n \le 42$.

(Tentative) Conclusion

- Many interesting sequences $(P_n(\mathbf{x}))_{n \ge 0}$ of combinatorial polynomials are (or appear to be) coefficientwise Hankel-totally positive.
- In some cases this can be proven by the Flajolet–Viennot method of continued fractions.
  - Flajolet and Viennot emphasized J-type continued fractions because they are more general.
  - But S-type continued fractions, when they exist, often have simpler coefficients; and they are the most direct tool for proving Hankel-total positivity.
  - Roughly speaking:

J-type c.f. $\Longleftrightarrow$ general orthogonal polynomials $\Longleftrightarrow$ Hamburger moment problem

S-type c.f. $\Longleftrightarrow$ orthogonal polynomials on $[0, \infty)$ $\Longleftrightarrow$ Stieltjes moment problem
$\Longleftrightarrow$ Hankel-total positivity

  - But sometimes J-type continued fractions exist when S-type don't, and they too can be used to prove coefficientwise Hankel-total positivity.
- For the other cases, **new methods of proof will be needed.**
- Deepest cases seem to be $I_n(y)$ and $J_n(y)$:
  - For $I_n(y)$, even the log-convexity $I_{n-1} I_{n+1} \succeq I_n^2$ is an open problem. (Bijective proof??)
  - For $J_n(y)$, even the nonnegativity $J_n \succeq 0$ is an open problem! We really need to know what $J_n(y)$ is counting!

Dedicated to the memory of Philippe Flajolet (1948–2011)