

Functional Analysis Problem Sheet 4

Only do [24] if you are confident using complex analysis

[23] Apply the Gram–Schmidt orthonormalization procedure to the basis

$$\{(2, 3), (1, 2)\} \text{ of } \mathbb{R}^2.$$

[24] Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc, and write $H_2(D)$ for the linear subspace of $L_2(D)$ consisting of all functions holomorphic in D . Prove that $H_2(D)$ is a closed set, and therefore a Hilbert space. Finally, prove that the functions $\{\sqrt{\frac{n}{\pi}} \cdot z^{n-1}\}$ for $n = 1, 2, \dots$ is a complete orthonormal set in $H_2(D)$.

[Hints: show that the n th Fourier coefficient of f is equal to

$$a_n = \lim_{r \rightarrow 1} \frac{1}{\sqrt{\pi n}} \frac{r^{2n}}{2i} \int_{|z|=r} \frac{f(z)}{z^n} dz.$$

On the other hand, if $f(z) = \sum b_n z^n$, then

$$b_{n-1} = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^n} dz,$$

so $a_n = \sqrt{\frac{\pi}{n}} b_{n-1}$. Deduce from this that Parseval's formula holds.]

[25] Let H be a Hilbert space and let u, v be orthogonal unit vectors in H . What is $\|u - v\|$?

[26] Show that a set $\{u_n\}$ spans the Hilbert space H if and only if $(v, u_n) = 0$ for all n implies that $v = 0$ for every $v \in H$.

[27] Let $S = \{(\cos 2\pi n x)/\sqrt{2}\}_{n \geq 0}$ and $T = \{(\sin 2\pi n x)/\sqrt{2}\}_{n > 0}$ in the Hilbert space $L_2[0, 1]$. Show that S and T are orthonormal systems, but that neither of them spans $L_2[0, 1]$. What are the closed linear spans of S and T ?

[28] Let $\{\phi_1, \phi_2, \dots\}$ be a maximal orthonormal basis in a Hilbert space H . Show that the best approximation to $x = \sum_{i=1}^{\infty} c_i \phi_i$ lying in the subspace spanned by $\{\phi_1, \dots, \phi_n\}$ is precisely $\sum_{i=1}^n c_i \phi_i$.

[29] Use Gram–Schmidt on the linearly independent set $\{1, x, x^2, x^3, \dots\}$ in $L_2[0, 1]$ to produce the first two elements of the corresponding orthonormal basis. Using this (and [28]) find the best approximation to $f(x) = x^3 + 1$ in the space spanned by $\{1, x\}$.

TBW/Functional Analysis