Functional Analysis Problem Sheet 3
Only do [18] if you know about cardinals.

[18] The dimension of a linear space X is the cardinality of any Hamel basis for X. Prove that this is well–defined, and that there is a bijective linear map $X \to Y$ if and only if X and Y have the same dimension. Find the dimension of ℓ_∞. Show (using [20]) that the dimension of an infinite–dimensional Banach space is at least c.
[19] Let X be an infinite–dimensional Banach space. Prove that there exists an infinite, strictly decreasing sequence (Y_n) of infinite–dimensional closed linear subspaces of X. [Hint: take Y_1 to be the kernel of some non–trivial linear functional x_1^*. Then take Y_2 to be the kernel of some non–trivial $x_2^* \in Y_1^*$ and so on.]
[20] Let X be an infinite–dimensional Banach space. Prove that ℓ_∞ is isomorphic to a linear subspace of X. [Hint: use problem [19] and choose $x_n \in Y_{n-1} \backslash Y_n$, $Y_0 = X$ in such a way that $\|x_n\| = 2^{-n}$. Then consider the map $(c_n) \mapsto \sum c_n x_n \in X$.]
[21] Show that a Banach space in which every linear subspace is closed must be finite–dimensional. [Hint: use [20]].
[22] and start looking at the review problems...

TBW/Functional Analysis