Functional Analysis Problem Sheet 2

[6] Prove that the spaces ℓ_p $(1 \leq p \leq \infty)$ are normed linear spaces, and that for $p < q$ there is a strict inclusion $\ell_p \subset \ell_q$. (Use results from sheet 1).

[7] Prove that if a sequence (f_n) of continuous functions on $[0,1]$ converges in the sup norm to f, then f is also continuous.

[8] Prove that $C[0,1]$ with the sup norm is a Banach space.

[9] Prove that ℓ_∞ is a Banach space.

[10] Prove that a normed linear space in which every absolutely convergent series is convergent is a Banach space.

[11] Norms $\|\cdot\|^{(1)}$ and $\|\cdot\|^{(2)}$ are said to be *equivalent* if the identity map

$$(X, \|\cdot\|^{(1)}) \to (X, \|\cdot\|^{(2)})$$

is a topological isomorphism. Prove that on $C[0,1]$ the sup norm is not equivalent to any p–norm, $1 \leq p < \infty$.

[12] Let X be a normed linear space. A continuous linear map $X \to \mathbb{R}$ is called a *continuous linear functional*. Denote the space of all continuous linear functionals by X^*, called the *dual* of X. Prove that X^* is a Banach space.

[13] Prove that ℓ_p^* is isomorphic to ℓ_q where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

[14] Let c_0 denote the space of sequences (x_n) with $x_n \to 0$ as $n \to \infty$, with the $\|\cdot\|_\infty$ norm (the *null sequences*). Prove that c_0^* is isomorphic to ℓ_1.

[15] Let $e_1 = (1, 0, \dots)$, $e_2 = (0, 1, 0, \dots)$ and so on. Show that

$$\sum_{k=1}^{\infty} \frac{e_k}{k \log(k+1)}$$

is convergent but not absolutely convergent in c_0. What is the sum of the series?

[16] If S is a dense subset of a normed space X, prove that every element of X can be written as the sum of an absolutely convergent series whose terms are all in S.

[17] Let X be a normed space. Prove that X is a Banach space if and only if every Cauchy sequence in $\{x \in X : \|x\| = 1\}$ is convergent.

TBW/Functional Analysis