Functional Analysis Problem Sheet 1
Problems marked with a dagger will involve some bookwork.

[1] Let V be a vector space with a function $(\cdot, \cdot) : V \times V \to \mathbb{R}$. Let $\|f\| = (f, f)^{1/2}$. Prove the Schwartz inequality

$$|(f, g)| \leq \|f\|\|g\|$$

in each of the following cases: (a) $V = \mathbb{R}$, $(f, g) = fg$; (b) $V = \mathbb{R}^n$, $(f, g) = \sum f_i g_i$ ("dot product"); (c) V =Riemann–integrable square–integrable functions $[0, 1] \to \mathbb{R}$, $(f, g) = \int fg$; (d) V =square–summable sequences, $(f, g) = \sum f_i g_i$.

[2] For each of the following linear spaces, determine the dimension: (a) the set of vectors $x = (x_i)$ in $\mathbb{R}^n$ with $\sum x_i = 0$; (b) the set of continuous functions $[0, 1] \to \mathbb{R}$; (c) the set of polynomials of degree $\leq r$ on $[0, 1]$; (d) the set of polynomials on $[0, 1]$.

[4†] Let $p \in [1, \infty]$ and let q have $\frac{1}{p} + \frac{1}{q} = 1$. Prove that q also lies in $[1, \infty]$. Prove Hölder's inequality in the following forms: For vectors:

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \left(\sum_{i=1}^{n} |x_i|^q\right)^{1/q} \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

For sequences (x_i) with $\sum |x_i|^q < \infty$. $\sum |y_i|^p < \infty$:

$$\left|\sum_{i=1}^{\infty} x_i y_i\right| \leq \left(\sum_{i=1}^{\infty} |x_i|^q\right)^{1/q} \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}.$$

For Riemann–integrable functions f, g with $\int_0^1 |f|^p < \infty$, $\int_0^1 |g|^q < \infty$:

$$\int_0^1 |fg| \leq \left(\int_0^1 |f|^p\right)^{1/p} \left(\int_0^1 |g|^q\right)^{1/q}.$$

[5†] Prove Minkowski's inequality in the following forms. For vectors:

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

For sequences (x_i) with the property that $\sum |x_i|^p < \infty$ ("p–summable"):

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}.$$

For functions f with $\int_0^1 |f|^p < \infty$:

$$\left(\int_0^1 |f + g|\right)^{1/p} \leq \left(\int_0^1 |f|^p\right)^{1/p} + \left(\int_0^1 |g|^p\right)^{1/p}.$$

[6] Set a calculator into "Radians" mode and enter any number between 0 and $\frac{\pi}{2}$. What happens if you repeatedly press the `cos` button? Can you explain why this is happening?

TBW/Functional Analysis