Functional Analysis Supplement 1

I still cannot see what was intended by the problem we looked at last Friday, but here are the details on the relationship between range and kernels of adjoints.

Let $T \in \mathcal{B}(H_1, H_2)$ be a bounded linear operator, and define T^* by

$$(Tx, y)_{H_1} = (x, T^*y)_{H_2}.$$

Write $R(T)$ for the range, and $N(T)$ for the kernel (nullity) of T.

Notice that the kernel of an operator or the orthocomplement of a space is guaranteed to be closed, but the range might not be. This is the best way to remember the form of the statements below.

[1] $R(T)^\perp = N(T^*)$.

Proof: Since $x \in N(T^*)$, $T^*x = 0$, so $(z, T^*x) = 0$ for all $z \in H_1$, hence $(Tz, x) = 0$ for all z, so $x \in R(T)^\perp$. (This shows that $N(T^*) \subset R(T)^\perp$).

Conversely, if $x \in R(T)^\perp$ then $(Tz, x) = 0$ for all z, so $(z, T^*x) = 0$ for all z, so $T^*x = 0$. (This shows that $N(T^*) \supset R(T)^\perp$).

[2] From [1], we get $R(T^*)^\perp = N(T)$ immediately.

[3] $N(T^*)^\perp = \overline{R(T)}$.

Proof: This is much more subtle than [1]. First, if $x \in \overline{R(T)}$, then we can find (z_n) such that $T(z_n) \to x$. Then for any $y \in N(T)$,

$$(x, y) = \lim(Tz_n, y) = \lim(z_n, T^*y) = 0$$

so $x \in N(T^*)^\perp$. (This shows that $\overline{R(T)} \subset N(T^*)^\perp$.)

Conversely, if $x \notin \overline{R(T)}$ (which we are assuming is not all of H_2), then by Hahn–Banach there is a linear functional $\xi \in H_2^*$ such that $\xi(x_0) \neq 0$ and $\xi(Tx) = 0$ for all x. Now apply Riesz's theorem: there exists $z \in H_2$ such that $\xi(x) = (x, z)$. So, $(x_0, z) \neq 0$ and $(Tx, z) = (x, T^*z) = 0$ for all x, so $z \in N(T^*)^\perp$, and hence $(x_0, z) \neq 0$ means that $x_0 \notin N(T^*)^\perp$. (This shows that $x \notin \overline{R(T)} \implies x_0 \notin N(T^*)^\perp$, so $\overline{R(T)} \supset N(T^*)^\perp$.)

[4] An immediate consequence of [3] is $N(T)^\perp = \overline{R(T^*)}$.

TBW/Functional Analysis