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The action is *expansive* if there is some $\delta > 0$ with

$$\sup_{\mathbf{n} \in \mathbb{Z}^d} \rho(\beta^{\mathbf{n}} x, \beta^{\mathbf{n}} y) \leq \delta \implies x = y.$$

Subdynamics: (Boyle & Lind) $A \subset \mathbb{R}^d$ is *expansive for* β, or β is *expansive along* A, if there exist constants $\delta > 0$ and $t > 0$ with

$$\sup_{\mathbf{n}, d(\mathbf{n}, A) < t} \rho(\beta^{\mathbf{n}} x, \beta^{\mathbf{n}} y) \leq \delta \implies x = y \text{ for all } x, y \in X$$

where $d(\mathbf{n}, A)$ denotes the distance from the point $\mathbf{n}$ to the set A in the Euclidean metric on $\mathbb{R}^d$.

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Special case: Write G_k for the Grassmannian of k-dimensional subspaces of $\mathbb{R}^d$; this is a compact $k(d-k)$-dimensional manifold in the usual topology (subspaces are close if their intersections with the unit $(d-1)$-sphere S_{d-1} are close in the Hausdorff topology). Write

$$\mathsf{N}_k(\beta) = \{V \in \mathsf{G}_k \mid V \text{ is not expansive for } \beta\}.$$

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So $\mathsf{N}_{d-1}(\beta)$ is a fundamental geometrical invariant of a topological $\mathbb{Z}^d$-action.

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Complete description of $\mathsf{N}_{d-1}(\beta)$ for β algebraic in terms of 'adelic amoebas'.

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Let α be the $\mathbb{Z}^2$-action obtained as the invertible extension of the commuting maps $x \mapsto 2x$, $x \mapsto 3x$ on the circle.

Equivalent formulation: α is the shift action of $\mathbb{Z}^2$ on

$$X = \{x \in \mathbb{T}^{\mathbb{Z}^2} \mid x_{\mathbf{n}+\mathbf{e}_1} = 2x_{\mathbf{n}}, x_{\mathbf{n}+\mathbf{e}_2} = 3x_{\mathbf{n}} \text{ for all } \mathbf{n} \in \mathbb{Z}^2\}.$$

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A 'variety' condition: $\mathbf{n}$ with $2^{n_1}3^{n_2} \sim 1$ means that $\alpha^{\mathbf{n}}$ behaves almost like an isometry (in a real eigen-direction), so we expect $2^x 3^y = 1$ to be in $\mathsf{N}_1(\alpha)$. Morally this is a vanishing of real Lyapunov exponents. These lines make up $\mathsf{N}_1^{\mathsf{v}}(\alpha)$.

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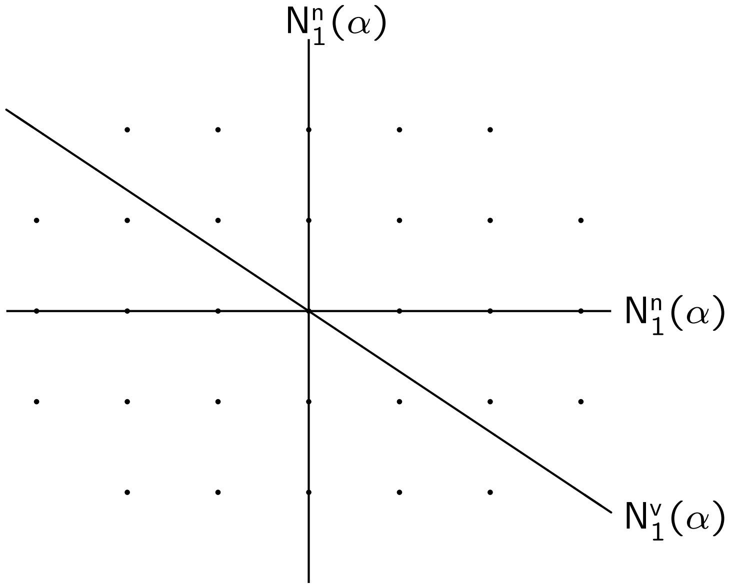
An 'algebraic' condition: a map like $x \mapsto 2x$ has small invariant subgroups in X corresponding to powers of 3. So we expect $2^x = 1$ and $3^y = 1$ to be in $\mathsf{N}_1(\alpha)$. These lines make up $\mathsf{N}_1^{\mathsf{n}}(\alpha)$.

The set $N_1(\alpha)$ for $\times 2, \times 3$:



Ledrappier's example

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This is a totally disconnected analogue of $\times 2, \times 3$, and it may be described as the shift action of $\mathbb{Z}^2$ on the compact group

$$X = \{x \in \{0,1\}^{\mathbb{Z}^2} \mid x_{\mathbf{n}+\mathbf{e}_1} + x_{\mathbf{n}} + x_{\mathbf{n}+\mathbf{e}_2} = 0 \pmod 2 \text{ for all } \mathbf{n} \in \mathbb{Z}^2\}.$$

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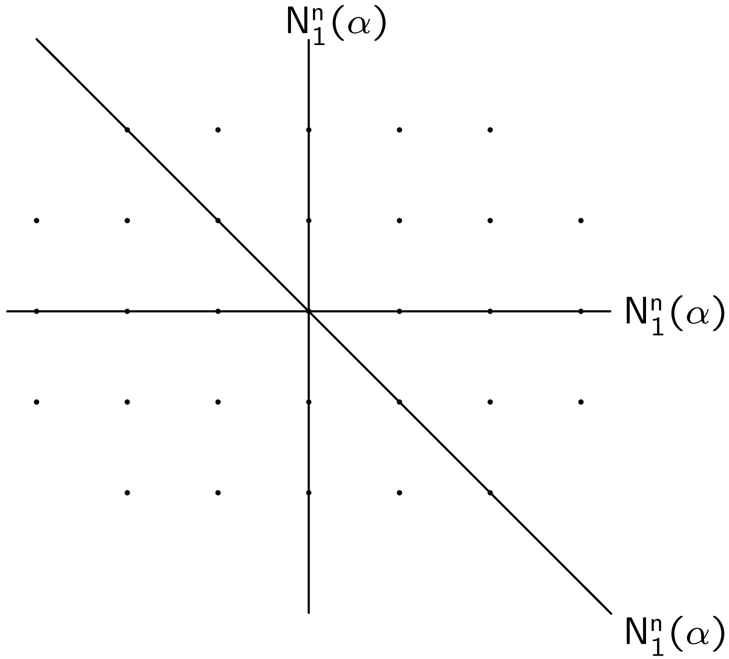
This has no real Lyapunov exponents so $\mathsf{N}_1^{\mathsf{v}}(\alpha)$ is empty.

The line $x = 0$ is in $\mathsf{N}_1(\alpha)$ since a point x with $x_{\mathbf{n}} = 0$ for all $\mathbf{n}$ to the right of $x = -K$ for some large K can be non-zero (but close to zero no matter how it is moved in the vertical direction).

The set $N_1(\alpha)$ for Ledrappier's example:



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Einsiedler & Lind have shown that any such system can be described in terms of products of local fields.

Periodic points: Write $F_{\mathbf{n}}(\beta) = \{x \in X \mid \beta^{\mathbf{n}}x = x\}$ for the set of points fixed by the homeomorphism $\beta^{\mathbf{n}}$. The combinatorial data of all these numbers may be thought of as a map

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There is a 'formula' to compute this, which subsumes a range of things including:

- there are $|\det(A^n - I)|$ points of period n for a toral endomorphism defined by a matrix A;

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• and finally an algebraic miracle that occurs in rank one which makes periodic point counts multiply nicely (Miles).

Periodic points for $\times 2, \times 3$:

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211	227	235	239	241	121	485	971	1943	3887	7775
49	65	73	77	79	5	161	323	647	1295	2591
5	11	19	23	25	13	53	107	215	431	863
23	7	1	5	7	1	17	35	71	143	287
29	13	5	1	1	1	5	11	23	47	95
31	5	7	1	1	∞	1	1	7	5	31

Periodic points for Ledrappier's example:

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16	32	32	32	32	16	64	128	256	512	1024
32	1	16	16	16	1	32	64	128	256	512
32	16	4	8	8	4	16	32	64	128	256
32	16	8	1	4	1	8	16	32	64	128
32	16	8	4	1	1	4	8	16	32	64
16	1	4	1	1	∞	1	1	4	1	16

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The connection goes via the dynamical zeta function:

$$\zeta_T(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} |F_n(T)|.$$

If $|F_n(T)| < \infty$ for all $n \geq 1$ and grows at most exponentially, then this defines a complex function in some disc.

In our setting there is a fixed $\mathbb{Z}^d$-action α, so write $\zeta_{\mathbf{n}}$ for the zeta function of the map $\alpha^{\mathbf{n}}$. Define $\mathsf{Q}(\alpha)$ to be the set of $\mathbf{n} \in \mathbb{Z}^d$ for which $\zeta_{\mathbf{n}}$ is a rational function.

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Notice that any $\mathbf{n} \in \mathbb{Z}^d$ with the property that $F_j(\alpha^{\mathbf{n}})$ is infinite for some $j \geq 1$ is not a member of $\mathsf{Q}(\alpha)$.

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There are more exotic absences: in the $\times 2, \times 3$ example $\zeta_{(1,0)}$ has a natural boundary, for example.

Example: In $\times 2, \times 3$,

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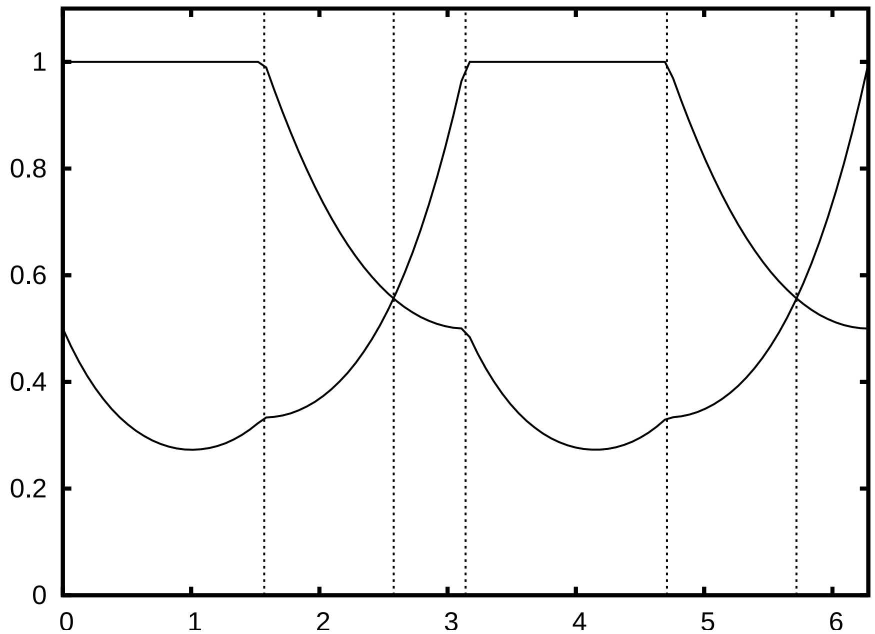
The issue here is to show that the zeta functions $\zeta_{(1,0)}$ and $\zeta_{(0,1)}$ in the two rational non-expansive lines are not rational, and the point is that $\mathsf{Q}(\alpha)$ is not able to detect all of N_1.

Given a rational function $h \in \mathbb{C}(z)$, denote the set of poles and zeros of h by $PZ(h) \subset \mathbb{C}$. For α an algebraic $\mathbb{Z}^d$-action of entropy rank one, define

$$\Omega_\alpha = \left\{ (\hat{\mathbf{n}}, |z|^{1/\|\mathbf{n}\|}) \mid z \in PZ(\zeta_{\mathbf{n}}), \mathbf{n} \in \mathsf{Q}(\alpha) \right\} \subset \mathsf{S}_{d-1} \times \mathbb{R}$$

where $\hat{\mathbf{n}}$ denotes the unit vector in the direction of $\mathbf{n}$.

For the $\times 2, \times 3$ example the set $\overline{\Omega}_\alpha$ looks like this:



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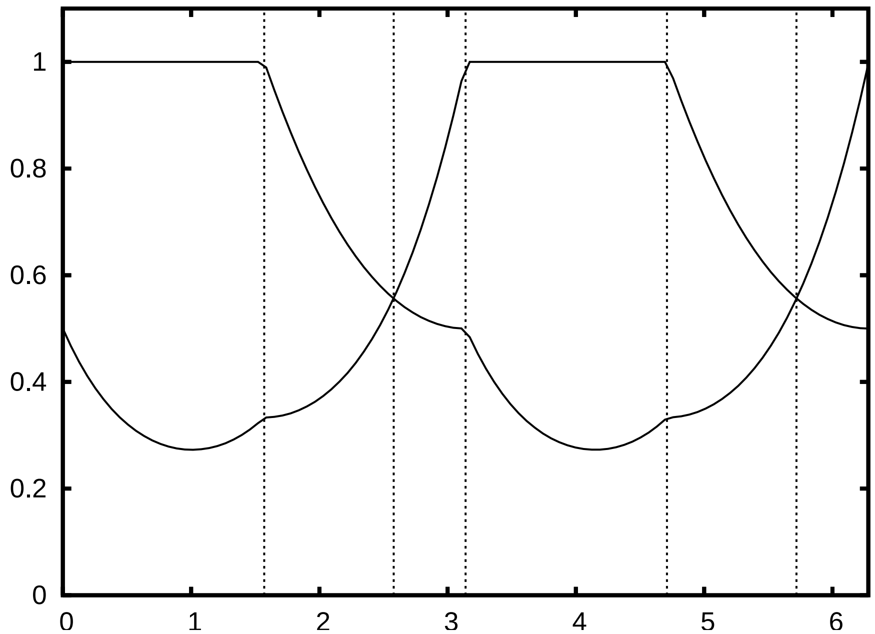

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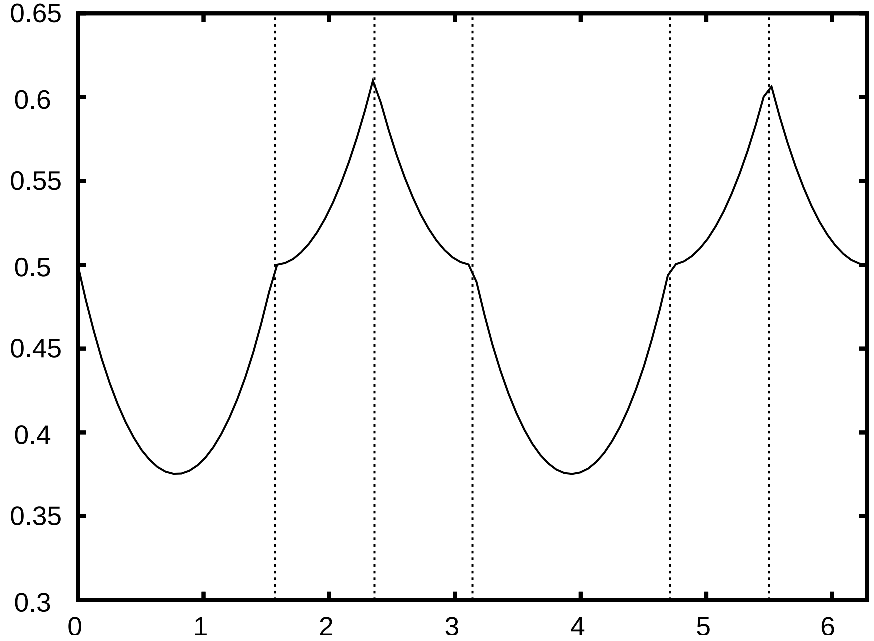

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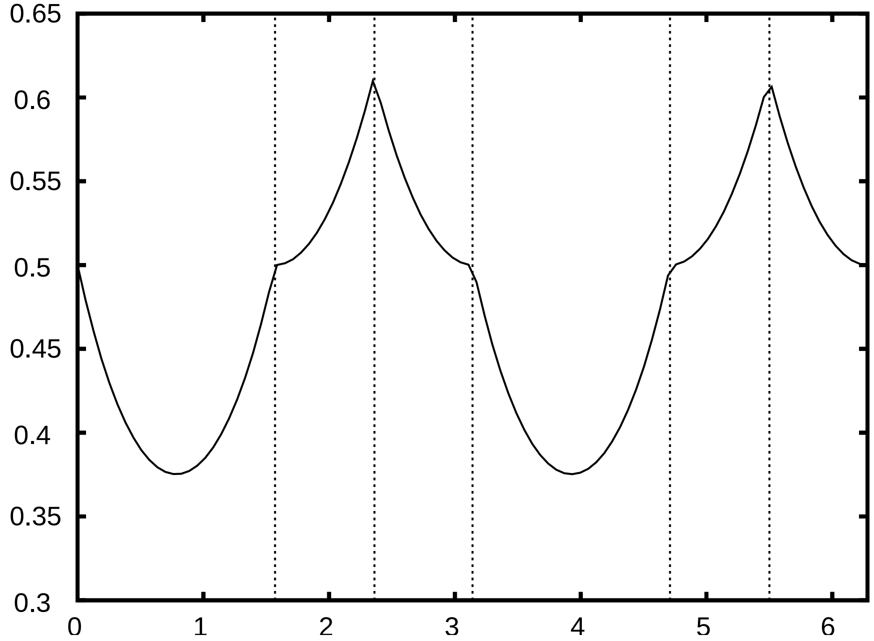

Again the non-expansive directions are shown with dashed lines.

Theorem: Let α be an ergodic algebraic $\mathbb{Z}^d$-action with rank one. Then there exist continuous functions $f_1, \dots, f_r \in \mathcal{C}(\mathsf{S}_{d-1}, \mathbb{R})$ such that

$$\overline{\Omega}_\alpha = \bigcup_{k=1}^{r} \{(\mathbf{v}, f_k(\mathbf{v})) \mid \mathbf{v} \in \mathsf{S}_{d-1}\}.$$

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- $\mathsf{N}_1^{\mathsf{n}}(\alpha) = \{\mathbf{v} \in \mathsf{S}_{d-1} \mid f_k \text{ is not smooth at } \mathbf{v} \text{ for some } k\}$.

Steps in the proof: Use a structure theorem for rank one systems due to Einsiedler & Lind and the algebraic ‘miracle’ due to Miles to express the functions f_j in a more or less explicit way.

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The proof has no inner meaning: it somehow amounts to computing both sides.

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Damjanović and Katok show that for $r \geq 6$ there are such actions on the r-torus: Let

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Then a and

$$b = 2a^5 - 6a^4 - 3a^3 - 6a^2 - 6a$$

are fundamental units for the ring of integers in k, and $M = \mathbb{Z}[a]$ carries a $\mathbb{Z}^2$ action of multiplication by a and by b.

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- a and b are multiplicatively independent, so $\alpha^{\mathbf{n}}$ is ergodic for every $\mathbf{n} \in \mathbb{Z}^2$;

- hence $\zeta_{\mathbf{n}}$ is rational for every $\mathbf{n} \in \mathbb{Z}^2$, and the whole action is not expansive, so the results above do not apply – in particular, $\mathsf{N}_1(\alpha)$ consists of all directions.

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These points correspond to directions $\mathbf{n} \in \mathbb{R}^2$ where the product of some subset of the eigenvalues of the integer matrix defining the automorphism $\widehat{\alpha_M^{\mathbf{n}}}$ is 1.

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Thus the portrait obtained from $\overline{\Omega}_\alpha$ contains those directions in which the action is non-expansive transverse to the two-dimensional stable foliation arising from the common two-dimensional eigenspace on which the actions behaves like a rotation.

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Just as before any non-expansive subgroup lies in a non-expansive subgroup Γ with $\mathbb{Q}^{\times}_{>0}/\Gamma \cong \mathbb{Z}$ (that is, lies in '$\mathsf{N}_{\infty-1}(\alpha)$').

Any such subgroup acts non-expansively, since it must omit some prime p, and will therefore act like an isometry on that direction in X.

On the other hand, $\overline{\Omega}_\alpha$ does not detect this rich non-expansive structure:

On the other hand, $\overline{\Omega}_\alpha$ does not detect this rich non-expansive structure:

Since $\widehat{X}$ is a field, every map α^r for $r \in \mathbb{Q}^\times_{>0}$ has $\zeta_{\alpha^r}(z) = \frac{1}{1-z}$, so the set $\overline{\Omega}_\alpha$ comprises the graph of the constant function 1.