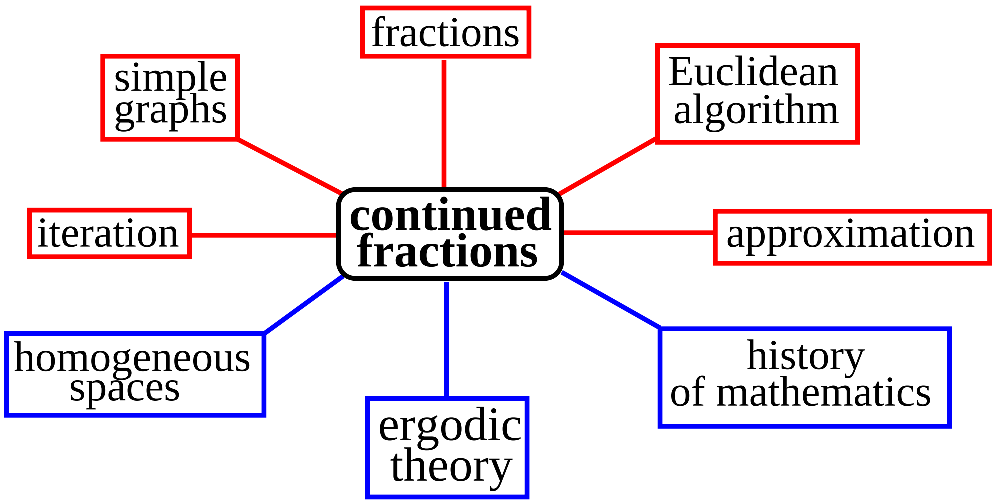
8/3/06, Park Farm Hotel

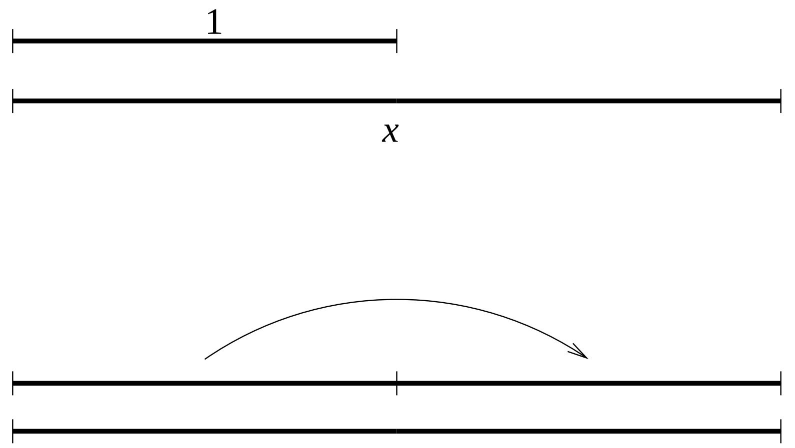
Continued fractions as a bridging topic



Question: using a pencil only, can you find the ratio between two lengths very accurately?

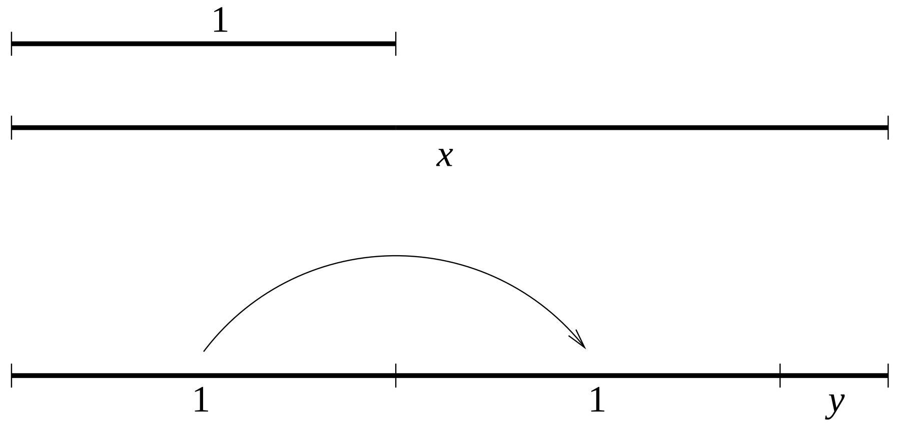
For example, can you find the ratio between the height and the width of a sheet of A4?

This could be easy:



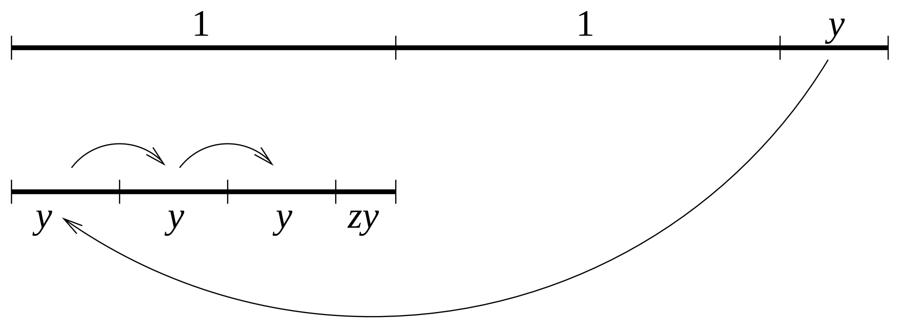
so $x = 2$ and the ratio is $1 : 2$.

It could be a bit more involved:



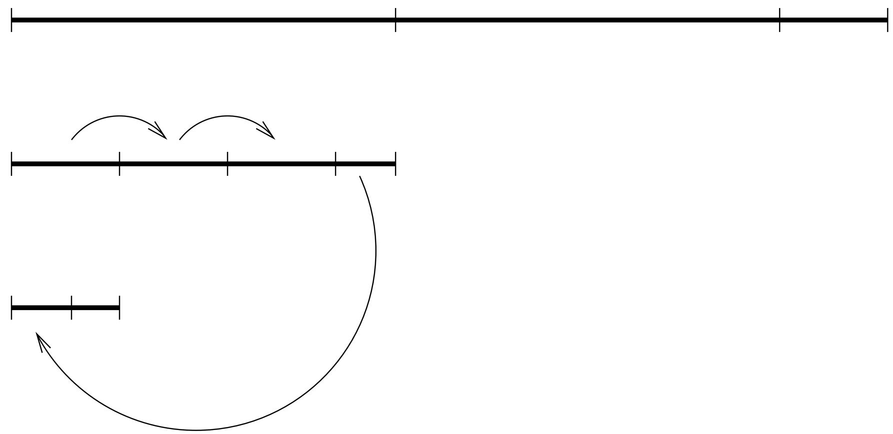
so the ratio is $1 : x$ where $x = 2 + y$.

Now compare the remainder y with 1:



which amounts to writing $1 = (3 + z)y$, so $y = \frac{1}{3+z}$.

Then compare the new remainder z with the previous remainder y:



to deduce that $z = \frac{1}{1+\cdots}$

and so on. In this example the process stops after 4 steps, giving

$$x = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}}}$$

which is a *continued fraction* for $2\frac{9}{32}$.

Continued fractions go back (implicitly) to Euclid (approx. 300 B.C.)

Raphael's Euclid from his “School of Philosophers” fresco

The Euclidean Algorithm (division)

Compute the gcd of 67 and 24:

$$67 = 2 \times 24 + 19$$

$$24 = 1 \times 19 + 5$$

$$19 = 3 \times 5 + 4$$

$$5 = 1 \times 4 + 1.$$

First conclusion: the gcd of 67 and 24 is 1 (the last remainder).

Second conclusion:

$$\frac{67}{24} = 2 + \frac{19}{24},\ \frac{24}{19} = 1 + \frac{5}{19},\ \frac{19}{5} = 3 + \frac{4}{5},\ \frac{5}{4} = 1 + \frac{1}{4}.$$

Hence

$$\frac{67}{24} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{4}}}}.$$

The 2,1,3,1,4 are the *terms* or *partial quotients* of the continued fraction.

The fractions obtained by stopping early are called *convergents*:

$$2 < \frac{67}{24}$$

$$2 + \frac{1}{1} = 3 > \frac{67}{24}$$

$$2 + \cfrac{1}{1 + \frac{1}{3}} = \frac{11}{4} < \frac{67}{24}$$

$$2 + \cfrac{1}{1 + \cfrac{1}{3 + \frac{1}{1}}} = \frac{14}{5} > \frac{67}{24}$$

Leonhard Euler *"De Fractionibus Continuis Dissertation"*, (1737)

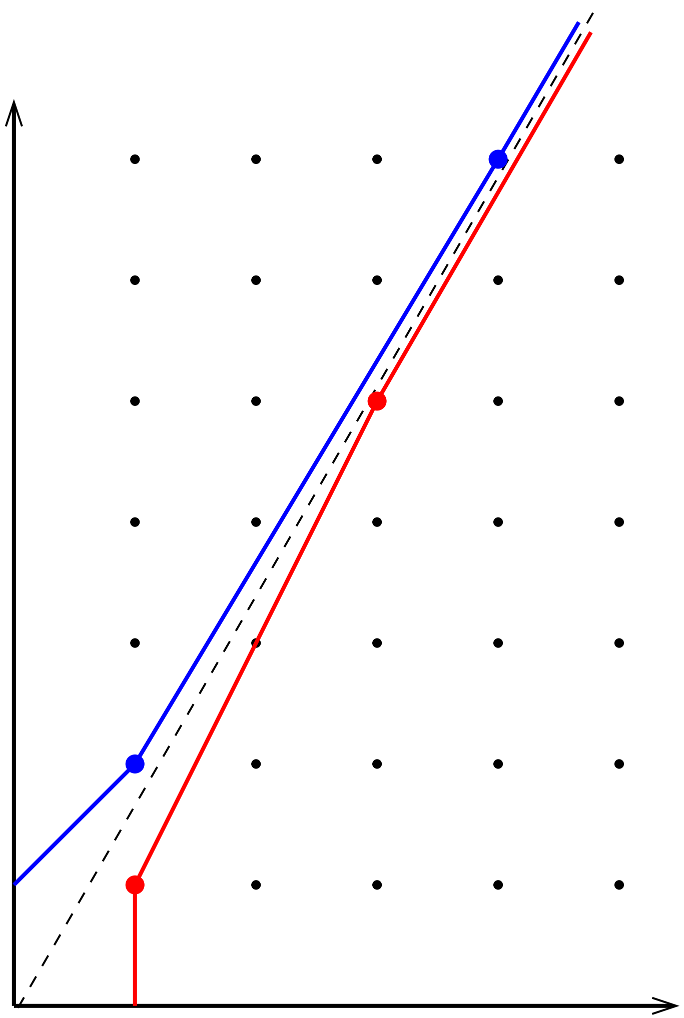
Euler developed the theory of infinite continued fractions, which has huge significance both practically and theoretically.

Geometrical interpretation of continued fractions due to Klein (1895)

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}$$

The convergents are $\frac{1}{1}$, $\frac{2}{1}$, $\frac{5}{3}$, $\frac{7}{4}$, $\frac{19}{11}$, $\frac{26}{15}, \ldots$.

Now draw the graph of $y = \sqrt{3}x$ and imagine two threads joining $(0,0)$ to "(∞,∞)" catching on pins at all integer lattice points...



The red thread catches at $(1, 1)$, $(3, 5)$, $(11, 19), \ldots$;

the blue thread catches at $(1, 2)$, $(4, 7)$, $(15, 26), \ldots$.

The triangle made by the origin and any two successive catch points has area $\frac{1}{2}$.

Patterns in continued fractions

Lagrange's Theorem: Any quadratic irrational $a + b\sqrt{D}$ with a, b rational and D an integer has a continued fraction whose terms eventually become periodic.

For example, if $\sqrt{2} = 1 + \frac{1}{a_1}$ then $a_1 = \frac{1}{\sqrt{2}-1} = \sqrt{2} + 1 = 2 + \frac{1}{a_2}$ (since $\sqrt{2} + 1$ is between 2 and 3).

Hence $a_2 = \frac{1}{a_1 - 2} = \frac{1}{\sqrt{2}-1} = \sqrt{2} + 1 = a_1$, so we deduce that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

Notice that this shows (incidentally) that $\sqrt{2}$ is irrational.

Euler asked if $e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$ is rational.

Deeper patterns

Euler proved that

$$\frac{e-1}{e+1} = \cfrac{1}{2+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cdots}}}},$$

where the terms continue in an arithmetic progression.

Similarly, e itself has terms $1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \cdots$ and so on.

Similar patterns exist for $\sqrt{e}$, $\sqrt[3]{e}$, etc.

Deeper problems

It is fair to say that not much else is known about specific numbers.

Open problem 1: Are the terms in the continued fraction for $\sqrt[3]{2}$ bounded?

$$\sqrt[3]{2} = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cdots}}}}}}.$$

It is known that the terms cannot increase very fast, but nothing else is known.

Open problem 2: Is there a pattern in the continued fraction for π,

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}?$$

The convergents are the approximations to π that have been used for thousands of years: 3, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, $\ldots$.

The very large term 292 means that the convergent $\frac{355}{113}$ is a much better approximation than it has any right to be, given the size of its denominator.

There does not seem to be a pattern...

but...

some slightly different expansions do have structure, including

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

and

$$\frac{\pi}{2} = 1 - \cfrac{1}{3 - \cfrac{2 \cdot 3}{1 - \cfrac{1 \cdot 2}{3 - \cfrac{4 \cdot 5}{1 - \cfrac{3 \cdot 4}{3 - \cfrac{6 \cdot 7}{1 - \cfrac{5 \cdot 6}{3 - \cdots}}}}}}}.$$