Topological rigidity on connected groups

Princeton March 2005

(Siddhartha Bhattacharya, TW)

ETDS, 25, 365-373, 2005
(arXiv:math.DS/0302039)

Setting: α an action of $\mathbb{Z}^d$ on a compact abelian group X; e is used to denote the identity element of any group. Write $\mathsf{X} = (X, \alpha)$ for such an algebraic dynamical system, and call the system X connected, mixing and so on if X is connected, α is mixing, and so on.

The system X is *mixing* if

$$\lim_{\mathbf{n}\to\infty} \lambda_X(A_1 \cap \alpha(\mathbf{n})(A_2)) = \lambda_X(A_1) \cdot \lambda_X(A_2)$$

for all measurable sets $A_1, A_2 \subset X$, where λ denotes Haar measure.

A map $\phi : \mathsf{X}_1 \to \mathsf{X}_2$ between algebraic dynamical systems is *equivariant* if

$$\phi \circ \alpha_1(\mathbf{n}) = \alpha_2(\mathbf{n}) \circ \phi$$

for all $\mathbf{n} \in \mathbb{Z}^d$, and is *affine* if there is a continuous group homomorphism

$$\psi : X_1 \to X_2$$

and an element $y \in X_2$ with

$$\phi(x) = \psi(x) \cdot y.$$

Topological (measurable) rigidity is a property X_1 and X_2 that forces an equivariant continuous (resp. measurable) map to coincide everywhere (resp. almost everywhere) with an affine map.

We fix X_1 throughout to be a mixing, connected algebraic $\mathbb{Z}^d$-action, so will speak of rigidity as a property of the target system X_2.

Algebraic description

Let $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$; $f \in R_d$ is written

$$f(\mathbf{u}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \mathbf{u}^{\mathbf{n}}$$

with $\mathbf{u}^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$, $f_{\mathbf{n}} \in \mathbb{Z}$ for all

$$\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d,$$

and $f_{\mathbf{n}} = 0$ for all but finitely many $\mathbf{n} \in \mathbb{Z}^d$.

If $\mathsf{X} = (X, \alpha)$ is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, then the countable dual group $M = \widehat{X}$ is a module over the ring R_d under the operation

$$f \cdot a = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \widehat{\alpha}(\mathbf{n})(a)$$

for $f \in R_d$ and $a \in M$. The module M is called the *dual module* of X. Conversely, a countable module M over R_d determines an algebraic $\mathbb{Z}^d$-action $\mathsf{X}_M = (X_M, \alpha_M)$ by setting

$$\widehat{\alpha_M}(\mathbf{n})(a) = \mathbf{u}^{\mathbf{n}} \cdot a$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in M$.

Rigidity

Measurable rigidity typically arises for $d \geq 2$.

Topological rigidity is a different phenomena – it is driven by weaker properties.

For a $\mathbb{Z}$-action generated by an automorphism θ on a connected finite-dimensional compact abelian group, the topological centralizer of the action admits non-affine maps if and only if θ is not ergodic. Ergodic automorphisms of infinite-dimensional groups may have non-affine maps in their centralizers. For any expansive connected algebraic $\mathbb{Z}^d$-action X, the topological centralizer of α consists of affine maps (expansiveness is a condition that implies Noetherian; for $d = 1$ it forces X to be finite-dimensional; i.e. a solenoid).

Theorem Let X_1, X_2 be connected mixing Noetherian algebraic $\mathbb{Z}^d$-actions. Then the following properties are equivalent.

1. Every equivariant continuous map

   $$\mathsf{X}_1 \to \mathsf{X}_2$$

   is affine.

2. X_2 has finite topological entropy.

For $d \geq 2$, X_2 might be infinite-dimensional, so lifting techniques cannot be applied directly. Write $\mathbb{T} \subset \mathbb{C}$ for the multiplicative unit circle.

Example 1: define a $\mathbb{Z}^2$-action by setting

$$X \subset \mathbb{T}^{\mathbb{Z}^2}$$

by $x \in X$ if and only if

$$x(m+1,n)\cdot x(m,n)\cdot x(m,n+1)=1$$

for all $m,n \in \mathbb{Z}$, and α is the shift action of $\mathbb{Z}^2$ on X. The system X is mixing and has finite entropy, so every continuous equivariant map from X to itself is affine. In contrast, the *measurable* centraliser of X contains many non-affine maps.

Example 2: The shift automorphism of $\mathbb{T}^{\mathbb{Z}}$ defines an ergodic $\mathbb{Z}$-action of infinite entropy that is not topologically rigid: if

$$f : \mathbb{T} \to \mathbb{T}$$

is any map, then the shift map commutes with the map ϕ defined by $(\phi(x))_k = f(x_k)$. The module corresponding to this action is Noetherian.

On the other hand, an ergodic automorphism of $\mathbb{T}^{\mathbb{Z}}$ that splits into a direct product of automorphisms of finite-dimensional tori *is* topologically rigid. The module corresponding to this action is not Noetherian. It is not known whether such an action can have finite topological entropy.

Example 3: Let F_d denote the field of fractions of R_d, considered as a R_d-module. Let X_1 denote the corresponding action. Notice that F_d is torsion-free as a R_d-module, and X_1 has infinite entropy. For any $\mathbf{n} \in \mathbb{Z}^d$, multiplication by $\mathbf{u}^{\mathbf{n}} - 1$ is an automorphism of F_d. By duality, the map $x \mapsto \alpha_1(\mathbf{n})(x) - x$ is a continuous automorphism of X_1 for any $\mathbf{n} \in \mathbb{Z}^d$. In particular, X_1 does not have any non-trivial periodic orbits. Now let X_2 be any mixing connected algebraic $\mathbb{Z}^d$-action with a dense set of periodic orbits (any Noetherian system has this property). Since continuous equivariant maps take periodic orbits to periodic orbits, it follows that any continuous equivariant map from X_2 to X_1 is trivial.

Machinery

A prime ideal $\mathfrak{p} \subset R_d$ is *associated with* M if there exists $m \in M$ with

$$\mathfrak{p} = \{f \in R_d \mid f \cdot m = 0\}.$$

The set of prime ideals associated with M is denoted Asc(M). If M is Noetherian, then Asc(M) is finite. The torsion submodule of M, Tor(M), is the set of $m \in M$ with $r \cdot m = 0$ for some non-zero $r \in R_d$; M is *torsion* if Tor$(M) = M$.

Standard results

X_M is mixing if and only if for all $\mathfrak{p} \in \mathsf{Asc}(M)$ and for every non-zero $\mathbf{n} \in \mathbb{Z}^d$, $\mathbf{u}^{\mathbf{n}} - 1 \notin \mathfrak{p}$.

For M Noetherian the following conditions are equivalent.

1. X_M has no non-trivial closed α_M-invariant subgroup H with the property that the restriction of α_M to H is an algebraic factor of the shift action of $\mathbb{Z}^d$ on $(\mathbb{T}^n)^{\mathbb{Z}^d}$ for some $n > 0$.
2. M is torsion.
3. X_M has finite topological entropy.

Replacing lifting: van Kampen's theorem

Need to split continuous maps into a 'linear' part (a character) and a 'non-linear' part in a unique way.

van Kampen: Let X be a compact connected abelian group and let $f : X \to \mathbb{T}$ be a continuous map with $f(e) = 1$. Then there exist a character $\phi \in \widehat{X}$ and a continuous map $S(f) : X \to \mathbb{R}$ such that

$$S(f)(e) = 0,$$

$$f(x) = \phi(x) \cdot e^{2\pi i S(f)(x)} \text{ for all } x \in X.$$

Moreover, ϕ and $S(f)$ are uniquely defined by those properties.

Proof

For any locally compact abelian group A, denote by A^X the group of continuous maps

$$h : X \to A, \quad h(e) = e,$$

with point-wise multiplication. The action α induces the structure of an R_d-module on A^X by

$$p \cdot h(x) = \sum_{\mathbf{n} \in \mathbb{Z}^d} p(\mathbf{n}) \cdot h \circ \alpha(\mathbf{n})(x).$$

Then:

1) $\widehat{X}$ can be regarded as a submodule of $\mathbb{T}^X$ with this structure. 2) If X is connected, $\mathbb{R}^X$ and $\mathbb{T}^X/\widehat{X}$ are isomorphic as R_d-modules.

For $f, g : \mathbb{Z}^d \to \mathbb{C}$, g of finite support,

$$f * g(\mathbf{i}) = \sum_{\mathbf{j} \in \mathbb{Z}^d} f(\mathbf{i} - \mathbf{j}) \cdot g(\mathbf{j}).$$

In addition to van Kampen's theorem, a simple version of the L^2 zero-divisor problem is needed:

If $f \in L^2(\mathbb{Z}^d)$ has $f * g = 0$ for some non-zero function g with finite support, then f is identically zero.

Lemma: If X is mixing and connected then Tor($\mathbb{T}^X$) $\subset \widehat{X}$.

Suppose that X$_2$ has finite entropy, and let f be an equivariant continuous map X$_1 \to$ X$_2$. Define $f_0 : X_1 \to X_2$ by

$$f_0(x) = f(x) - f(e). \qquad (1)$$

Since f is equivariant, so is f_0.

Now $\widehat{X_2}$ is a torsion module, so any character ϕ lies in the torsion submodule of $\mathbb{T}^{X_2}$. Since f_0 is equivariant and $f_0(e) = 1$, $h \mapsto h \circ f_0$ is an R_d-module homomorphism $\mathbb{T}^{X_2} \to \mathbb{T}^{X_1}$. Hence $\phi \circ f_0$ is an element of the torsion submodule of $\mathbb{T}^{X_1}$. By Lemma, $\phi \circ f_0$ lies in $\widehat{X_1}$. So $\phi \mapsto \phi \circ f_0$ is a group homomorphism from $\widehat{X_2}$ to $\widehat{X_1}$. By duality, there exists a continuous homomorphism $\theta : X_1 \to X_2$ such that $\phi \circ f_0 = \phi \circ \theta$ for all $\phi \in \widehat{X_2}$. Characters separate points, so $f_0 = \theta$. Hence $f = f(e) + f_0$ is affine.

If X_2 has infinite entropy, then it is possible to construct a continuous family of maps $h_t : X_1 \to X_2$, all equivariant and continuous, with $h_0 = 0$ and $h_1 \neq 0$. Hence some h_t is not an affine map.