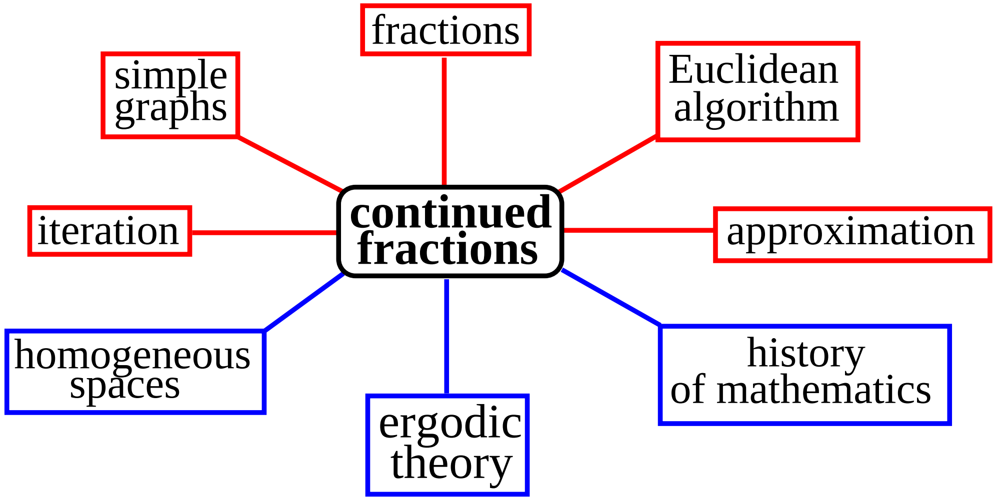
8/3/06, Park Farm Hotel

8/3/06, Park Farm Hotel

Continued fractions as a bridging topic

Continued fractions as a bridging topic



continued fractions

fractions

simple graphs

Euclidean algorithm

iteration

approximation

homogeneous spaces

ergodic theory

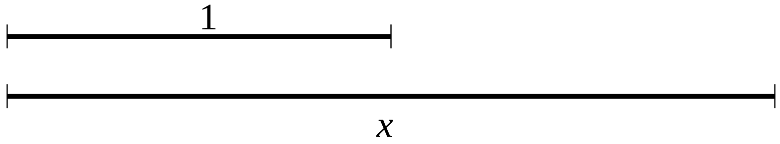
history of mathematics

Question: using a pencil only, can you find the ratio between two lengths very accurately?

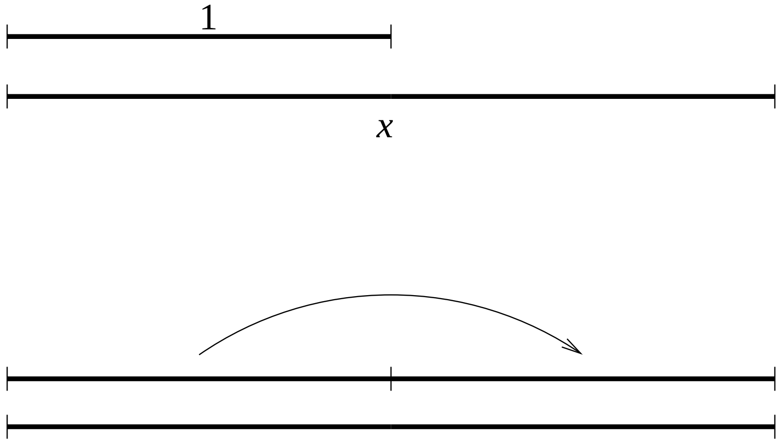
Question: using a pencil only, can you find the ratio between two lengths very accurately?

For example, can you find the ratio between the height and the width of a sheet of A4?

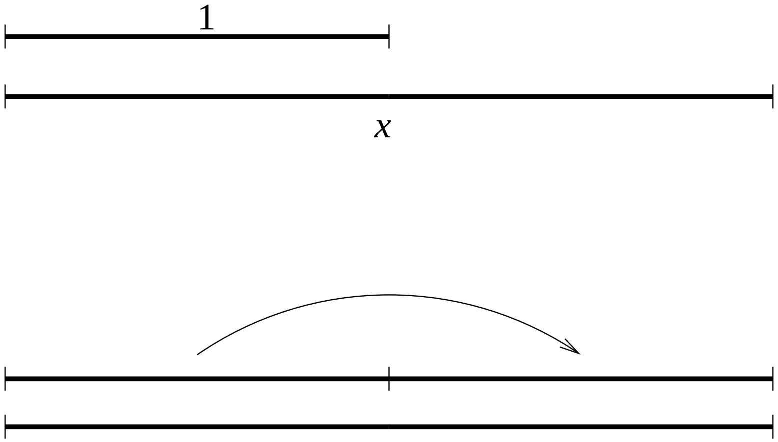
This could be easy:



This could be easy:



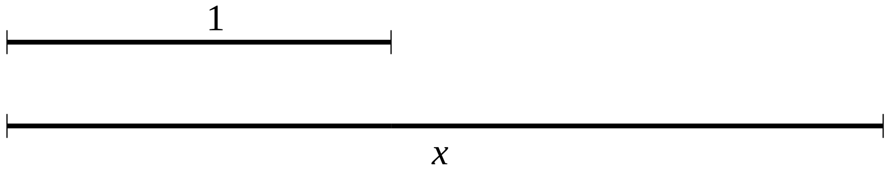
This could be easy:



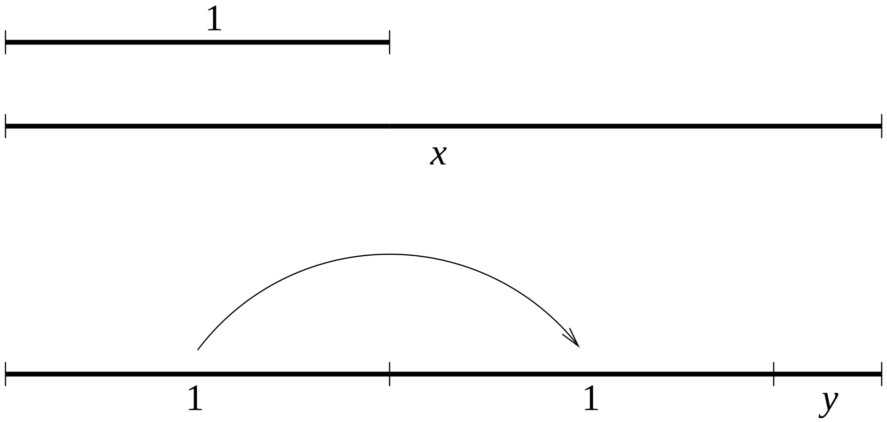
so $x = 2$ and the ratio is $1 : 2$.

It could be a bit more involved:

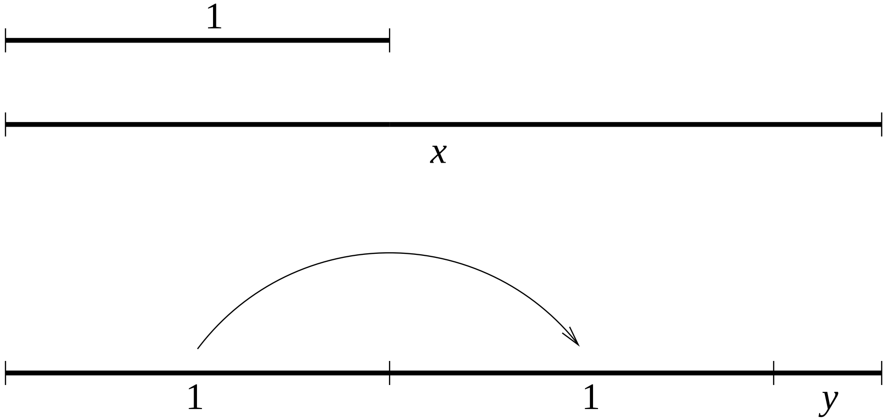
It could be a bit more involved:



It could be a bit more involved:

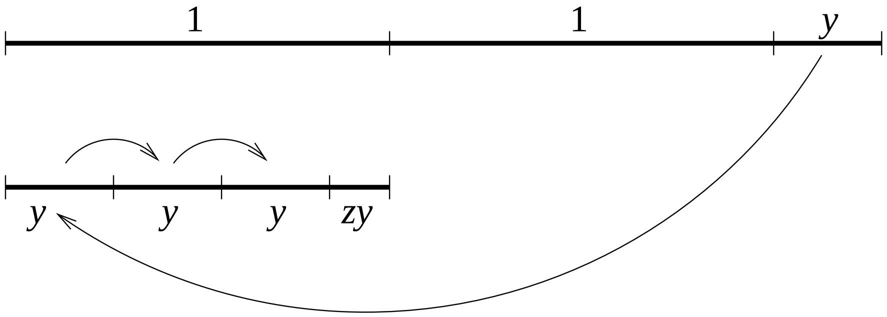


It could be a bit more involved:

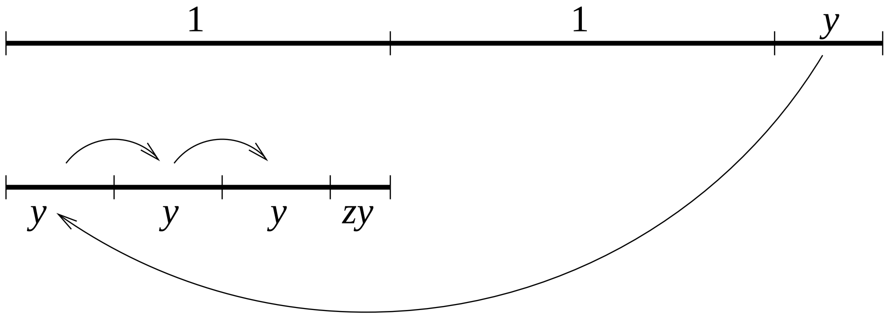


so the ratio is $1 : x$ where $x = 2 + y$.

Now compare the remainder y with 1:



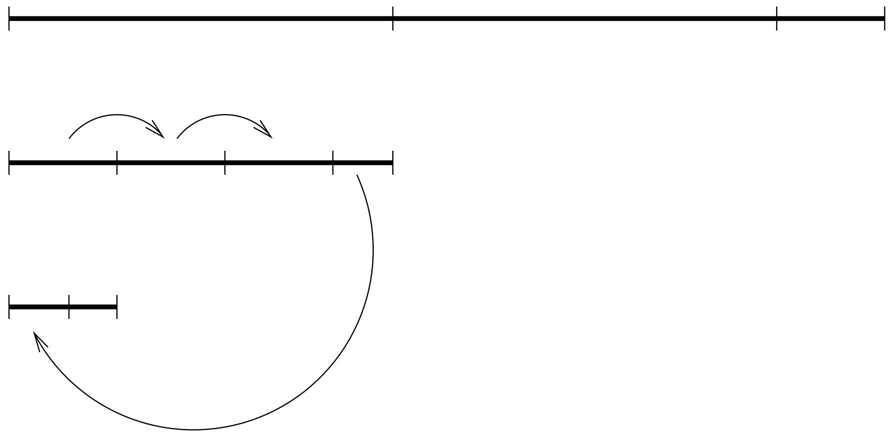
Now compare the remainder y with 1:



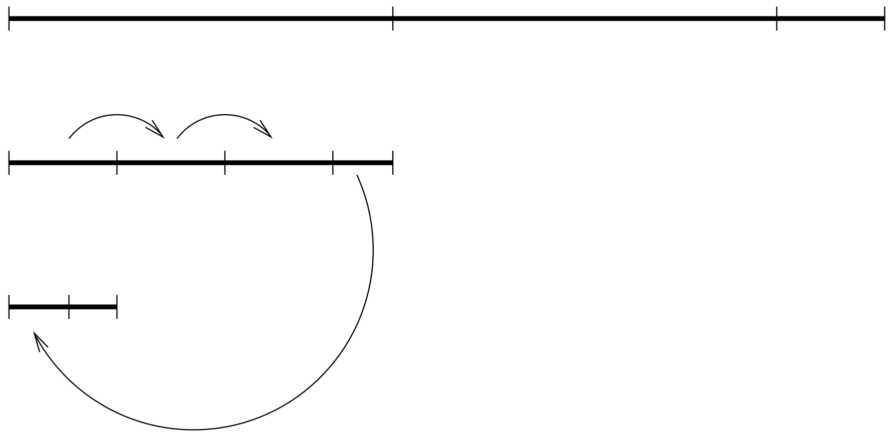
which amounts to writing $1 = (3+z)y$, so $y = \frac{1}{3+z}$.

Then compare the new remainder z with the previous remainder y:

Then compare the new remainder z with the previous remainder y:



Then compare the new remainder z with the previous remainder y:



to deduce that $z = \frac{1}{1+\cdots}$

and so on. In this example the process stops after 4 steps, giving

$$x = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}}}$$

which is a *continued fraction* for $2\frac{9}{32}$.

Continued fractions go back (implicitly) to Euclid (approx. 300 B.C.)

Continued fractions go back (implicitly) to Euclid (approx. 300 B.C.)

Raphael's Euclid from his "School of Philosophers" fresco

The Euclidean Algorithm (division)

The Euclidean Algorithm (division)

Compute the gcd of 67 and 24:

The Euclidean Algorithm (division)

Compute the gcd of 67 and 24:

$$67 = 2 \times 24 + 19$$

The Euclidean Algorithm (division)

Compute the gcd of 67 and 24:

$$67 = 2 \times 24 + 19$$

$$24 = 1 \times 19 + 5$$

The Euclidean Algorithm (division)

Compute the gcd of 67 and 24:

$$67 = 2 \times 24 + 19$$

$$24 = 1 \times 19 + 5$$

$$19 = 3 \times 5 + 4$$

The Euclidean Algorithm (division)

Compute the gcd of 67 and 24:

$$67 = 2 \times 24 + 19$$

$$24 = 1 \times 19 + 5$$

$$19 = 3 \times 5 + 4$$

$$5 = 1 \times 4 + 1.$$

The Euclidean Algorithm (division)

Compute the gcd of 67 and 24:

$$67 = 2 \times 24 + 19$$

$$24 = 1 \times 19 + 5$$

$$19 = 3 \times 5 + 4$$

$$5 = 1 \times 4 + 1.$$

First conclusion: the gcd of 67 and 24 is 1 (the last remainder).

Second conclusion:

Second conclusion:

$$\frac{67}{24} = 2 + \frac{19}{24},\ \frac{24}{19} = 1 + \frac{5}{19},\ \frac{19}{5} = 3 + \frac{4}{5},\ \frac{5}{4} = 1 + \frac{1}{4}.$$

Second conclusion:

$$\frac{67}{24} = 2 + \frac{19}{24}, \ \frac{24}{19} = 1 + \frac{5}{19}, \ \frac{19}{5} = 3 + \frac{4}{5}, \ \frac{5}{4} = 1 + \frac{1}{4}.$$

Hence

$$\frac{67}{24} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{4}}}}.$$

Second conclusion:

$$\frac{67}{24} = 2 + \frac{19}{24},\ \frac{24}{19} = 1 + \frac{5}{19},\ \frac{19}{5} = 3 + \frac{4}{5},\ \frac{5}{4} = 1 + \frac{1}{4}.$$

Hence

$$\frac{67}{24} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{4}}}}.$$

The 2,1,3,1,4 are the *terms* or *partial quotients* of the continued fraction.

The fractions obtained by stopping early are called *convergents*:

$$2 < \frac{67}{24}$$

The fractions obtained by stopping early are called *convergents*:

$$2 < \frac{67}{24}$$

$$2 + \frac{1}{1} = 3 > \frac{67}{24}$$

The fractions obtained by stopping early are called *convergents*:

$$2 < \tfrac{67}{24}$$

$$2 + \tfrac{1}{1} = 3 > \tfrac{67}{24}$$

$$2 + \frac{1}{1 + \frac{1}{3}} = \tfrac{11}{4} < \tfrac{67}{24}$$

The fractions obtained by stopping early are called *convergents*:

$$2 < \tfrac{67}{24}$$

$$2 + \tfrac{1}{1} = 3 > \tfrac{67}{24}$$

$$2 + \cfrac{1}{1 + \tfrac{1}{3}} = \tfrac{11}{4} < \tfrac{67}{24}$$

$$2 + \cfrac{1}{1 + \cfrac{1}{3 + \tfrac{1}{1}}} = \tfrac{14}{5} > \tfrac{67}{24}$$

Leonhard Euler *"De Fractionibus Continuis Dissertation"*, (1737)

Euler developed the theory of infinite continued fractions, which has huge significance both practically and theoretically.

Geometrical interpretation of continued fractions due to Klein (1895)

Geometrical interpretation of continued fractions due to Klein (1895)

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}$$

Geometrical interpretation of continued fractions due to Klein (1895)

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}$$

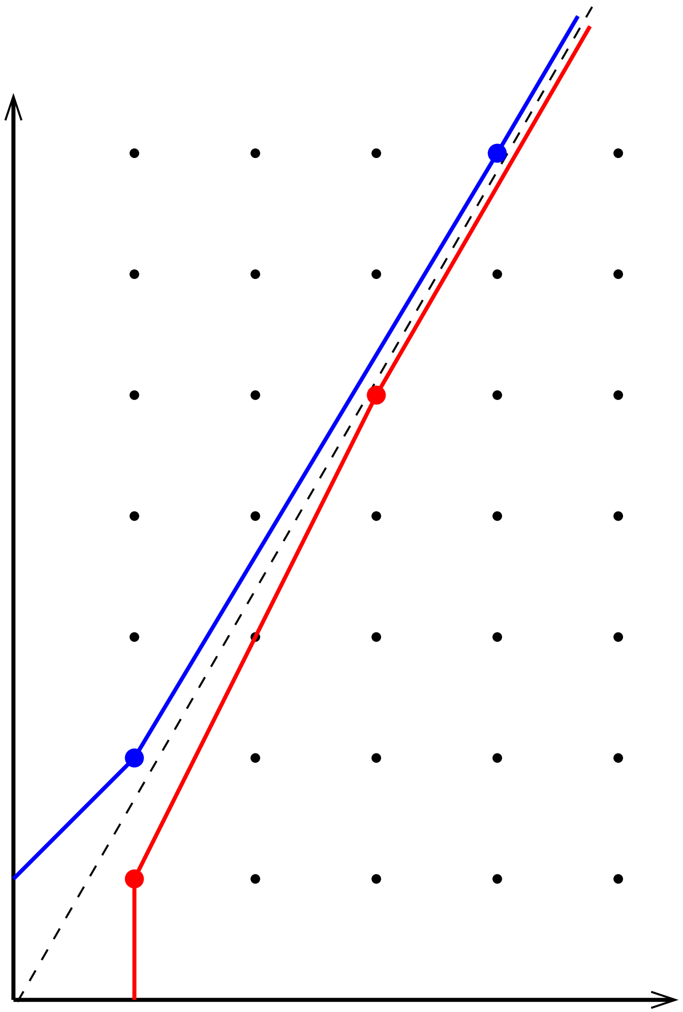
The convergents are $\frac{1}{1}$, $\frac{2}{1}$, $\frac{5}{3}$, $\frac{7}{4}$, $\frac{19}{11}$, $\frac{26}{15}, \ldots$.

Geometrical interpretation of continued fractions due to Klein (1895)

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}$$

The convergents are $\frac{1}{1}$, $\frac{2}{1}$, $\frac{5}{3}$, $\frac{7}{4}$, $\frac{19}{11}$, $\frac{26}{15}, \ldots$.

Now draw the graph of $y = \sqrt{3}x$ and imagine two threads joining $(0,0)$ to "(∞,∞)" catching on pins at all integer lattice points...



The red thread catches at $(1,1)$, $(3,5)$, $(11,19),\ldots$;

the blue thread catches at $(1,2)$, $(4,7)$, $(15,26),\ldots$.

The red thread catches at $(1, 1)$, $(3, 5)$, $(11, 19), \ldots$;

the blue thread catches at $(1, 2)$, $(4, 7)$, $(15, 26), \ldots$.

The triangle made by the origin and any two successive catch points has area $\frac{1}{2}$.

Patterns in continued fractions

Lagrange's Theorem: Any quadratic irrational $a + b\sqrt{D}$ with a, b rational and D an integer has a continued fraction whose terms eventually become periodic.

Patterns in continued fractions

Lagrange's Theorem: Any quadratic irrational $a + b\sqrt{D}$ with a, b rational and D an integer has a continued fraction whose terms eventually become periodic.

For example, if $\sqrt{2} = 1 + \frac{1}{a_1}$ then $a_1 = \frac{1}{\sqrt{2}-1} = \sqrt{2} + 1 = 2 + \frac{1}{a_2}$ (since $\sqrt{2} + 1$ is between 2 and 3).

Patterns in continued fractions

Lagrange's Theorem: Any quadratic irrational $a + b\sqrt{D}$ with a, b rational and D an integer has a continued fraction whose terms eventually become periodic.

For example, if $\sqrt{2} = 1 + \frac{1}{a_1}$ then $a_1 = \frac{1}{\sqrt{2}-1} = \sqrt{2} + 1 = 2 + \frac{1}{a_2}$ (since $\sqrt{2} + 1$ is between 2 and 3).

Hence $a_2 = \frac{1}{a_1 - 2} = \frac{1}{\sqrt{2}-1} = \sqrt{2} + 1 = a_1$,

so we deduce that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

so we deduce that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

Notice that this shows (incidentally) that $\sqrt{2}$ is irrational.

so we deduce that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

Notice that this shows (incidentally) that $\sqrt{2}$ is irrational.

Euler asked if $e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$ is rational.

Deeper patterns

Euler proved that

$$\frac{e-1}{e+1} = \cfrac{1}{2+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cdots}}}},$$

where the terms continue in an arithmetic progression.

Deeper patterns

Euler proved that

$$\frac{e-1}{e+1} = \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cdots}}}},$$

where the terms continue in an arithmetic progression.

Similarly, e itself has terms $1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \cdots$ and so on.

Deeper patterns

Euler proved that

$$\frac{e-1}{e+1} = \cfrac{1}{2+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cdots}}}},$$

where the terms continue in an arithmetic progression.

Similarly, e itself has terms $1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \cdots$ and so on.

Similar patterns exist for $\sqrt{e}$, $\sqrt[3]{e}$, etc.

Deeper problems

Deeper problems

It is fair to say that not much else is known about specific numbers.

Open problem 1: Are the terms in the continued fraction for $\sqrt[3]{2}$ bounded?

$$\sqrt[3]{2} = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cdots}}}}}}.$$

Open problem 1: Are the terms in the continued fraction for $\sqrt[3]{2}$ bounded?

$$\sqrt[3]{2} = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cdots}}}}}}.$$

It is known that the terms cannot increase very fast, but nothing else is known.

Open problem 2: Is there a pattern in the continued fraction for π,

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}?$$

Open problem 2: Is there a pattern in the continued fraction for π,

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}?$$

The convergents are the approximations to π that have been used for thousands of years: 3, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}, \ldots$.

The very large term 292 means that the convergent $\frac{355}{113}$ is a much better approximation than it has any right to be, given the size of its denominator.

The very large term 292 means that the convergent $\frac{355}{113}$ is a much better approximation than it has any right to be, given the size of its denominator.

There does not seem to be a pattern...

but...

some slightly different expansions do have structure, including

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

and

$$\frac{\pi}{2} = 1 - \cfrac{1}{3 - \cfrac{2 \cdot 3}{1 - \cfrac{1 \cdot 2}{3 - \cfrac{4 \cdot 5}{1 - \cfrac{3 \cdot 4}{3 - \cfrac{6 \cdot 7}{1 - \cfrac{5 \cdot 6}{3 - \cdots}}}}}}}.$$