"A Beautiful Mind": Why is beauty the metaphor?

The title of Sylvia Nasar's biography of the schizophrenic genius John Nash is *'A beautiful mind'* (a phrase suggested to her by Lloyd Shapley). The film does not really portray quite how difficult and unpleasant Nash could be to people around him, even when relatively well, so the question I would like to address is this. What could be 'beautiful' about the mind of John Nash?

beautiful *adj., Excelling in grace of form, charm of colouring, and other qualities which delight the eye, and call forth admiration.*

Clearly this is not beauty in the visual–artistic sense. I suggest it is a *necessary metaphor* for an elusive quality in Mathematics, and that this notion of beauty is central to the subject.

beautiful *adj., Impressing with charm the intellectual or moral sense, through inherent fitness or grace, or exact adaptation to a purpose; hence sometimes applied to things that, in other aspects, are even repulsive, as a beautiful operation in surgery.*

A computer search for the word 'beautiful' among Mathematical research papers finds examples such as *'Cauchy's singular integral operator and its beautiful spectrum'*, *'The fundamental group as a beautiful functor'* and *'Beauty and truth. Is quantum gauge theory beautiful enough to be true?'*. The phrase 'the ugly truth' is probably more familiar: why should the plausibility of *quantum gauge theory* (whatever that is) depend on how 'beautiful' it is?

There are almost as many answers to the question of why beauty is such a pervasive notion in Mathematics as there are mathematicians; I can give you my perspective.

The mathematician does not study pure mathematics because it is useful; [s]he studies it because [s]he delights in it and [s]he delights in it because it is beautiful. Henri Poincaré (1854–1912)

First, what is Mathematics? The *utility* of Mathematics in everything from pricing insurance, to making mobile phones secure, to the design of satellites, is undisputed and unparalleled. The physical world seems to obey Mathematical laws (or perhaps we can only understand the part of nature that does so). Pure Mathematics, however, is not studied with a particular problem in mind — indeed it is not *about* the physical universe at all. As Ian Stewart puts it, *"Over the centuries, the collective minds of mathematicians have created their own universe. I don't know where it is situated — I don't think there is a 'where' in any meaningful sense of the word — but I assure you this mathematical universe seems real enough when you're in it. It is by necessity described in symbols and pictures, but the symbols are no more that world than musical notation is music. And, not despite its peculiarities but because of them, it has provided humans with their deepest insights into the world around them."*

Last time, I asked: 'What does mathematics mean to you?' And some people answered: 'The manipulation of numbers, the manipulation of structures.' And if I had asked what music means to you, would you have answered: 'The manipulation of notes?' Serge Lang.

So Pure Mathematics studies a universe of the imagination, in which the objects and truths can only rarely be depicted. A vast array of visual, psychological, and linguistic metaphors have been developed to deal with this. Mathematicians will use phrases amongst themselves like 'a big group', 'a neat argument', 'a scary method', 'a direct attack', 'a nasty lemma', even 'a sexy result', projecting familiar ways of describing things onto this universe of the imagination. The most positive things that can be said of a piece of Mathematics are that it is 'deep' (indicating that it is a very profound fact or method, only arrived at after much effort) and that it is 'beautiful'. These all

Mathematics in this sense is a form of poetry, which has the same relation to the prose of practical mathematics as poetry has to prose in any other language. The element of poetry, the delight of exploring the medium for its own sake, is an essential ingredient in the creative process. J. Bronowski (1908–1974)

are metaphors for what is being seen in the universe of the imagination, and one of the most powerful of these metaphors is beauty.

What then might make a Mathematical argument or result 'beautiful'? Like pornography, this is much easier to *recognize* than it is to *define*. A beautiful proof must surely involve real ingenuity, and surprise us at some level with its elegance or unexpected simplicity. It might also contribute to the wider fabric of Mathematics, but it need not be particularly long or important.

Proofs are to mathematics what spelling (or even calligraphy) is to poetry. Mathematical works do consist of proofs, just as poems do consist of [letters]. Vladimir Arnold

The mathematician Paul Erdős imagined a divine book in which the most beautiful and profound mathematical proofs were recorded, and his highest compliment was to describe a proof as coming 'from the book'. Every now and then, either in their reading or, if they are lucky, in their own work, all mathematicians at some point feel they have touched beauty, or seen a proof 'from the book'. This notion of beauty can be a compass to a mathematician, sailing on an ocean of ideas with no obvious marks or buoys. The wonderful, mysterious, Mathematics that we do not yet know will hold many surprises, but for now we can safely say that the pursuit of beauty as we perceive it in Mathematics is a good compass to use.

The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics. G.H.21 Hardy (1877–1947).

An ancient proof of Pythagoras' Theorem 

The sublime must always be great; the beautiful can also be small. I. Kant (1724–1804)

Two final comments, both a little ironic. The first is that John Nash has several different public faces. His short period of work in game theory resulted many years later in a shared Nobel Prize for Economics, which thrust him into the public eye. He also had a short but highly influential career as a research mathematician, and he proved a series of profound results in differential geometry, partial differential equations and embedding problems.

My work always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose the beautiful. Herman Weyl (1885–1955)

The second is an extraordinary paradox: the universe is stubbornly mathematical. Pure Mathematics, pursued sometimes in a spirit of pure intellectual enquiry with aesthetic motives, keeps on telling us deep truths about the world around us.

To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature ... If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in. Richard Feynman (1918–1988)

A short reading list

G.H. Hardy, *A Mathematician's Apology*, Cambridge University Press, Cambridge (1967).

P. Hoffman, *The Man Who Loved Only Numbers: The Story of Paul Erdős and the Search for Mathematical Truth*, Fourth Estate, London (1998).

K.R. Jamison, *Touched with Fire: Manic-Depressive Illness and the Artistic Temperament*, The Free Press, New York (1993).

Sylvia Nasar, *A Beautiful Mind*, Faber and Faber, London (1988).

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