Orbit counting without hyperbolicity

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Let $f : X \to X$ be a map that is hyperbolic (suitably interpreted to include Axiom A, expansive group automorphisms, group endomorphisms with expansive invertible extension and so on). Let $h = h(f)$ denote the topological entropy of f.

A *closed orbit* τ of *length* $|\tau| = n$ is a set of the form

$$\{x, f(x), f^2(x), \ldots, f^n(x) = x\}$$

with cardinality n.

Then

$$\pi_f(X) = \#\{\tau \mid |\tau| \leq X\} \sim \frac{e^{h(X+1)}}{X(e^h - 1)}, \qquad (1)$$

like the Prime Number Theorem, and

$$\sum_{|\tau| \leq X} \frac{1}{e^{h|\tau|}} \sim \log X + C. \qquad (2)$$

like Mertens' Theorem. (These are results of Parry, Pollicott, Sharp and others).

Without hyperbolicity, less is known. For ergodic toral automorphisms that are not hyperbolic, Waddington shows

$$\pi(X) \sim \frac{e^{h(X+1)}}{h} \sum_{\rho \in U} K(\rho) \frac{\rho^{x+1}}{\rho e^h - 1}, \qquad (3)$$

for some integers $K(\rho)$ and U the (finite) set of eigenvalues of modulus one.

In a different direction, Knieper found asymptotic upper and lower bounds for the function counting closed geodesics on rank-1 manifolds of non-positive curvature of the form

$$A\frac{e^{hX}}{X} \leq \#\{\text{length} \leq X\} \leq Be^{hX}$$

for constants $A, B > 0$.

Simple test case: Consider the map

$$\phi : x \mapsto 2x$$

on the ring $\mathbb{Z}[\frac{1}{3}]$. Write $X = \widehat{\mathbb{Z}[\frac{1}{3}]}$ for the dual (character) group, and $f = \widehat{\phi}$ for the dual map. The pair (X, f) is the map we will study here, using the following properties:

- Locally, the action of f is isometric to the map $(s, t) \mapsto (2s, 2t)$ on an open set in $\mathbb{R} \times \mathbb{Q}_3$.

- In this local picture, the real coordinate is stretched by 2 while the action on the 3-adic coordinate is an isometry.

- The number of points of period n under f is $|2^n - 1| \cdot |2^n - 1|_3$.

- The topological entropy of f is $\log 2$.

The map f is an extension of the circle-doubling map by a cocycle in $\mathbb{Z}_3$, the 3-adic integers. The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\imath} \mathbb{Z}[\tfrac{1}{3}] \longrightarrow \mathbb{Z}[\tfrac{1}{3}]/\mathbb{Z} \longrightarrow 0$$

(where $\imath$ is the inclusion map) commutes with $x \mapsto 2x$. The dual is the short exact sequence

$$0 \longrightarrow \mathbb{Z}_3 \longrightarrow X \xrightarrow{\widehat{\imath}} \mathbb{T} \longrightarrow 0$$

which commutes with the dual of $x \mapsto 2x$. This expresses $f : X \to X$ as an extension of the circle-doubling map by a cocycle taking values in $\mathbb{Z}_3$. This extension kills certain periodic orbits – how many?

Write

$$\mathcal{F}_n(T) = \#\{x \in X \mid T^n x = x\},$$

$$\mathcal{L}_n(T) = \#\{x \in X \mid T^n x = x$$

$$\text{and } \#\{T^k x\}_{k\in\mathbb{N}} = n\},$$

so the number of closed orbits of length n is

$$\mathcal{O}_n(T) = \mathcal{L}_n(T)/n. \tag{4}$$

The results stated earlier use a meromorphic extension of the zeta function

$$\exp\left(\sum_{n=1}^{\infty} \mathcal{F}_n(T) z^n/n\right).$$

In our case, notice that

$$|2^n - 1|_3 = \begin{cases} \frac{1}{3}|n|_3 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

so

$$\frac{2^n - 1}{3n} \leq \mathcal{F}_n(T) \leq 2^n - 1.$$

A small change – the radius of convergence of the zeta function is not affected. But...

Lemma. The zeta function of f has natural boundary $|z| = \frac{1}{2}$.

Idea of proof: show that a related function has logarithmic singularities as z approaches any 3^kth root of unity from inside the unit circle.

So the standard approach will not work. Instead try to control deviation from the standard case.

Recall $g : \mathbb{T} \to \mathbb{T}$ is the map $x \mapsto 2x \bmod 1$, and f is the non-hyperbolic $\mathbb{Z}_3$ extension of g. Instead of a Prime Number Theorem we find something like Tchebychef:

Theorem

$$\pi_f(X) \leq \pi_g(X) \text{ for all } X \geq 1,$$

and

$$\limsup_{X\to\infty} \frac{X\pi_f(X)}{2^{X+1}} \leq 1, \quad \liminf_{X\to\infty} \frac{X\pi_f(X)}{2^{X+1}} \geq \frac{1}{3}.$$

The proof goes as follows: Let $b_n = 2^n - 1$ and $a_n = b_n|2^n - 1|_3$, so

$$\pi_f(X) = \sum_{n \le X} \mathcal{O}_n(f) = \sum_{n \le X} \frac{1}{n} \sum_{d|n} \mu(n/d) a_d.$$

We first claim that

$$\mathcal{O}_n(f) \le \mathcal{O}_n(g) \text{ for all } n \ge 1.$$

This does not follow *a priori* from

$$\mathcal{F}_n(f) \le \mathcal{F}_n(g) \text{ for all } n \ge 1.$$

For odd n, $a_n = b_n$ so $\mathcal{O}_n(f) = \mathcal{O}_n(g)$ (since all factors of n are also odd).

For even n,

$$\begin{aligned} \mathcal{L}_n(f) &= \sum_{d|n} \mu(n/d) a_d \\ &\le a_n + \sum_{d|n, d<n} a_d \\ &\le \tfrac{1}{3} b_n + \textstyle\sum_{d|n, d<n} b_d \\ &\le \sum_{d|n} \mu(n/d) b_d = \mathcal{L}_n(g) \end{aligned}$$

so $\mathcal{O}_n(f) \le \mathcal{O}_n(g)$ again.

That gives the upper bound.

Turning to the lower bound, write

$$\delta(X) = \pi_g(X) - \pi_f(X) \geq 0,$$

so

$$\delta(X) = \sum_{n \leq X} (\mathcal{O}_n(g) - \mathcal{O}_n(f))$$

$$= \sum_{2|n \leq X} (\mathcal{O}_n(g) - \mathcal{O}_n(f)) \leq \sum_{2|n \leq X} \mathcal{O}_n(g).$$

So we need to estimate the size of

$$\sum_{2|n \leq X} \mathcal{O}_n(g),$$

the number of orbits of even length under g. Notice that

$$\mathcal{F}_n(g^2) = 4^n - 1$$

and g^2 (the map $x \mapsto 4x$ mod 1 on the circle) has

$$\sum_{n \leq X} \mathcal{O}_n(g^2) \sim \frac{4^{X+1}}{3X}.$$

Now use a trick:

$$\mathcal{O}_n(g^2) = \begin{cases} 2\,\mathcal{O}_{2n}(g) + \mathcal{O}_n(g) & \text{if } n \text{ is odd,} \\ 2\,\mathcal{O}_{2n}(g) & \text{if } n \text{ is even.} \end{cases}$$

So

$$\begin{aligned} 2\sum_{n\le X} \mathcal{O}_{2n}(g) &= \sum_{2\nmid n\le X} \Big(\mathcal{O}_n(g^2) - \mathcal{O}_n(g)\Big) + \\ &\qquad \sum_{2\mid n\le X} \mathcal{O}_n(g^2) \\ &= \sum_{n\le X} \mathcal{O}_n(g^2) - \sum_{2\nmid n\le X} \mathcal{O}_n(g). \end{aligned}$$

Now $\sum_{n\le X} \mathcal{O}_n(g^2) \sim \frac{4^{X+1}}{3X}$, so

$$\sum_{n\le X} \mathcal{O}_{2n}(g) \sim \frac{2}{3}\cdot\frac{4^X}{X} - \frac{1}{2}\sum_{2\nmid n\le X} \mathcal{O}_n(g).$$

On the other hand, $\sum_{n\le X} \mathcal{O}_n(g) \sim \frac{2^{X+1}}{X}$, so the last term is of lower order. It follows that

$$\sum_{n\le X} \mathcal{O}_{2n}(g) \sim \frac{2}{3}\cdot\frac{4^X}{X},$$

so

$$\pi_g(X) - \pi_f(X) \leq \sum_{2|n\leq X} \mathcal{O}_n(g)$$

$$\sim \frac{2}{3}\cdot\frac{4^{X/2}}{X/2} = \frac{2}{3}\cdot\frac{2^{X+1}}{X}.$$

Thus

$$\limsup_{X\to\infty} \frac{X\pi_f(X)}{2^{X+1}} \leq 1, \qquad \liminf_{X\to\infty} \frac{X\pi_f(X)}{2^{X+1}} \geq \frac{1}{3}.$$

What is the true picture?

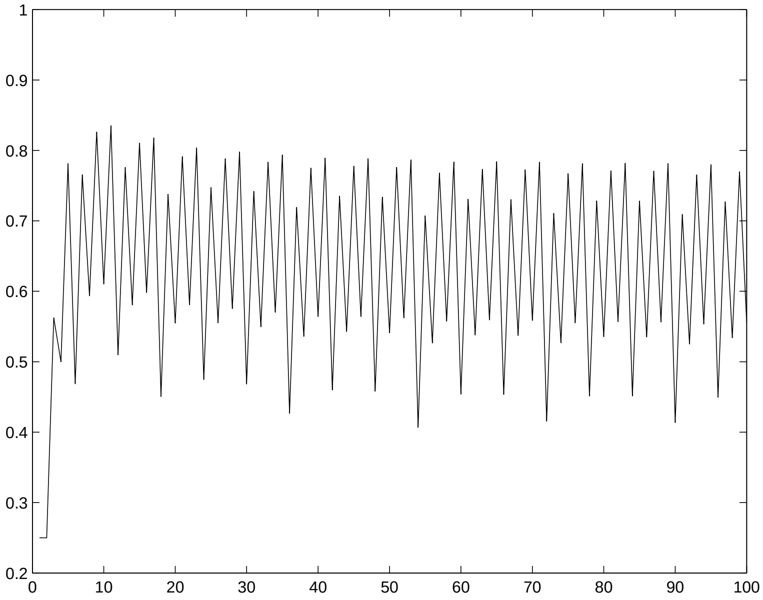

Figure 1: Graph of $\pi_f(x)(x/2^{(x+1)})$

Infinitely many limit points are expected.

Mertens: for the circle-doubling map,

$$\sum_{n\le X}\frac{\mathcal{O}_n(g)}{2^n}=\log X+O(1).$$

Theorem

$\frac{1}{2}\log X+O(1)\le\sum_{n\le X}\frac{\mathcal{O}_n(f)}{2^n}\le\log X+O(1).$

The upper bound is done, since

$$\mathcal{O}_n(f)\le\mathcal{O}_n(g)$$

for all $n\ge 1$.

Lower bound:

$$\sum_{n\le X}\frac{\mathcal{O}_n(f)}{2^n}=\sum_{2\nmid n\le X}\frac{\mathcal{O}_n(f)}{2^n}+\sum_{2|n\le X}\frac{\mathcal{O}_n(f)}{2^n}$$

$$\ge\sum_{2\nmid n\le X}\frac{\mathcal{O}_n(g)}{2^n}=A(X).$$

This may be estimated in a similar way: First,

$$A(X)+\sum_{2|n\le X}\frac{\mathcal{O}_n(g)}{2^n}=\sum_{n\le X}\frac{\mathcal{O}_n(g)}{2^n}=\log X+O(1).$$

Then the orbit-iterate trick gives:

$$\begin{aligned}
2\sum_{2|n\le X}\frac{\mathcal{O}_n(g)}{2^n} &= 2\sum_{m\le X/2}\frac{\mathcal{O}_{2m}(g)}{2^{2m}} \\
&= \sum_{2\not| m\le X/2}\frac{\mathcal{O}_m(g^2)-\mathcal{O}_m(g)}{2^{2m}} \\
&\qquad + \sum_{2|m\le X/2}\frac{\mathcal{O}_m(g^2)}{2^{2m}} \\
&= \sum_{m\le X/2}\frac{\mathcal{O}_m(g^2)}{2^{2m}} \\
&\qquad - \sum_{2\not| m\le X/2}\frac{\mathcal{O}_m(g)}{2^{2m}} \\
&\ge \log\frac{X}{2}+O(1)=\log X+O(1)
\end{aligned}$$

since $\mathcal{O}_n(g)\le 2^n$ implies that

$$\sum_{2\not| m\le X/2}\frac{\mathcal{O}_m(g)}{2^{2m}}\le\sum_{m\le X/2}\frac{\mathcal{O}_m(g)}{2^{2m}}<\infty.$$

Numerical evidence suggests that the lower asymptotic is genuinely lower than $\log X$.