Disjointness and entropy geometry

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Theorem: If algebraic $\mathbb{Z}^d$-actions of entropy rank one on zero-dimensional groups look different, then they are disjoint.

Setting: let $\mathsf{X} = (X, \alpha)$ be an algebraic $\mathbb{Z}^d$-action. This means X is a compact abelian group, α an action of $\mathbb{Z}^d$ by automorphisms. Such an action automatically preserves Haar measure λ_X.

A joining of

$$\mathsf{X}_i = (X_i, \alpha_i), \quad 1 \le i \le n,$$

is a measure μ on $X_1 \times \cdots \times X_n$ invariant under $\alpha_1 \times \cdots \times \alpha_n$ with the property that the projection of μ onto the ith coordinate is λ_i for each i. Write $J(\mathsf{X}_1, \ldots, \mathsf{X}_n)$ for the collection of all joinings. The systems are mutually disjoint if the only joining is the product measure, so $J(\mathsf{X}_1, \ldots, \mathsf{X}_n) = \{\lambda_1 \times \cdots \times \lambda_n\}$. For $n = 2$ this property is simply called disjointness.

Kalinin and Katok (ETDS 2002) found very precise results for the case $X = \mathbb{T}^k$, a torus, of the form: $\mathsf{X}_1, \mathsf{X}_2$ not disjoint implies strong algebraic connection between the systems.

Algebra: An algebraic $\mathbb{Z}^d$-action is an action of $\mathbb{Z}^d$ generated by d commuting automorphisms of a compact abelian metrizable group X. Duality gives a one–to–one correspondence between countable modules $M, N, \dots$ over $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ and algebraic $\mathbb{Z}^d$-actions $\mathsf{X}_M = (X_M, \alpha_M), \mathsf{X}_N, \dots$ Write monomials (units) in R_d in the form $\mathbf{u}^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$.

Subdynamics: $N(\alpha)$ denotes the set of non-expansive vectors $\mathbf{v} \in \mathbb{R}^d \backslash \{0\}$. That is, $\mathbf{v} \in N(\alpha)$ if and only if for every $\epsilon > 0$ there exists a pair of points $x \neq y$ in X with the property that

$$\rho\left(\alpha^{\mathbf{n}} x, \alpha^{\mathbf{n}} y\right) \leq \epsilon \text{ for all } \mathbf{n} \in \{\mathbf{m} \in \mathbb{Z}^d \mid \mathbf{v} \cdot \mathbf{m} < 0\}$$

where ρ is the metric on X. The whole action is expansive if there is an $\epsilon > 0$ with the property that

$$\rho\left(\alpha^{\mathbf{n}} x, \alpha^{\mathbf{n}} y\right) \leq \epsilon \text{ for all } \mathbf{n} \in \mathbb{Z}^d \implies x = y.$$

Markov shifts: Let α be an expansive algebraic $\mathbb{Z}^d$-action on a zero-dimensional group X. Then X is an algebraic Markov shift: There are integers q and s and a module of relations $J \subset (R_d/(q))^s$ such that

$$X \cong J^{\perp} \subset ((\mathbb{Z}/q\mathbb{Z})^s)^{\mathbb{Z}^d},$$

where $\cong$ denotes an algebraic isomorphism of $\mathbb{Z}^d$-actions and $J^{\perp}$ denotes the annihilator of the submodule J in the dual group $((\mathbb{Z}/q\mathbb{Z})^s)^{\mathbb{Z}^d}$ of the R_d-module $(R_d/(q))^s$. The $\mathbb{Z}^d$-action on X corresponds to the natural shift action on $J^{\perp}$. Having chosen such a presentation of the system, there is an associated (non-canonical) state partition

$$\xi = \xi(q, s, J)$$

comprising the q^s cylinder sets obtained by specifying the $\mathbf{0}$ coordinate (some of these sets may be empty).

For a measurable partition η of X, write

$$\eta^A = \bigvee_{\mathbf{n} \in A \cap \mathbb{Z}^d} \alpha^{-\mathbf{n}} \eta$$

for the join of η over any set $A \subset \mathbb{R}^d$. The conditional entropy of A given B with respect to η and μ is defined to be $H_\mu(\eta^A | \eta^B)$. For a fixed η (for instance the state partition for a fixed presentation), write $H_\mu(A|B)$ for this conditional entropy.

Coding: A set $A \subset \mathbb{R}^d$ codes $B \subset \mathbb{R}^d$ if for every $\mathbf{m} \in B \cap \mathbb{Z}^d$ there exists a polynomial

$$f(\mathbf{u}) = \sum_{\mathbf{n} \in A \cap \mathbb{Z}^d} f_{\mathbf{n}} \mathbf{u}^{\mathbf{n}}$$

such that $(\mathbf{u}^{\mathbf{m}} - f)M = 0_M$. Viewing X_M as a shift, this means that knowledge of the coordinates $(x_{\mathbf{m}})_{\mathbf{m} \in A}$ of a point $x \in X_M$ determines uniquely the coordinates $(x_{\mathbf{m}})_{\mathbf{m} \in B}$. Notice that

- A codes $B \implies H_\mu(B|A) = 0$;
- A codes $B \implies A + \mathbf{n}$ codes $B + \mathbf{n}$ for every $\mathbf{n} \in \mathbb{Z}^d$;
- A codes B, and $A \cup B$ codes C

  $$\implies$$

  A codes $B \cup C$.

Entropy geometry: (See recent paper of Einsiedler in Monatshefte; a technicality is that we now assume $d = 2$.) Let μ be an invariant measure on the zero-dimensional expansive algebraic system $\mathsf{X} = (X, \alpha)$ presented as a Markov shift.

For $\mathbf{v} \in \mathbb{Z}^2 \backslash \{0\}$ define

$$\mathsf{H}_{\mathbf{v}} = \{\mathbf{n} \in \mathbb{Z}^2 \mid \mathbf{v} \cdot \mathbf{n} < 0\}.$$

The half-space entropy of $\mathbf{v}$ is

$$\mathsf{h}_{\mu}(\mathbf{v}) = H_{\mu}(\xi^{\mathbf{v}^{\perp}} | \xi^{\mathsf{H}_{\mathbf{v}}}) \tag{1}$$

where ξ is the state partition (for a fixed presentation) and

$$\mathbf{v}^{\perp} = \{\mathbf{t} \in \mathbb{Z}^2 \mid \mathbf{v} \cdot \mathbf{t} = 0\}.$$

If $\mathcal{C}$ is an α-invariant σ-algebra, then similarly define

$$\mathsf{h}_{\mu}(\mathbf{v} | \mathcal{C}) = H_{\mu}(\xi^{\mathbf{v}^{\perp}} | \xi^{\mathsf{H}_{\mathbf{v}}} \vee \mathcal{C}).$$

Lemma: The conditional half-space entropy, despite appearances, does not depend on the choice of presentation

$$X \cong J^{\perp} \subset ((\mathbb{Z}/q\mathbb{Z})^s)^{\mathbb{Z}^d}.$$

Example: (Ledrappier's example) Let

$$X_1 = \Big\{x \in \mathbb{F}_2^{\mathbb{Z}^2} \mid x_{\mathbf{n}} + x_{\mathbf{n}+\mathbf{e}_1} + x_{\mathbf{n}+\mathbf{e}_2} = 0$$

$$\text{for all } \mathbf{n} \in \mathbb{Z}^2\Big\},$$

with α_1 the $\mathbb{Z}^2$-action defined by the natural shift action, and $\lambda = \lambda_{X_1}$ the Haar measure. Then $\mathbf{v} \in N(\alpha_1)$ if and only if $\mathbf{v}$ is parallel to an outward normal of the convex hull of the set

$$L = \{(0,0),(0,1),(1,0)\}.$$

Similarly, the half-space entropy $\mathsf{h}_\lambda(\mathbf{v})$ is positive if and only if $\mathbf{v}$ is parallel to an outward normal of the convex hull of the set L.

General picture: For a polynomial $f \in R_2$ with $f(\mathbf{u}) = \sum_{\mathbf{n} \in \mathbb{Z}^2} f_{\mathbf{n}} \mathbf{u}^{\mathbf{n}}$, the Newton polygon $\mathcal{N}(f)$ of f is the convex hull of the support $\{\mathbf{n} \mid f_{\mathbf{n}} \neq 0\}$.

In Ledrappier's example, the set of points whose convex hull determines the non-expansive directions is exactly the support of the polynomial $1 + u_1 + u_2$ generating the module of relations. The same holds more generally when the entropy co-rank is one.

For expansive systems X_M of entropy rank one on zero-dimensional groups:

- There is an annihilating polynomial $f \in R_d$ with $fM = 0_M$ and each vertex coefficient of f is coprime to q.

- For every direction $\mathbf{v}$, $\mathsf{h}_\mu(\mathbf{v}) < \infty$.

- If $\mathbf{v}$ is not an outward normal vector to an edge of $\mathcal{N}(f)$, then $\mathsf{h}_\mu(\mathbf{v}) = 0$.

- Hence, $\mathsf{h}_\mu(\mathbf{v}) > 0$ only for $\mathbf{v}$ in finitely many directions, all of them rational.

Theorem: (Einsiedler) If μ is any α-invariant measure on X, and $\mathcal{C}$ any α-invariant σ-algebra, then

$$h_\mu(\alpha^{\mathbf{n}}|\mathcal{C}) = \sum_{\mathbf{v}\cdot\mathbf{n}>0} (\mathbf{v}\cdot\mathbf{n})\mathsf{h}_\mu(\mathbf{v}|\mathcal{C})$$

where the sum is taken over all primitive integer vectors $\mathbf{v}$ with $\mathbf{v}\cdot\mathbf{n} > 0$.

Abramov–Rokhlin formula.

Theorem: Let ϕ : X $\to$ Y be a continuous surjective map between zero-dimensional expansive entropy rank one algebraic $\mathbb{Z}^2$-systems, and assume that ϕ sends the invariant measure μ on X to the invariant measure ν on Y. Then, for any non-zero vector $\mathbf{v} \in \mathbb{Z}^2$,

$$\mathsf{h}_\mu(\mathbf{v}) = \mathsf{h}_\nu(\mathbf{v}) + \mathsf{h}_\mu(\mathbf{v}|\phi^{-1}(\mathcal{B}_Y))$$

where $\mathcal{B}_Y$ denotes the Borel σ-algebra on Y.

A simple example:

Recall Ledrappier's Example,

$$X_1 = \Big\{ x \in \mathbb{F}_2^{\mathbb{Z}^2} \mid x_{\mathbf{n}} + x_{\mathbf{n}+\mathbf{e}_1} + x_{\mathbf{n}+\mathbf{e}_2} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^2 \Big\},$$

and its sibling

$$X_2 = \Big\{ x \in \mathbb{F}_2^{\mathbb{Z}^2} \mid x_{\mathbf{n}} + x_{\mathbf{n}+\mathbf{e}_1} + x_{\mathbf{n}-\mathbf{e}_2} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^2 \Big\}.$$

These systems are disjoint.

Let μ be a joining of the two systems. A polynomial which annihilates the module corresponding to $X = X_1 \times X_2$ is the product

$$(1 + u_1 + u_2)(1 + u_1 + u_2^{-1})$$

$$= u_2^{-1} + u_1 u_2^{-1} + u_1^2 + u_2 + u_1 u_2.$$

Write $\mathcal{B}_i$ for the Borel σ-algebra and $\mathcal{N}_i$ for the trivial σ-algebra on X_i, ξ_i for the state partition in X_i for $i = 1, 2$, and $\xi = \xi_1 \times \xi_2$ for the state partition in X.

By the Abramov–Rokhlin formula for half-space entropies,

$$\mathsf{h}_\mu(\mathbf{e}_2) = \mathsf{h}_{\lambda_2}(\mathbf{e}_2) + \mathsf{h}_\mu(\mathbf{e}_2 | \mathcal{N}_1 \times \mathcal{B}_2) \geq \log 2,$$

where we use the fact that

$$\mathsf{h}_{\lambda_2}(\mathbf{e}_2) = h_{\lambda_2}(\alpha^{\mathbf{e}_2}) = \log 2.$$

Similarly

$$\mathsf{h}_\mu(\mathbf{e}_1 + \mathbf{e}_2)$$

$$= \mathsf{h}_{\lambda_1}(\mathbf{e}_1 + \mathbf{e}_2) + \mathsf{h}_\mu(\mathbf{e}_1 + \mathbf{e}_2 | \mathcal{B}_1 \times \mathcal{N}_2) \geq \log 2,$$

so by the entropy formula the entropy of the map $\alpha^{\mathbf{e}_2}$ satisfies

$$h_\mu(\alpha^{\mathbf{e}_2}) = \mathsf{h}_\mu(\mathbf{e}_2) + \mathsf{h}_\mu(\mathbf{e}_1 + \mathbf{e}_2)$$

$$\geq \log 4 = h_\lambda(\alpha^{\mathbf{e}_2}).$$

That is, the joining measure μ is maximal for the transformation $\alpha^{\mathbf{e}_2}$. Since $\alpha^{\mathbf{e}_2}$ is itself an ergodic automorphism of a compact group with finite entropy, it follows from Berg's Theorem that $\mu = \lambda = \lambda_{X_1} \times \lambda_{X_2}$. Thus the systems X_1 and X_2 are disjoint.

Similar ideas prove the following:

Theorem: Let $\mathsf{X}_1, \ldots, \mathsf{X}_n$ be a collection of irreducible algebraic zero-dimensional $\mathbb{Z}^d$-actions, all with entropy rank one. If

$$N(\alpha_j) \backslash \bigcup_{k>j} N(\alpha_k) \neq \emptyset \text{ for } j = 1, \ldots, n$$

then the systems are mutually disjoint.