Why Mahler measures arise in dynamics

Mahler Measure in Mobile January 5-8, 2006

1. Entropy in algebraic dynamics

2. Convergence problems

3. Lehmer's problem

Notation:

Write

$$m(f) = m_1(f) = \int_0^1 \log|f(e^{2\pi i t})|dt$$

for $f \in \mathbb{Z}[u]$; more generally

$$m_d(f) = \int_0^1 .. \int_0^1 \log|f(e^{2\pi i t_1}, .., e^{2\pi i t_d})|dt_1..dt_d$$

for $f \in \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm d}]$.

Topological entropy $h(T)$ is a numerical invariant attached to a continuous map

$$T : X \to X$$

on a compact space; it measures the asymptotic growth in complexity of the orbits of the map.

In the special case

$$X = \text{compact metric group},$$

and

$$T = \text{continuous group automorphism},$$

it can be computed in terms of Haar measure.

Using the metric, write

$$B_n = \bigcap_{j=0}^{n-1} T^{-j} B_\epsilon(0)$$

and λ for Haar measure.

Theorem: [Bowen]

$$h(T) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \lambda \left(B_n \right).$$

Toral automorphisms

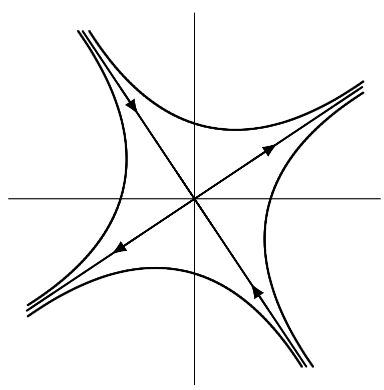
How big is $\lambda(B_n)$?

It is enough to look locally near the identity (this is a consequence of the homogeneous nature of groups: the local action of T is the same everywhere.) Indeed (Bowen) it is enough to compute this in a covering space.

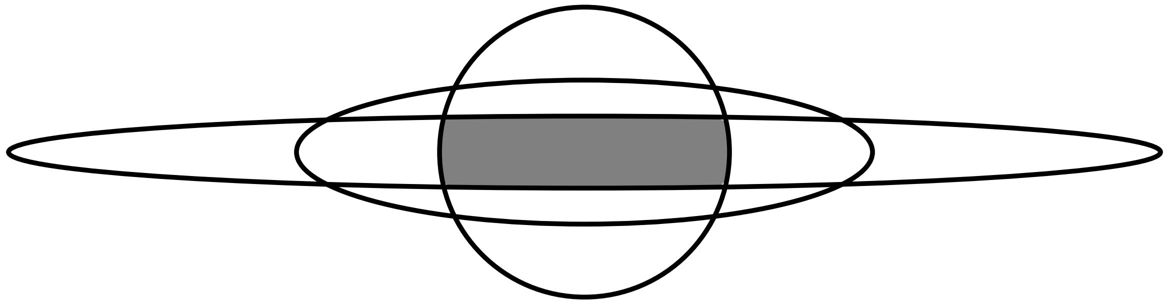
Consider the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

close to the identity in the 2-torus $\mathbb{T}^2$.



Use axes adapted to the eigenvectors to estimate $\lambda(B_n)$:



Thus $\lambda(B_n)$ is expected to be roughly ρ^{-n}, where ρ is the dominant eigenvalue of the matrix.

Refining this argument shows that

$$h(A) = \sum_{\text{eigenvalues}} \log^+ |\rho|.$$

for a toral automorphism defined by a matrix $A \in GL(d, \mathbb{Z})$ (Sinai, Rokhlin for $d = 2$; Arov in general).

Remark: So $h(A)$ is the Mahler measure of the characteristic polynomial of A by Jensen's formula.

A non-trivial convergence issue arises when periodic points are related to entropy.

For the toral automorphism corresponding to A in $GL(d, \mathbb{Z})$, the number of points of period k (points fixed by the kth iterate) is given by

$$\left|\mathbb{Z}^d/(A^k - I)\mathbb{Z}^d\right| = |\det(A^k - I)|$$

assuming that no eigenvalue is a unit root.

Lemma: The growth rate of the periodic points is the entropy:

$$\lim_{k\to\infty} \frac{1}{k} \log|\det(A^k - I)| = h(A).$$

The possibility that there are eigenvalues of unit modulus makes this non-trivial (it is, for example, a consequence of Baker's theorem).

This is again related to a property of Mahler measure. Writing the characteristic polynomial of A as

$$f(u) = \prod_i (u - \rho_i)$$

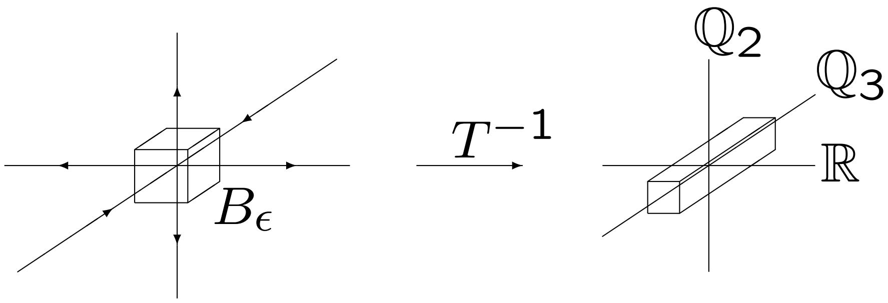
expresses this in terms of a Cauchy approximation to $m(f)$, since:

$$\begin{aligned}\frac{1}{k}\log|1-\rho_i^k| &= \frac{1}{k}\sum_{j=1}^{k}\log|e^{2\pi ij/k}-\rho_i| \\ &\stackrel{?}{\longrightarrow} \int_0^1 \log|e^{2\pi it}-\rho_i|dt.\end{aligned}$$

Solenoids

Essentially the same argument works for automorphisms of solenoids (compact, abelian, finite-dimensional abelian groups).

Let $T : X \to X$ be the automorphism dual to $x \mapsto \frac{3}{2}x$ on $\mathbb{Z}[\frac{1}{6}]$. Locally the group is isometric to $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$, and T^{-1} shrinks the real and 2-adic directions and dilates in the 3-adic direction:



Similar reasoning gives Abramov's formula:

$$h(\text{map dual to } x \mapsto \tfrac{a}{b}x) = \sum_{p \le \infty} \log^{+} |\tfrac{a}{b}|_p$$

$$= \log \max\{|a|, |b|\}.$$

This is also a Mahler measure in a natural way.

The map $x \mapsto \frac{a}{b}x$ is associated in a simple way with the polynomial $bu_1 - a$, and

$$\begin{aligned} m(bu_1 - a) &= \int_0^1 \log|be^{2\pi it} - a|dt \\ &= \log|a| + \int_0^1 \log|\tfrac{b}{a}e^{2\pi it} - 1|dt \\ &= \log|a| + \begin{cases} 0 & \text{if } |\frac{b}{a}| < 1; \\ \log|\frac{b}{a}| & \text{if } |\frac{b}{a}| \geq 1 \end{cases} \\ &= \log\max\{|a|, |b|\}. \end{aligned}$$

More generally, the automorphism dual to the action of $A \in GL(d, \mathbb{Q})$ on $\mathbb{Q}^d$ has entropy

$$\sum_{p \leq \infty} \sum_i \log^+ |\rho_{i,p}|_p$$

where the sum is over the eigenvalues in $\overline{\mathbb{Q}_p}$ (Yuzvinskii's formula). This is again a Mahler measure of an associated integral polynomial.

Commuting automorphisms

Algebraic actions of $\mathbb{Z}^d$ are associated via duality with modules over the ring

$$\mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}],$$

and the entropy of the system attached to the cyclic module

$$\mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]/\mathfrak{p}$$

is given by $m(f)$ if $\mathfrak{p} = \langle f \rangle$, 0 if $\mathfrak{p}$ is non-principal.

This is a difficult result, using machinery both on the dynamics side and in number theory. In particular, it uses a convergence result due to Lawton; in a special case this says that

$$m_2(f(u_1, u_2)) = \lim_{N \to \infty} m_1(f(u_1, u_1^N)),$$

where m_d is the d-dimensional Mahler measure. As with the last convergence question, this is easy if the polynomial does not vanish on a pair (z, w) with $|z| = |w| = 1$.

Lehmer's problem

Lehmer's problem asks if

$$\inf\{m(f) : f \in \mathbb{Z}[x], m(f) > 0\} > 0.$$

This is a significant problem in dynamics for the following reason. Viewed as measurable dynamical systems, ergodic compact group automorphisms (for toral automorphisms, ergodic means no eigenvalues are unit roots) are measurably isomorphic to Bernoulli shifts (iid processes). This means that they are classified up to measurable isomorphism by their entropy.

The 'interesting' part of the entropy of a group automorphism is carried by a solenoid in the sense that the entropy of any automorphism takes the form

$$\lim_{N\to\infty} m(F_N)$$

for some sequence (F_N) of polynomials in $\mathbb{Z}[u]$.

Thus the set of entropies of group automorphisms is either all of $[0, \infty]$ OR the countable set $\{m(f) : f \in \mathbb{Z}[u]\} \cup \{\infty\}$, depending on the answer to Lehmer's problem.

In particular, the question of whether or not group automorphisms provide measurable models of all Bernoulli shifts or only those with special entropies is exactly Lehmer's problem.