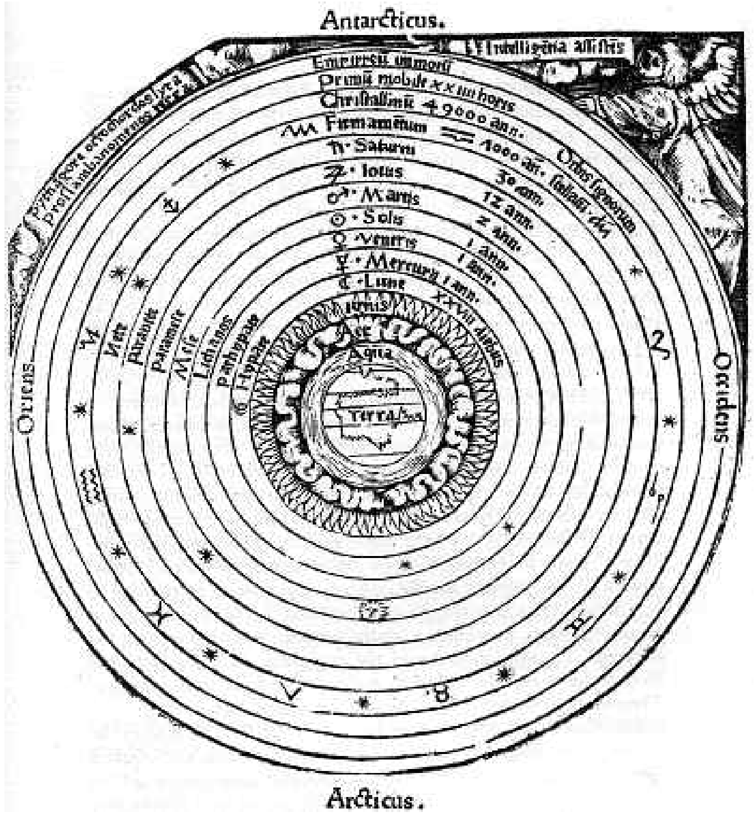
29/11/05, UEA

14th Century planetary orbits to 21st Century number theory: What is ergodic theory?

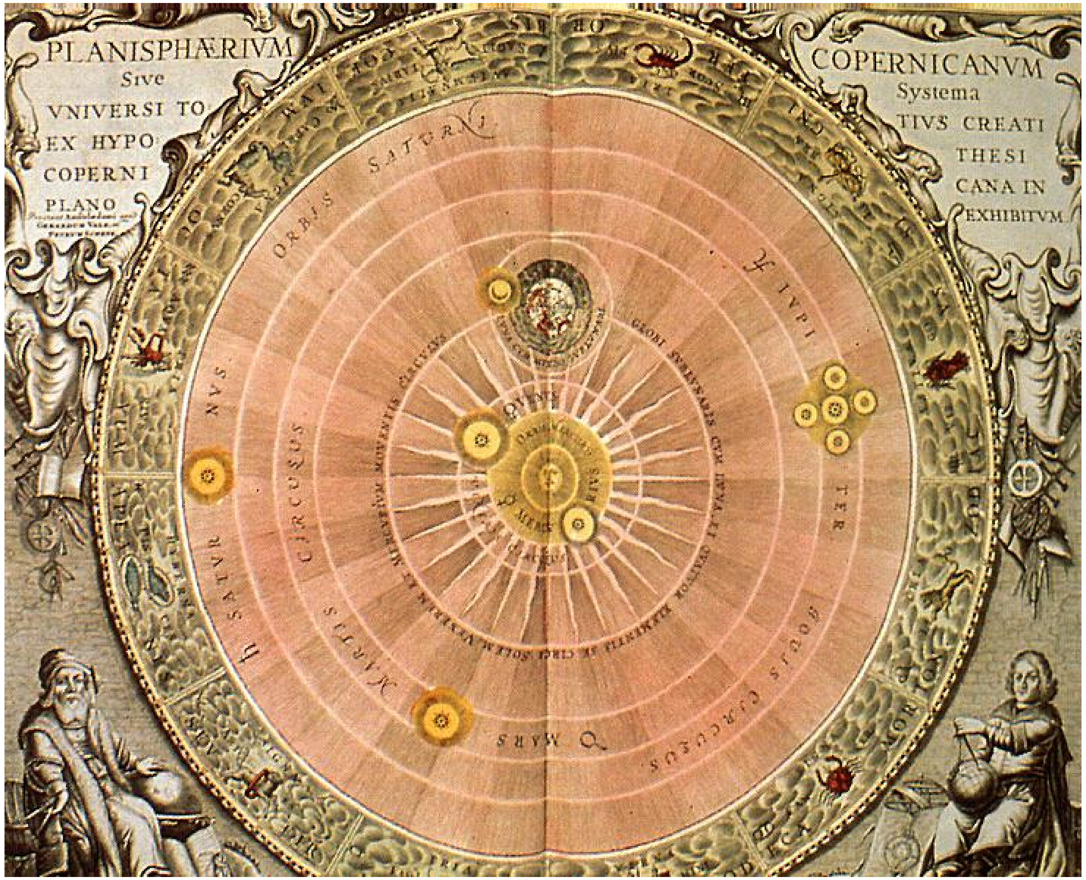
- Historical scope: ‘modern’ mathematics goes back to 350B.C.
- Mathematics is often a collective endeavour.
- Good questions have lasting impact.
- Mathematics is filled with surprising cross-connections.

Nicole d'Oresme (1323 – 1382)

The Solar system according to Aristotle (around 350 B.C.)



The Solar system according to Copernicus (1514; published 1543)



Oresme's c.v.

Master of Theology (1355).

Grand Master of the Collège de Navarre (1356).

Canon of Rouen Cathedral (1362).

Invented a 'coordinate' geometry long before Descartes.

Argued against Aristotle's idea of a stationary Earth in 1377, 200 years before Copernicus,

and even better,

Argued that no experiment can decide whether the heavens move East to West or the Earth West to East.

Oresme sometimes thought like a 14th Century theologian:

“... if anyone should make a mechanical clock, would he not make all the wheels move as harmoniously as possible?”

... so surely the Creator would make the planets have commensurable orbits?

Oresme could also make great imaginative leaps:

He answered his own question by arguing that the irrationality of ratios will not rob the heavens of their beauty, nor contradict the regularity of the orbits.

This is a profound idea for three reasons (at least).

Dynamical systems may evolve in highly constrained ways (obeying physical laws) but within that be ‘random’ and never repeating.

Commensurable orbits arise with vanishing probability.

At a deeper level, Oresme’s imagined solar system would exhibit *measure rigidity*.

In any system with a ‘symmetry’ (in this case, the fact that the solar system moves through time according to fixed laws), an *invariant measure* is a complete description of the statistical behaviour of the system.

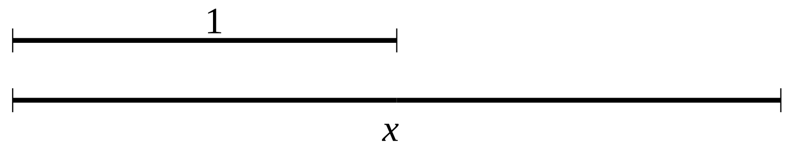
Sample question: how long does a planet spend on average over eternity in a specific part of space and with a specific range of velocities?

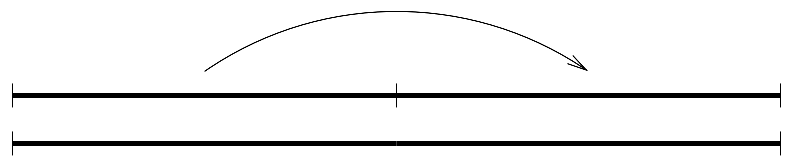
In the clockwork solar system, there are many invariant measures.

If all the orbits are incommensurable, there is exactly one invariant measure, and this phenomenon is called *measure rigidity*.

Exercise: using a pencil only, can you find the ratio between the lengths of two sticks very accurately?

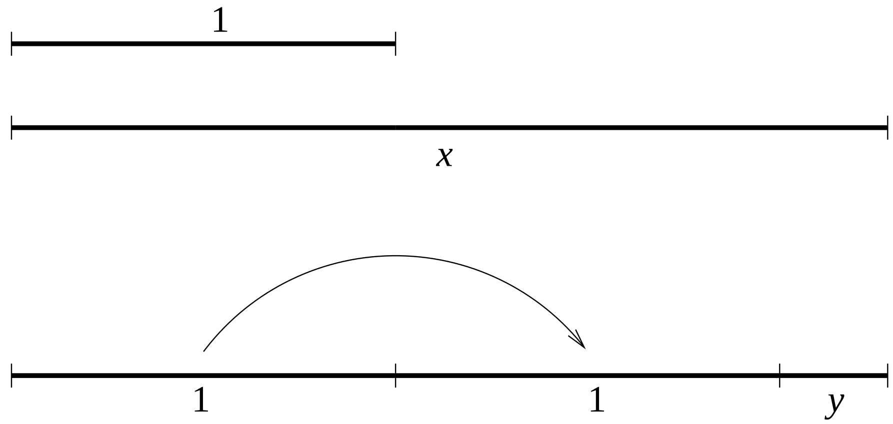
This could be easy:





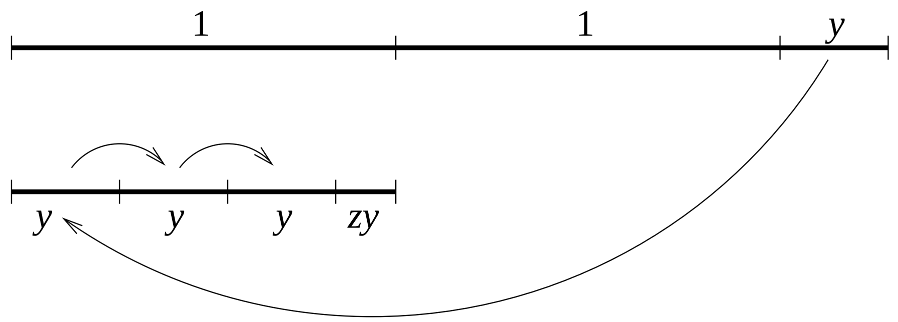
so $x = 2$ and the ratio is $1:2$.

It could be a bit more involved:



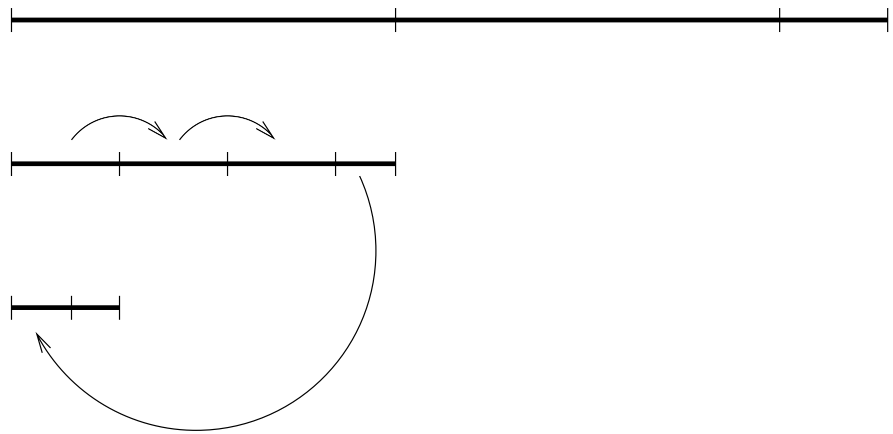
so the ratio is $1 : x$ where $x = 2 + y$.

Now compare the remainder y with 1:



which amounts to writing $1 = (3 + z)y$, so $y = \frac{1}{3+z}$.

Then compare the new remainder z with the previous remainder y:



to deduce that $z = \frac{1}{1+\cdots}$

and so on. In this example the process stops after 4 steps, giving

$$x = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}}}$$

which is a *continued fraction* for $2\frac{9}{32}$.

Usually the process will continue for ever – but it will always rapidly give a very good approximation. This process is of great importance because it can be applied to any number (ratio) at all.

A 16th Century question:

Given any number u, write $\langle u \rangle$ for the distance to the nearest whole number. Thus $\langle 2.3 \rangle = 0.3$, $\langle 4.9 \rangle = 0.1$ and so on. How small can you make $\langle nu \rangle$ by choosing a whole number n carefully?

Can you find a list of whole numbers $q_1, q_2, \ldots$, larger and larger, with $\langle q_n u \rangle$ getting smaller and smaller?

Or even with $\langle q_n u \rangle < \frac{1}{q_n}$?

This was shown to be possible in the 16th Century, using the theory of continued fractions.

For example, if $u = 1.4142135 \cdots = \sqrt{2}$, then

$$\begin{aligned} \langle u \rangle &< 1, \\ \langle 2u \rangle &< \tfrac{1}{2}, \\ \langle 5u \rangle &< \tfrac{1}{5}, \\ \langle 12u \rangle &< \tfrac{1}{12}, \\ \langle 29u \rangle &< \tfrac{1}{29}, \\ \langle 70u \rangle &< \tfrac{1}{70}, \end{aligned}$$

and so on.

In contrast, $\langle 69u \rangle$ is about 0.4, much larger than $\frac{1}{69}$. The numbers $1, 2, 5, 12, \ldots$ arise when a stick of length 1 is compared to a stick of length $\sqrt{2}$.

The importance of continued fractions is that they produce such a special list of whole numbers for *any* original number u at all, so this start to tell us something universal about real numbers.

Littlewood's Question (1930)

John Littlewood (1885 – 1977) enjoyed one of the most famous long collaborations in mathematics, with the more famous G.H. Hardy.

Littlewood was a formidable mathematician, hampered for much of his life by serious depression.

Littlewood asked what was possible for a *pair* of real numbers x and y.He argued as follows: if you choose a 'good' sequence $q_1, q_2, \ldots$ for the first number x, then

$$q_n \langle q_n x \rangle < 1$$

by *design*.On the other hand, the second number y will have

$$\langle q_n y \rangle \leq \frac{1}{2}$$

by *accident*. Can we be slightly more ingenious in choosing the numbers q_n to do a little better?

Littlewood's Conjecture: For any pair of numbers x and y, there is a sequence $q_1, q_2, \ldots$ with

$$q_n \langle q_n x \rangle \langle q_n y \rangle \to 0.$$

This has turned out to be a good question:

It concerns a basic property of all numbers.

It does not seem easy (it is still open).

Partial solutions have required the development of powerful new methods.

It is a bridge between dynamical systems and number theory.

It is a central testing ground for the state of the art in several other parts of mathematics.

Until 2003 the only progress on Littlewood involved special results: the set of pairs (x, y) for which it was known to hold formed a small set exhibiting possibly anomalous behaviour.

Ergodic theory studies the statistical behaviour of dynamical systems with symmetries. In Oresme's case, the symmetry is the evolution in time according to fixed physical laws.

In other settings more complicated symmetries are studied, described using an algebraic structure called a *Lie group.*

The theory of continued fractions can be encoded in terms of a family called $SL_2(\mathbb{R})$ of 2×2 matrices – a 3 dimensional space – with 1 dimension of symmetry (like time).

This system has many wildly behaved invariant measures – it has no rigidity.

Since the 1960s a series of deep results of the following form have emerged:

A ‘big enough’ symmetry group, and a ‘dynamical’ hypothesisimpliesmeasure rigidity (the invariant measures can be described).

This is a new phenomenon: in Oresme's solar system, the measure rigidity comes from the incommensurable orbits. In the 'big enough symmetry group' setting, a totally new mechanism is at work, where the rigidity cannot be seen (and does not hold) for any one direction of symmetry, but only holds for several of them combined.

It has been developed notably by Einsiedler (Ohio State), Furstenberg (Jerusalem), Katok (Penn. State), Kitchens (IUPUI Indiana), Lindenstrauss (Princeton), Margulis (Yale), Raghunathan (TIFR Mumbai), Ratner (UC Berkeley), Schmidt (Vienna), and many others.

Littlewood's problem can be encoded in terms of an eight-dimensional family of 3×3 matrices called $SL_3(\mathbb{R})$, with a 2-dimensional space of symmetries.

Margulis made several far-reaching conjectures about the extent of measure rigidity for such groups.

In 2003 Einsiedler, Katok and Lindenstrauss proved a version of Margulis' conjecture strong enough to make the first real progress on Littlewood: the set of exceptions to Littlewood forms a tiny set (a countable union of compact sets of zero box dimension).

Removing the 'dynamical' hypothesis would give the full result.

This work has given a context in which to view Littlewood's conjecture as something that is expected to be true, and added another strand to the links between number theory and ergodic theory.

Source of images:

Nicole D'Oresme: `http://www.nicole-oresme.com/`

Aristotle: `http://ise.uvic.ca/`

Copernicus: `http://aeea.nmns.edu.tw/`

Littlewood: `http://www-history.mcs.st-andrews.ac.uk/`