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- Historical scope: ‘modern’ mathematics goes back to 350B.C.

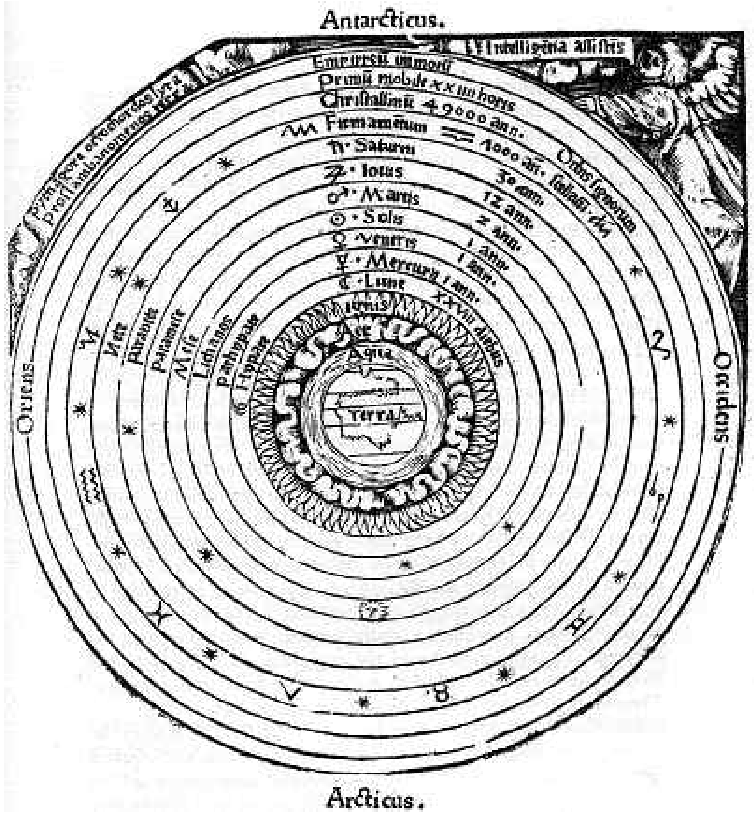
- Mathematics is often a collective endeavour.

- Good questions have lasting impact.

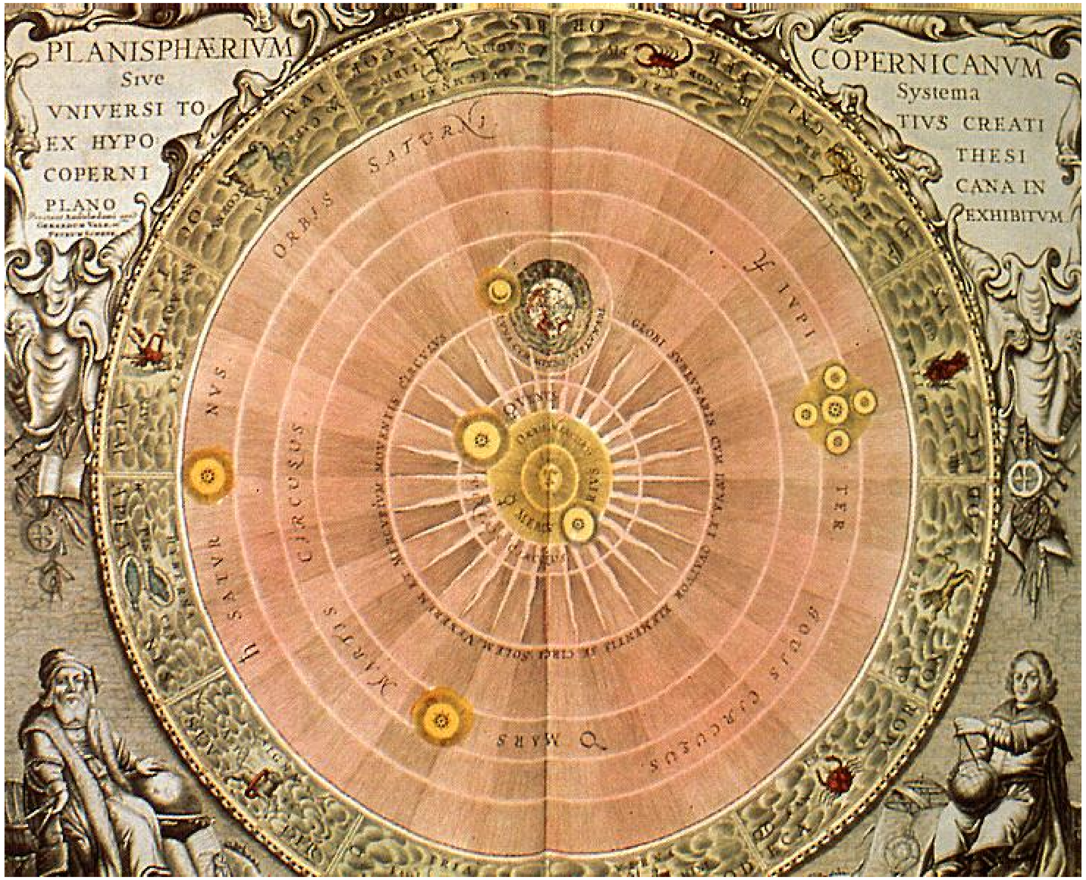
- Mathematics is filled with surprising cross-connections.

Nicole d’Oresme (1323 – 1382)

The Solar system according to Aristotle (around 350 B.C.)



The Solar system according to Copernicus (1514; published 1543)



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Argued that no experiment can decide whether the heavens move East to West or the Earth West to East.

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Oresme could also make great imaginative leaps:

He answered his own question by arguing that the irrationality of ratios will not rob the heavens of their beauty, nor contradict the regularity of the orbits.

This is a profound idea for three reasons (at least).

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At a deeper level, Oresme’s imagined solar system would exhibit *measure rigidity*.

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Sample question: how long does a planet spend on average over eternity in a specific part of space and with a specific range of velocities?

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If all the orbits are incommensurable, there is exactly one invariant measure, and this phenomenon is called *measure rigidity*.

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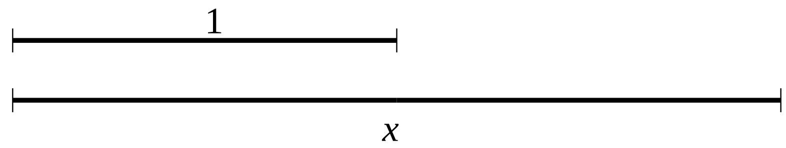

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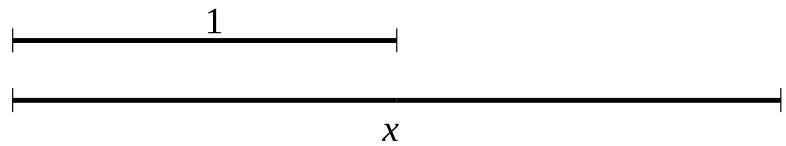

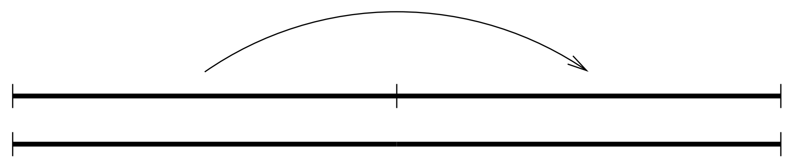

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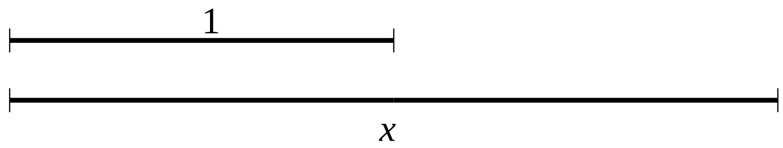

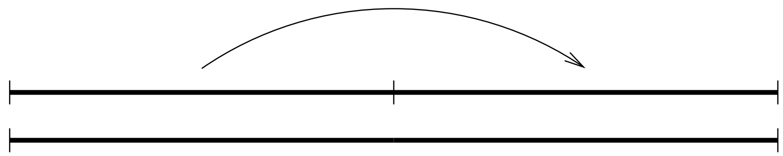

so $x = 2$ and the ratio is $1 : 2$.

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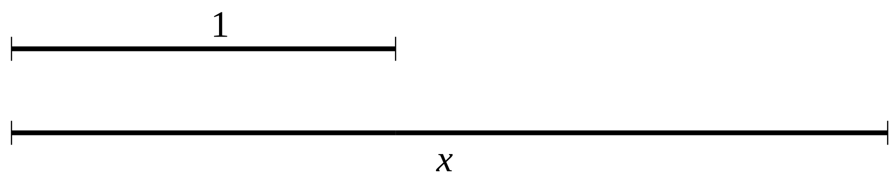

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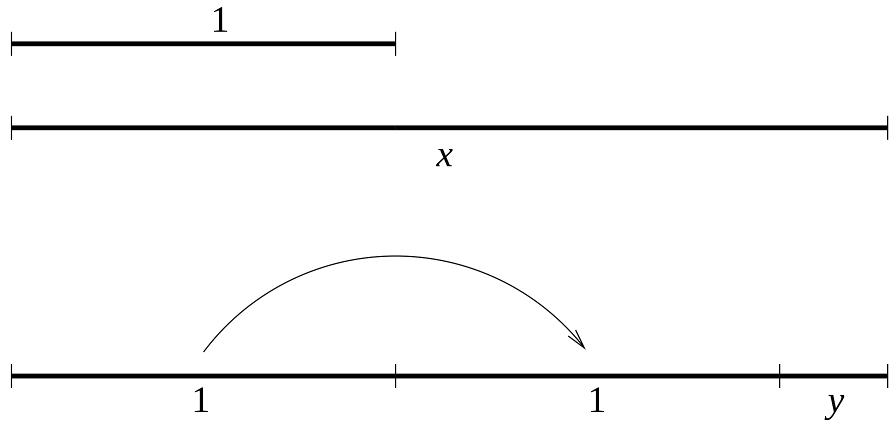

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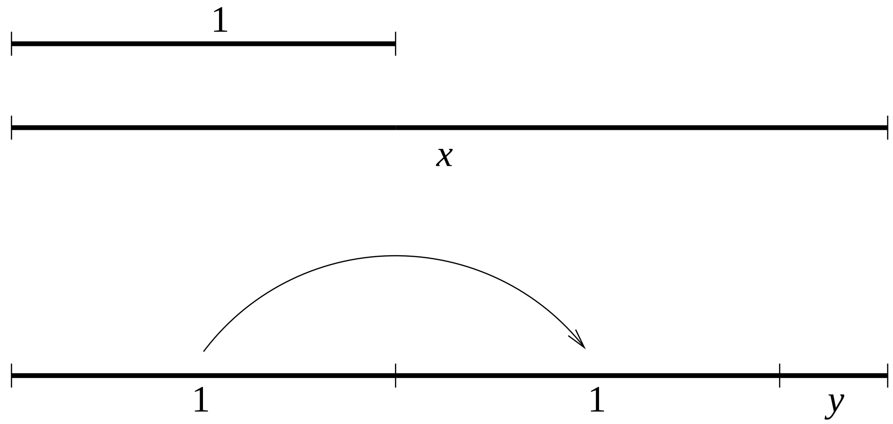
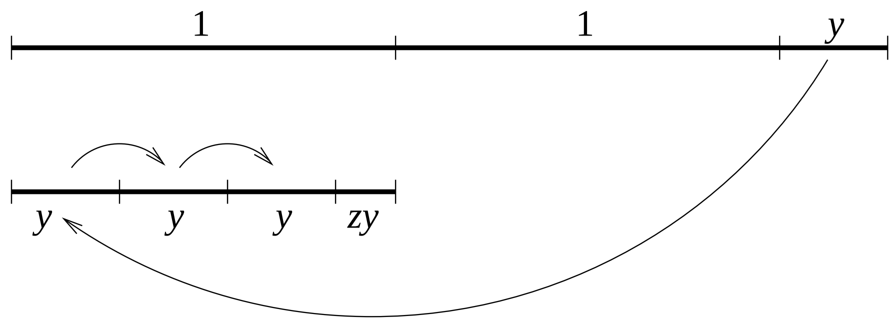


so the ratio is $1 : x$ where $x = 2 + y$.

Now compare the remainder y with 1:



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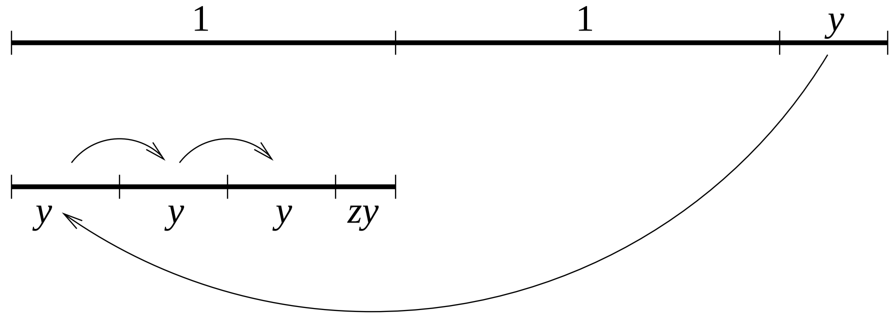

which amounts to writing $1 = (3 + z)y$, so $y = \frac{1}{3+z}$.

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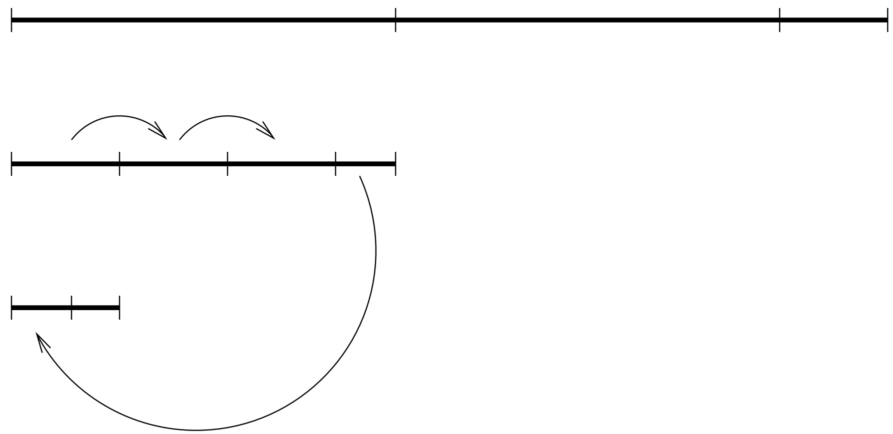

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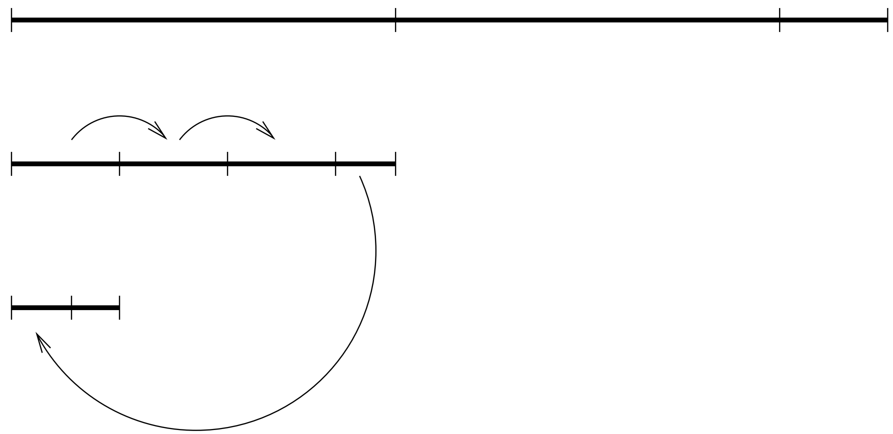

to deduce that $z = \frac{1}{1+\cdots}$

and so on. In this example the process stops after 4 steps, giving

$$x = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}}}$$

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Usually the process will continue for ever – but it will always rapidly give a very good approximation. This process is of great importance because it can be applied to any number (ratio) at all.

A 16th Century question:

Given any number u, write $\langle u \rangle$ for the distance to the nearest whole number. Thus $\langle 2.3 \rangle = 0.3$, $\langle 4.9 \rangle = 0.1$ and so on. How small can you make $\langle nu \rangle$ by choosing a whole number n carefully?

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This was shown to be possible in the 16th Century, using the theory of continued fractions.

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$$\begin{aligned}
\langle u \rangle &< 1, \\
\langle 2u \rangle &< \tfrac{1}{2}, \\
\langle 5u \rangle &< \tfrac{1}{5}, \\
\langle 12u \rangle &< \tfrac{1}{12}, \\
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and so on.

In contrast, $\langle 69u \rangle$ is about 0.4, much larger than $\frac{1}{69}$. The numbers $1, 2, 5, 12, \ldots$ arise when a stick of length 1 is compared to a stick of length $\sqrt{2}$.

The importance of continued fractions is that they produce such a special list of whole numbers for *any* original number u at all, so this start to tell us something universal about real numbers.

Littlewood's Question (1930)

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Littlewood was a formidable mathematician, hampered for much of his life by serious depression.

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It is a central testing ground for the state of the art in several other parts of mathematics.

Until 2003 the only progress on Littlewood involved special results: the set of pairs (x, y) for which it was known to hold formed a small set exhibiting possibly anomalous behaviour.

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This system has many wildly behaved invariant measures – it has no rigidity.

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measure rigidity (the invariant measures can be described).

This is a new phenomenon: in Oresme's solar system, the measure rigidity comes from the incommensurable orbits. In the 'big enough symmetry group' setting, a totally new mechanism is at work, where the rigidity cannot be seen (and does not hold) for any one direction of symmetry, but only holds for several of them combined.

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It has been developed notably by Einsiedler (Ohio State), Furstenberg (Jerusalem), Katok (Penn. State), Kitchens (IUPUI Indiana), Lindenstrauss (Princeton), Margulis (Yale), Raghunathan (TIFR Mumbai), Ratner (UC Berkeley), Schmidt (Vienna), and many others.

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Removing the ‘dynamical’ hypothesis would give the full result.

This work has given a context in which to view Littlewood's conjecture as something that is expected to be true, and added another strand to the links between number theory and ergodic theory.

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Source of images:

Nicole D'Oresme: `http://www.nicole-oresme.com/`

Aristotle: `http://ise.uvic.ca/`

Copernicus: `http://aeea.nmns.edu.tw/`

Littlewood: `http://www-history.mcs.st-andrews.ac.uk/`