

10/11/06, Loughborough

## **Entropy in algebraic dynamical systems**

.... and undergraduate complex variables.

**Setting:** There are many ‘algebraic dynamical systems’. For this talk, we mean the following.

- $X$  is a compact (metrizable) group;
- There are  $d$  commuting continuous automorphisms  $\alpha_1, \dots, \alpha_d$  of  $X$ ;
- equivalently, there is a  $\mathbb{Z}^d$ -action  $\alpha$  by automorphisms of  $X$ .

That is, each  $\mathbf{n} \in \mathbb{Z}^d$  defines an automorphism  $\alpha^{\mathbf{n}}$  of  $X$ , and

$$\alpha^{\mathbf{m}} \circ \alpha^{\mathbf{n}} = \alpha^{\mathbf{m}+\mathbf{n}}.$$

**Example 1:** The automorphism  $T$  of the 2-torus defined by the matrix

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

is an example. Here  $X = \mathbb{T}^2$ ,  $d = 1$ ,  $\alpha^n = T^n$ .

We know the (topological) entropy of this map is  $\log |\rho|$ , where  $\rho$  is the dominant eigenvalue.

**Example 2:** Let

$$f(x) = x^6 - 2x^5 - 5x^4 - 3x^3 - 5x^2 - 2x + 1$$

and write  $\mathbb{Q}(a)$ , where  $a$  is a complex root of  $f$ . Then  $a$  and

$$b = 2a^5 - 6a^4 - 3a^3 - 6a^2 - 6a$$

are fundamental units for the ring of integers in  $\mathbb{Q}(a)$ .

Since  $\{1, a, a^2, a^3, a^4, a^5\}$  is an integral basis for  $M = \mathbb{Z}[a]$ , the dual of  $M$  is a 6-torus. The automorphisms dual to multiplication by  $a$  and by  $b$  define an algebraic  $\mathbb{Z}^2$ -action on  $\mathbb{T}^6$ .

(This exotic example is due to Damjanović and Katok; it is mixing as a  $\mathbb{Z}^2$ -action but *genuinely partially hyperbolic*.)

More explicitly, Example 2 is the action defined by the commuting matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & 5 & 3 & 5 & 2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & -6 & -6 & -3 & -6 & 2 \\ -2 & 4 & 4 & 0 & 7 & -2 \\ 2 & -6 & -6 & -2 & -10 & 3 \\ -3 & 8 & 9 & 3 & 13 & -4 \\ 4 & -11 & -12 & -3 & -17 & 5 \\ -5 & 14 & 14 & 3 & 22 & -7 \end{pmatrix}$$

## Bowen's formula and homogeneity

Group automorphisms are very special in that the action on the space is geometrically the same everywhere.

As a result topological entropy coincides with the entropy of Haar measure, and this common value is also the decay rate of the volume of Bowen balls.

Using the metric, write

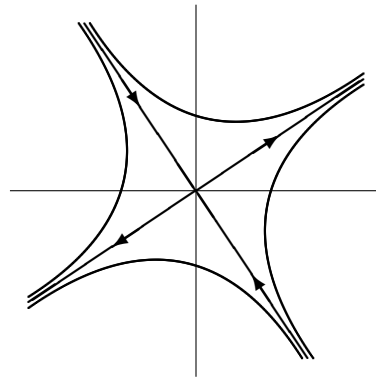
$$B_n = \bigcap_{j=0}^{n-1} T^{-j} B_\epsilon(0)$$

and  $\lambda$  for Haar measure.

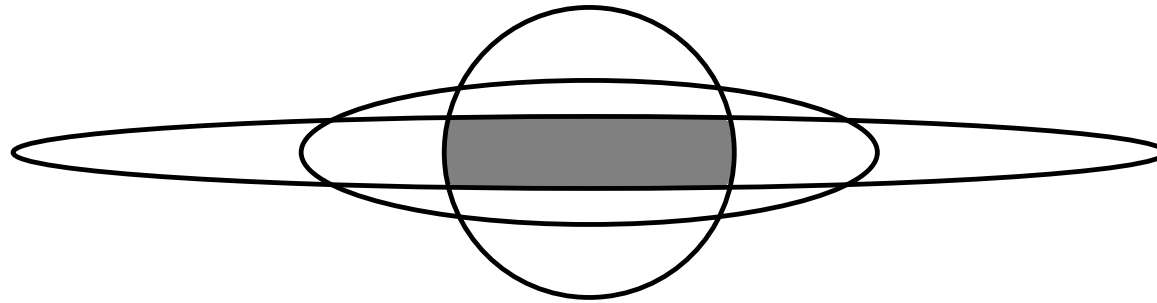
**Theorem:** [Bowen]

$$h(T) = \lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \lambda(B_n).$$

For the toral automorphism defined by  $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ , the action close to the identity looks like this:



Use axes adapted to the eigenvectors to estimate  $\lambda(B_n)$ :



Thus  $\lambda(B_n)$  is expected to be roughly  $\rho^{-n}C$ , where  $\rho$  is the dominant eigenvalue of the matrix, which explains why  $h(T) = \log \rho$ .

Refining this argument shows that

$$h(A) = \sum_{\text{eigenvalues}} \log^+ |\rho|.$$

for a toral automorphism defined by a matrix  $A \in GL(d, \mathbb{Z})$  (Sinai, Rokhlin for  $d = 2$ ; Arov in general).

This is a familiar formula from two places:

- Lyapunov exponents, smooth maps, hyperbolicity.
- Jensen's formula in complex analysis.

**Jensen's formula:** Let  $f$  be holomorphic on the closed unit disk, with zeros  $\lambda_1, \dots, \lambda_r$  inside the disk and none on the unit circle, and assume that  $f(0) \neq 0$ . Then

$$\int_0^1 \log |f(e^{2\pi it})| dt = \log |f(0)| - \sum_{i=1}^r \log |\lambda_i|.$$

More suggestively, if  $f$  is a monic polynomial with  $f(0) = \pm 1$  (the characteristic polynomial of the matrix defining a toral automorphism, for example) with zeros  $\lambda_1, \dots, \lambda_r$ , then

$$\int_0^1 \log |f(e^{2\pi it})| dt = \sum_{i=1}^r \log^+ |\lambda_i|.$$

(For zeros of unit modulus an argument is needed)

## Recast the Sinai, Rokhlin, Arov, Bowen formula:

Start with a polynomial  $f(u) = \sum_{s=0}^k f_s u^s \in \mathbb{Z}[u]$ , and associate to it the compact group

$$X_f = \{x \in \mathbb{T}^{\mathbb{Z}} \mid \sum_{s=0}^k f_s x_{s+n} \equiv 0 \pmod{1} \text{ for all } n \in \mathbb{Z}\}.$$

Since the condition defining  $X_f$  is homogeneous, the shift map  $\alpha_f$  defined by

$$\left(\alpha_f(x)\right)_k = x_{k+1}$$

is an automorphism of  $X_f$ .

**Theorem:**  $h(\alpha_f) = m(f) = \int_0^1 \log |f(e^{2\pi it})| dt$ , the logarithmic Mahler measure of  $f$ .

This includes most (but not all) toral automorphisms.

If  $f$  is not monic or  $|f(0)| \neq 1$  then  $X_f$  is a solenoid.

The shift in emphasis from (say) a toral automorphism defined by a matrix and the formula involving eigenvalues to a dynamical system defined by a polynomial and the formula involving Mahler measure opens the door to higher-rank systems.

Start as before with a polynomial

$$f(\mathbf{u}) = \sum_{\mathbf{s} \in S(f)} f_{\mathbf{s}} \mathbf{u}^{\mathbf{s}} \in \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}],$$

and associate to it the compact group

$$X_f = \{x \in \mathbb{T}^{\mathbb{Z}^d} \mid \sum_{\mathbf{s} \in S(f)} f_{\mathbf{s}} x_{\mathbf{s}+\mathbf{n}} \equiv 0 \pmod{1} \text{ for all } \mathbf{n} \in \mathbb{Z}^d\}.$$

Since the condition defining  $X_f$  is homogeneous, the shift action  $\alpha_f$  defined by

$$\left(\alpha_f^{\mathbf{n}}(x)\right)_{\mathbf{k}} = x_{\mathbf{k}+\mathbf{n}}$$

is a  $\mathbb{Z}^d$ -action by automorphisms of  $X_f$ .

Entropy has the expected definition for an action of  $\mathbb{Z}^d$ ,

$$h(\alpha) = \lim_{\epsilon \searrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{n^d} \lambda \left( \bigcap_{0 \leq n_i \leq n-1} \alpha^{-\mathbf{n}} B_\epsilon(0) \right).$$

**Theorem:**

$$h(\alpha_f) = m(f) = \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_d})| dt_1 \cdots dt_d,$$

the logarithmic Mahler measure of  $f$ .

This is a difficult result, and in particular some convergence issues become much more complicated for  $d > 1$ .

**Example 3:** Let  $d = 2$  and  $f(u_1, u_2) = 1 + u_1 + u_2$ . Then the shift  $\mathbb{Z}^2$ -action on the group defined by the 'sum to zero in each triangle' rule with alphabet the circle has topological entropy

$$h = m(1 + u_1 + u_2) = \frac{3\sqrt{3}}{4\pi} \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{1}{n^2},$$

where  $\left(\frac{n}{3}\right)$  is the Legendre symbol.

Why might the formula be true?

A linear algebra argument shows that the log of the number of points fixed by the subgroup  $(n\mathbb{Z})^d$  (when this is finite) is given by

$$\sum_{z_i^n=1} \log |f(z_1, \dots, z_d)|.$$

Thus the growth rate

$$\lim_{n \rightarrow \infty} \frac{1}{n^d} \sum_{z_i^n=1} \log |f(z_1, \dots, z_d)|$$

(if it exists) looks like a Riemann (more precisely, Cauchy) approximation to the Mahler measure.

That is no use for the entropy formula itself, but now more is known about specification properties for algebraic dynamical systems, periodic point counting can be used for actions of other groups to compute the entropy.

What about systems like commuting toral automorphisms? These definitely have zero entropy for  $d > 1$  because each map has finite entropy.

Again the algebra is revealing: for  $d > 1$  the ring  $R_d = \mathbb{Q}[u_1, \dots, u_d]$  is not a p.i.d.

If  $X$  carries an algebraic  $\mathbb{Z}^d$ -action  $\alpha$ , then the dual group  $\widehat{X}$  carries the structure of a module  $M$  over  $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ .

The dual of the automorphism  $\alpha^{e_i}$  is  $m \mapsto u_i m$ .

If  $d > 1$  then modules over  $R_d$  can be very complicated.

New setting: let  $M$  be an  $R_d$ -module, associated by duality to an algebraic  $\mathbb{Z}^d$ -action  $\alpha_M$ .

**Theorem:** For a prime ideal  $P \subset R_d$ ,

$$h(\alpha_{R_d/P}) = \begin{cases} m(f) & \text{if } P = \langle f \rangle; \\ 0 & \text{if } P \text{ is non-principal.} \end{cases}$$

We allow  $f = 0$  and then  $m(f) = \infty$ .

## More recent work:

Two directions seem particularly interesting.

- 1) The entropy can also be expressed as an integral over the amoeba associated to the polynomial (Einsiedler, Lind, Ward – not precisely formulated let alone published).
- 2) Deninger and Schmidt have found a formula for the topological entropy of certain actions of amenable groups by automorphisms in terms of a Fuglede–Kadison determinant in the associated von Neumann algebra (math.DS/0605723, math.DS/0608539, math.DS/0502233).