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.... and undergraduate complex variables.

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- There are d commuting continuous automorphisms $\alpha_1, \dots, \alpha_d$ of X;
- equivalently, there is a $\mathbb{Z}^d$-action α by automorphisms of X.

That is, each $\mathbf{n} \in \mathbb{Z}^d$ defines an automorphism $\alpha^{\mathbf{n}}$ of X, and

$$\alpha^{\mathbf{m}} \circ \alpha^{\mathbf{n}} = \alpha^{\mathbf{m}+\mathbf{n}}.$$

Example 1: The automorphism T of the 2-torus defined by the matrix

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

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We know the (topological) entropy of this map is $\log|\rho|$, where ρ is the dominant eigenvalue.

Example 2: Let

$$f(x) = x^6 - 2x^5 - 5x^4 - 3x^3 - 5x^2 - 2x + 1$$

and write $\mathbb{Q}(a)$, where a is a complex root of f. Then a and

$$b = 2a^5 - 6a^4 - 3a^3 - 6a^2 - 6a$$

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Since $\{1, a, a^2, a^3, a^4, a^5\}$ is an integral basis for $M = \mathbb{Z}[a]$, the dual of M is a 6-torus. The automorphisms dual to multiplication by a and by b define an algebraic $\mathbb{Z}^2$-action on $\mathbb{T}^6$.

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(This exotic example is due to Damjanović and Katok; it is mixing as a $\mathbb{Z}^2$-action but *genuinely partially hyperbolic*.)

More explicitly, Example 2 is the action defined by the commuting matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & 5 & 3 & 5 & 2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & -6 & -6 & -3 & -6 & 2 \\ -2 & 4 & 4 & 0 & 7 & -2 \\ 2 & -6 & -6 & -2 & -10 & 3 \\ -3 & 8 & 9 & 3 & 13 & -4 \\ 4 & -11 & -12 & -3 & -17 & 5 \\ -5 & 14 & 14 & 3 & 22 & -7 \end{pmatrix}$$

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Using the metric, write

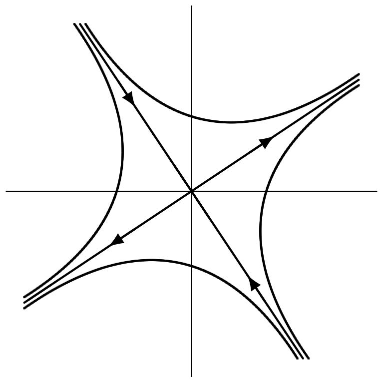
$$B_n = \bigcap_{j=0}^{n-1} T^{-j} B_\epsilon(0)$$

and λ for Haar measure.

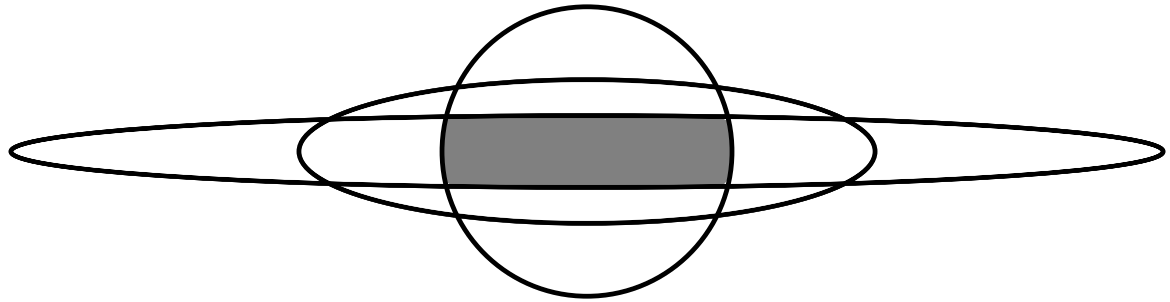
Theorem: [Bowen]

$$h(T) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \lambda \left(B_n \right).$$

For the toral automorphism defined by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, the action close to the identity looks like this:



Use axes adapted to the eigenvectors to estimate $\lambda(B_n)$:



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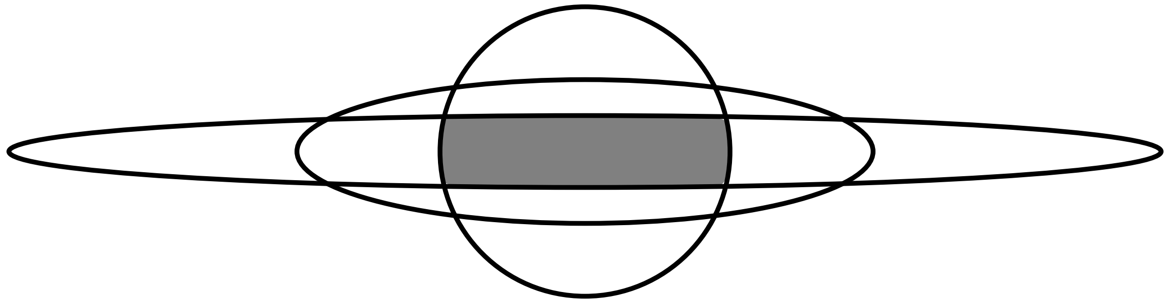

Thus $\lambda(B_n)$ is expected to be roughly $\rho^{-n}C$, where ρ is the dominant eigenvalue of the matrix, which explains why $h(T) = \log \rho$.

Refining this argument shows that

$$h(A) = \sum_{\text{eigenvalues}} \log^{+} |\rho|.$$

for a toral automorphism defined by a matrix $A \in GL(d, \mathbb{Z})$ (Sinai, Rokhlin for $d = 2$; Arov in general).

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- Lyapunov exponents, smooth maps, hyperbolicity.

- Jensen's formula in complex analysis.

Jensen's formula: Let f be holomorphic on the closed unit disk, with zeros $\lambda_1, \ldots, \lambda_r$ inside the disk and none on the unit circle, and assume that $f(0) \neq 0$. Then

$$\int_0^1 \log|f(e^{2\pi \mathrm{i} t})| \mathrm{d}t = \log|f(0)| - \sum_{i=1}^{r} \log|\lambda_i|.$$

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More suggestively, if f is a monic polynomial with $f(0) = \pm 1$ (the characteristic polynomial of the matrix defining a toral automorphism, for example) with zeros $\lambda_1, \dots, \lambda_r$, then

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(For zeros of unit modulus an argument is needed)

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Since the condition defining X_f is homogeneous, the shift map α_f defined by

$$\Big(\alpha_f(x)\Big)_k = x_{k+1}$$

is an automorphism of X_f.

Theorem: $h(\alpha_f) = m(f) = \int_0^1 \log|f(e^{2\pi \mathrm{i} t})| \mathrm{d}t$, the logarithmic Mahler measure of f.

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The shift in emphasis from (say) a toral automorphism defined by a matrix and the formula involving eigenvalues to a dynamical system defined by a polynomial and the formula involving Mahler measure opens the door to higher-rank systems.

Start as before with a polynomial

$$f(\mathbf{u}) = \sum_{\mathbf{s} \in S(f)} f_{\mathbf{s}} \mathbf{u}^{\mathbf{s}} \in \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}],$$

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is a $\mathbb{Z}^d$-action by automorphisms of X_f.

Entropy has the expected definition for an action of $\mathbb{Z}^d$,

$$h(\alpha) = \lim_{\epsilon \searrow 0} \liminf_{n \to \infty} -\frac{1}{n^d} \lambda \left(\bigcap_{0 \le n_i \le n-1} \alpha^{-\mathbf{n}} B_\epsilon(0) \right).$$

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Theorem:

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This is a difficult result, and in particular some convergence issues become much more complicated for $d > 1$.

Example 3: Let $d = 2$ and $f(u_1, u_2) = 1 + u_1 + u_2$. Then the shift $\mathbb{Z}^2$-action on the group defined by the 'sum to zero in each triangle' rule with alphabet the circle has topological entropy

$$h = m(1 + u_1 + u_2) = \frac{3\sqrt{3}}{4\pi} \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{1}{n^2},$$

where $\left(\frac{n}{3}\right)$ is the Legendre symbol.

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A linear algebra argument shows that the log of the number of points fixed by the subgroup $(n\mathbb{Z})^d$ (when this is finite) is given by

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Thus the growth rate

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{z_i^n=1} \log |f(z_1, \ldots, z_d)|$$

(if it exists) looks like a Riemann (more precisely, Cauchy) approximation to the Mahler measure.

That is no use for the entropy formula itself, but now more is known about specification properties for algebraic dynamical systems, periodic point counting can be used for actions of other groups to compute the entropy.

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If X carries an algebraic $\mathbb{Z}^d$-action α, then the dual group $\widehat{X}$ carries the structure of a module M over $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$.

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The dual of the automorphism $\alpha^{\mathbf{e}_i}$ is $m \mapsto u_i m$.

If $d > 1$ then modules over R_d can be very complicated.

New setting: let M be an R_d-module, associated by duality to an algebraic $\mathbb{Z}^d$-action α_M.

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Theorem: For a prime ideal $P \subset R_d$,

$$h(\alpha_{R_d/P}) = \begin{cases} m(f) & \text{if } P = \langle f \rangle; \\ 0 & \text{if } P \text{ is non-principal.} \end{cases}$$

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We allow $f = 0$ and then $m(f) = \infty$.

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1) The entropy can also be expressed as an integral over the amoeba associated to the polynomial (Einsiedler, Lind, Ward – not precisely formulated let alone published).

2) Deninger and Schmidt have found a formula for the topological entropy of certain actions of amenable groups by automorphisms in terms of a Fuglede–Kadison determinant in the associated von Neumann algebra (math.DS/0605723, math.DS/0608539, math.DS/0502233).