Orbit-counting in non-hyperbolic arithmetic dynamical systems

(with Everest, Miles, Stevens)

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and

$$\mathcal{M}_f(X) = \sum_{|\tau| \leq X} \frac{1}{e^{h|\tau|}} \sim \log X + C.$$

like Mertens' Theorem. (These are results of Parry, Pollicott, Sharp and others).

Without hyperbolicity, less is known. For ergodic toral automorphisms that are not hyperbolic, Waddington shows

$$\pi(X) \sim \frac{e^{h(X+1)}}{h} \sum_{\rho \in U} K(\rho) \frac{\rho^{x+1}}{\rho e^{h} - 1},$$

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What happens to these results for the simplest non-hyperbolic systems?

Setting: fix a number field $\mathbb{K}$, with set of places $P(\mathbb{K})$, infinite places $P_\infty(\mathbb{K})$, an element $\xi \in \mathbb{K}^*$ of infinite multiplicative order, and a finite set $S \subset P(\mathbb{K}) \setminus P_\infty(\mathbb{K})$ with $|\xi|_w \leq 1$ for $w \notin S \cup P_\infty(\mathbb{K})$.

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Associate to this a ring of S-integers,

$$R_S = \{x \in \mathbb{K} \mid |x|_w \le 1 \text{ for all } w \notin S \cup P_\infty(\mathbb{K})\}$$

and a map $T : X \to X$ dual to $x \mapsto \xi x$ on R_S.

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- $\mathbb{K} = \mathbb{Q}$, $S = \emptyset$, $\xi = 2$, map is $x \mapsto 2x \pmod 1$ on the circle; $S = \{2\}$ gives its invertible extension.

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- ξ a Salem number, $\mathbb{K} = \mathbb{Q}(\xi)$, $S = \emptyset$ gives a quasihyperbolic toral automorphism.

Any place w with $|\xi|_w = 1$ behaves like a non-hyperbolic eigenvalue.

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so the number of orbits of length n is

$$\mathcal{O}_T(n) = \frac{1}{n} \sum_{d|n} \mu(\tfrac{n}{d}) \prod_{w \in S \cup P_\infty(\mathbb{K})} |\xi^n - 1|_w$$

by Möbius inversion;

hence

$$\pi_T(N) = \sum_{n \leq N} \frac{1}{n} \sum_{d|n} \mu(\tfrac{n}{d}) \prod_w |\xi^n - 1|_w$$

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Simple estimates show that

$$\frac{1}{n} \log \mathcal{F}_T(n) \to h(T) \text{ as } n \to \infty,$$

so we divide out by the expected asymptotic, writing

$$\Pi_T(N) = \frac{N \pi_T(N)}{e^{h(T)(N+1)}}.$$

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The proof consists of extremely careful counting.

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For a hyperbolic map, X^* is trivial.

Mertens' Theorem is a bit more involved.

Theorem: For the case $\mathbb{K} = \mathbb{Q}$, there are constants $k_T \in \mathbb{Q}$, C_T such that

$$\mathcal{M}_T(N) = k_T \log N + C_T + O(1/N).$$

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For larger fields – in particular for fields large enough to contain Salem numbers – life is harder.

Theorem: For an S-integer map T with S finite, there are constants $k_T \in \mathbb{Q}$, C_T, $\delta > 0$ with

$$\mathcal{M}_T(N) = k_T \log N + C_T + O(N^{-\delta}).$$

Baby proof of Mertens' Theorem (no error term)

Recall that

$$\mathcal{M}_T(N) = \sum_{n\le N} \frac{1}{n} \sum_{d|n} \mu(n/d) \left(\frac{\prod_w |\xi^d - 1|_w}{e^{hn}} \right).$$

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Let

$$C(n) = \prod_{|\xi|_w \neq 1} |\xi^n - 1|_w$$

and

$$D(n) = \prod_{|\xi|_w = 1} |\xi^n - 1|_w.$$

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Define

$$F(N) = \sum_{n \le N} \frac{1}{n} D(n),$$

and

$$h^* = \prod_{\substack{|\xi|_w > 1, \\ w | \infty}} |\xi|_w$$

(the Archimedean part of the entropy).

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Lemma: Let g be an element of a compact abelian group G. Then the sequence (g^n) is uniformly distributed in the smallest closed subgroup of G containing g.

Define X^* to be the product of a circle for each infinite non-hyperbolic place, and the maximal compact subring in $\mathbb{K}_w$ for each finite non-hyperbolic place; a_T is ξ suitably embedded in each.

The function

$$x \mapsto \prod_{|\xi|_w=1} |x-1|_w$$

is continuous on X^*, so

$$\frac{1}{N}\sum_{n=1}^{N} D(n) \to k_T \text{ as } N \to \infty$$

by the Lemma, where

$$k_T = \int_{X^*} \prod_{|\xi|_w=1} |x-1|_w d\mu_{X^*}.$$

Thus

$$\begin{aligned} F(N) &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) \sum_{m=1}^{n} D(m) \\ &\quad + \frac{1}{N+1} \sum_{m=1}^{N} D(m) \\ &\sim k_T \log N, \end{aligned}$$

giving the result without error term.

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Lemma: If M is an integral S-unit, then the solutions of

$$|\xi^n - 1|_S = \frac{1}{M}$$

comprise $O(M^{1-1/d})$ cosets mod ρM for some fixed ρ and $d > 0$, independent of M.

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Write $\sum'$ for a sum taken only over integral S-units.

Lemma: For any $c > 0$,

$$\sum_{M>X}{}' \frac{\log M}{M^c} = O(1/X^e),$$

for any $e < c$.

If there is no infinite non-hyperbolicity life is easier:

Theorem: There are constants $K \in \mathbb{Q}$, c such that

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Theorem: Let a denote a complex algebraic number with $|a| = 1$ and a not a root of unity. Then for some $\delta > 0$ and constant ℓ,

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The proof uses Abel summation, the Lemmas above, and Baker's Theorem.

This allows the proof to be assembled: write

$$D(n) = \prod_{|\xi|_w=1} |\xi^n - 1|_w$$

as

$$\prod_{|\xi|_w=1, w|\infty} |\xi^n - 1|_w \times \prod_{|\xi|_w=1, w<\infty} |\xi^n - 1|_w$$

$$= f(a_1^n, \dots, a_r^n) \times \prod_{|\xi|_w=1, w<\infty} |\xi^n - 1|_w$$

where f is an integral polynomial in r variables, and $a_i \in \mathbb{S}^1$ for $i = 1, \dots, r$ are multiplicatively independent.

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Each part of this expression has been dealt with, giving the asymptotic for

$$\sum_{n<N} \frac{1}{n} D(n)$$

which is close enough to the quantity we are after.