Group automorphisms from a dynamical point of view

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Can you describe the space of compact group automorphisms
modulo dynamical equivalences?

Tom Ward (UEA) 20/4/11, Leicester

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- the algebraic structure is 'rigid';

- the dynamics is as homogeneous as possible (it locally looks the same everywhere).

More generally, if Γ is a discrete group with a homomorphism to the group of continuous automorphisms of G, then we can think of the action of Γ as a measurable (or topological) Γ-action T denote $T_\gamma : G \to G$ for each $\gamma \in \Gamma$.

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2) To avoid degeneracies, we always assume the action is 'ergodic' (there are no invariant L^2 functions $\equiv$ the dual automorphism is aperiodic $\equiv$ (morally) no iterate of T looks like the identity on part of the space).

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$$\text{Describe the space } \mathcal{G}/\!\sim.$$

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Less clearly, topological conjugacy $\implies$ measurable isomorphism.

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For instance, it is easy to check that $\begin{pmatrix} -1 & 3 \\ 3 & 2 \end{pmatrix}$ is not conjugate to $\begin{pmatrix} 0 & 1 \\ 11 & 1 \end{pmatrix}$.

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So on zero-dimensional groups, topological conjugacy has large equivalence classes.

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This means that A and B must be conjugate in the group $GL_d(\mathbb{Z})$ – algebraic isomorphism.

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In fact *much more* is true: the conjugacy ϕ itself must be a linear automorphism composed with rotation by a fixed point (Adler & Palais, 1965 for tori; Clark & Fokkink for solenoids).

The topological structure is surprisingly subtle. An obvious topological invariant to use is the dynamical zeta function,

$$\zeta_T(z) = \operatorname{exp} \sum_{n \geqslant 1} \frac{z^n}{n} |\{g \in G \mid T^n g = g\}|.$$

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Example: There are uncountably many topologically distinct 1-dimensional solenoidal automorphisms with zeta function $\frac{1-z}{1-2z}$ (Miles).

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Equivalently, T is measurably the same as a fair coin toss, or the shift map S on $A^{\mathbb{Z}}$ where A is the index set of ξ, with measure being the IID measure given on each coordinate by the probability vector $(m(B_a))_{a\in A}$, where $\xi = \{B_a \mid a \in A\}$.

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So we expect

$$h(T) = \sum_{i=s+1}^{d} \log |\lambda_i| = \sum_{i=1}^{d} \log^+ |\lambda_i|$$

... which can be written

$$h(T) = \int_0^1 \log|f(\mathrm{e}^{2\pi \mathrm{i} t})\mathrm{d}t$$

(by Jensen's theorem), the *Mahler measure* $m(f)$ of f, the characteristic polynomial.

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Theorem: In general, $h(T) = \log k + A$, where $k \in \mathbb{N} \cup \{\infty\}$ and A lies in the closure of the set $\{m(f) \mid m(f) > 0\}$ (Yuzvinskii).

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If the answer is no, then entropy defines a bijection $\mathcal{G}/\sim \longrightarrow \mathbb{R}_{>0}$.

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Example: There exist mixing $\mathbb{Z}^8$-actions by automorphisms that do not exhibit this rigidity (Bhattacharya). (They are not irreducible)

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Theorem: For $\mathbb{Z}^d$-actions by automorphisms ($d \geqslant 1$) of compact connected groups which are mixing and satisfy a descending chain condition on closed invariant subgroups, any equivariant continuous map must be affine (topological rigidity) if and only if the entropy of the target system is finite.