The article “Putting reality back into the equation” follows a sad and all too familiar trajectory. From a striking and provocative question about the decisions and behaviour that led us into the credit crunch, it rapidly descends into utter nonsense. Ironically it closely echoes one of the debates it alludes to, by being “not even wrong” – a charge levelled at parts of string theory as part of a robust discussion in Physics.

Rather than descend into a wearisome dissection of the statements in this article (something which I suspect colleagues in Economics, Physics, and above all Probability, would also be tempted to do), can I offer something more timely and positive?

The decision-making inside the banks and hedge funds that were part of the descent into collective financial lunacy did use mathematical tools – but these were essentially the calculus of Newton (or at least of Malliavan and Ito) together with the arithmetic of finite numbers, and great piles of hubris. Even the boldest remuneration committee in the brashest hedge fund does not deal with infinite quantities, despite appearances. Indeed, the psychologically numbing effect of dealing with large numbers, rendering them more or less indistinguishable from each other, may have had a part to play in our collective failure to prevent the credit crunch. A good example is the headlines concerning our national total debt reaching a trillion pounds – who has any real sense of what that quantity means?

The positive message I would like to respond with is also timely, because I invite the readers of the THES to put to one side the slurs cast on poor Cantor’s work by Chris Ormell. Instead, please start to appreciate what a rigorous, beautiful, profound, insight Cantor gave us all. It is also timely to remind ourselves that we are all part of a long and wonderful intellectual tradition capable of surviving HEFCE cuts, the vagaries of fashion, and the odd misplaced step.

The early part of Cantor’s work is easily presentable to a high-school audience. It is as much a vital part of our shared intellectual heritage as is the Mona Lisa, the Venus de Milo, the Great Sphinx, Great Zimbabwe, the King James Bible, the Tragical History of Doctor Faustus, and Hamlet. It is the first essential step into exactly the negation of the suggestion made in “Putting reality back into the equation”. Cantor’s work showed that we can speak with absolute precision about infinite quantities, that there are many of them, and – startlingly – they are part and parcel of the mathematical language that maps so powerfully onto the physical universe.

The story is all about counting. What do I mean when I claim to have three Smarties?

Definition 1: Possessing three Smarties means I have one, and another one, and another one.

This works fairly well, and indeed understanding that I have a first Smartie, a second one, and a third one gives us the notion of a successor, or a next item. It feels cumbersome to embark on, say, arithmetic, using this notion of three-ness. So let us embrace a more sophisticated and flexible idea of counting. Why should I be attached to “three” Smarties? Is this a different notion than “trois” Smarties? Is the three-ness of my Smarties a different property than the three-ness of your Jelly Babies? We need to get past three/3/trois and the Smarties and ask a question of Cantor. What does it mean for two sets (collections of things) to have the same number of elements, or the same cardinality? My Smartie collection has the same number, or the same cardinality, as your collection of Jelly Babies because I can match them up: first Smartie to first Jelly Baby, second to second and third to third. This matching up has paired each Smartie to a unique Jelly Baby, and vice-versa. With apologies for the jargon, let us call such a pairing-up a bijection.

Definition 2: Two sets (collections of things) have the same number of elements, or the same cardinality, or are equi-numerous, if there is a bijection (a pairing-up) between them.

We need some notation for this, so if A and B are sets we write $|A|=|B|$ to mean that they have the same cardinality.

This does not seem to help us count, but it does allow us to extract an abstract notion from a concrete one. We agree that I have three Smarties, and can use Definition 2 to test whether any other set of things has three elements. It does so if and only if its elements can be paired up with my test set of Smarties.

With an eye on my figure, I would now like to replace Smarties with a less calorific way to generate test sets. One way to do this is to use a piece of notation: the curly bracket. We write $\{a\}$ to mean “the set containing the single element a”. Thus my collection of Smarties might be written $\{a,b,c\}$ with a representing the first Smartie, b the second, and c the third. Now sets are elusive things, so we should proceed with caution and start modestly.

There is a special set – the empty set – written $\emptyset=\{\ \}$. This is a set with no elements, and so we may use Definition 2 to write the equation $|\emptyset|=0$. Via Definition 2, I can now say what it means for any collection to have no elements – we have given a meaning to the symbol 0.

While having a robust definition of the number 0 may be useful for some purposes – analysing government plans for Humanities teaching funding for example – we clearly need more. So we write down the set $\{\emptyset\}$. This set is not empty because it has a member, namely $\emptyset$. I can define the symbol 1 by saying that $|\{\emptyset\}|=1$.

We can now, in many different ways, write down sets that have one more element. For example, $\{\emptyset, \{\emptyset\}\}$ is a set with two elements, giving a definition for the symbol 2, and so on.

This gives us meanings for the symbols 0,1,2,3,... and a robust definition of what it means for a set to have any given number of elements. This is independent of the language used, the type of elements, the location of the set, and so on.

Enter Cantor again. I have implicitly been talking about finite sets, but Definition 2 really only uses the bijection (the rule for pairing up). Dare I imagine that I can use it to ask if any two sets have the same cardinality? There are some simple things to check, but the answer is emphatic and clear. Definition 2 does indeed allow us to assert with complete rigour that two infinite sets have the same number of elements.

We have defined the counting number 0,1,2,3,... so we have a first infinite set:

N = {0,1,2,3,...}.

There are many infinite sets that can be constructed from **N**, but let us compare **N** to something more literary. How many "words" (arbitrary strings of lower-case letters, with no account taking of meaning or pronouncability) are there? This huge set, call it **W**, contains things like *a, b, c,* but also things like

aa, ab, xyz, galileo, xxxx,

and so on. It also, strangely, contains the words

zero, one, two, three, four, fivehundredandseventhousandsixhundredandeight, ninetynine,

and so on. How many elements does **W** have? It seems to be huge – in particular, I can pair up each element of **N** with a unique element of **W** just by writing out the name of a number, and that seems to leave out most of the strange words in **W**.

Theorem 3: N and **W** have the same cardinality.

Following Definition 2, to prove this I must pair up the numbers in **N** with the words in **W** using a bijection. Here is one way to do it, exploiting the lexicographic order on words of a given length:

0 ↔*a*, 1↔*b*, 2↔*c*,3↔*d*,...,25↔*z*

We have only covered the words of length one – but we have only used the numbers up to 25. So we plod on:

26↔*aa*,27↔*ab*,27↔*ac*,...,51↔*az*

and then keep going:

52↔*ba*,53↔*bb*,54↔*bc*,...,77↔*bz*.

It would be a painful task to work out which number corresponds to *one, two, three, potato, elsewhere,* and so on – but every single word appears exactly once in this listing, and we have used every number exactly once. Theorem 3 is proved. However, the words describing each number form a (tiny) subset of **W**, so we have arrived at a startling observation.

Surprise 4: Removing (or adding) some elements from (or to) an infinite set does not necessarily change its cardinality.

In fact one possible way to define "being infinite" is to say that you are a set with the property that removing one element does not change your cardinality. For now we should just note Surprise 4, and record that our intuition needs to be tested rather carefully when dealing with infinite sets.

Enter Cantor again, this time with dazzling effect. Is the story over? Do we now understand infinite sets? Can we, with Swinburne, be thankful "*That even the weariest river Winds somewhere safe to sea"*?

Happily not, because Cantor proved two beautiful theorems whose consequences are still of central importance across mathematics. A proof of the first of these should be part of the cultural heritage of anyone with an interest in the history of ideas.

In order to state the first of Cantor's great ideas, we need some more notation. Let **I** denote all the real numbers between 0 and 1. This set contains things like 0.5, but also 0.137456..., the reciprocal of π, and so on.

Theorem 5: There is no bijection between **N** and **I**. Thus |**N**| cannot be equal to |**I**|.

This is proved by a "diagonal" argument of great power and reach. Imagine that there were such a bijection. This would be represented as an infinite list of pairings:

$0 \leftrightarrow 0.x_{0,1}x_{0,2}x_{0,3}x_{0,4}\ldots$

$1 \leftrightarrow 0.x_{1,1}x_{1,2}x_{1,3}x_{1,4}\ldots$

$2 \leftrightarrow 0.x_{2,1}x_{2,2}x_{2,3}x_{2,4}\ldots$

$3 \leftrightarrow 0.x_{3,1}x_{3,2}x_{3,3}x_{3,4}\ldots$

and so on. The notation on the right records the decimal expansion, so if the number 2 is paired to the decimal expansion 0.32453... then $x_{2,1}$ is 3, $x_{2,2}$ is 2, $x_{2,3}$ is 4, $x_{2,4}$ is 5 and so on. Each of the numbers $x_{i,j}$ is a decimal digit chosen from the list 0,1,2,3,4,5,6,7,8,9.

Now Cantor simply constructs a number that must have been omitted. Pick a rule that changes a decimal digit (one possibility is to replace 0 by 1, 1 by 2, 2 by 3, 3 by 4, 4 by 5, 5 by 6, 6 by 7, 7 by 8, 8 by 9, and 9 by 0), and create a number y whose decimal expansion reads

$y = 0.y_1y_2y_3y_4y_5$...

where y_1 is obtained by applying the rule to $x_{0,1}$, y_2 by applying it to $x_{1,2}$, y_3 from $x_{2,3}$, y_4 from $x_{3,4}$ and so on. Now y is a number in **I**, but it cannot be in the list we started with. It cannot be the first element (the number paired with 0) because its first decimal digit is wrong; it cannot be the second because its second digit is wrong, and so on. The only conclusion is that no matter how you attempt to create a bijection from **N** to **I**, you are doomed to fail.

On the other hand, it is easy to see that there is a bijection from **N** to a subset of **I**, since one may send 0 to 0 and each n to the reciprocal of n for n greater than 0. Thus we are led to the conclusion that |**N**| and |**I**| are both infinite, but one is larger than the other.

Corollary 6: |**N**| is strictly smaller than |**I**|.

We now have two different infinities, a smaller one and a larger one. A further insight of Cantor takes this much further – it is not difficult to prove, but does take some notation so I will not attempt it here.

Theorem 7: If A is any set, finite or infinite of any cardinality, then there is another set P(A) with the property that $|A|<|P(A)|$.

This means there is an infinite hierarchy of different infinities – and no legerdemain was needed to get there. The impression given that these assertions are somehow innately riddled with logical inconsistencies, or connected with subtle questions to do with the completeness of arithmetic in the sense of logic, is not for me to contradict. It is for you to grapple with. If the tiny part of Cantor's work I have described here makes sense to you, then you can form a view on this yourself.

A footnote: I should come clean about how near this discussion comes to deeper waters. A question which Corollary 6 might prompt, namely is it the case that any infinite subset of **I** must either have cardinality as small as |**N**| or as large as |**I**| itself, is genuinely subtle and has a most surprising answer – but that is another story.