OB-81 LECTURE NOTES (FIRST HALF OF COURSE)

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Contents

These notes are designed to accompany the lectures in the first half of OB81. Please let me know if you find errors. There will be a current corrected version on the web at

http://www.mth.uea.ac.uk/~h720/lecture_notes/

Don't panic if they look very difficult or very easy to you. For some of you this is useful revision (though please contact me if you are bored) and for some this is new material (though please contact me if you are really feeling unable to keep up). The main point of contact outside the lectures will be seminars every fortnight, but you can come and see me at any time (my office hours are Wednesday 10-11: at other times I am happy to meet with you but may be busy with other things).

1. Introduction (3 lectures)

Mathematics is very exact. You should develop the habit of reading your own mathematics and checking that it says what was intended. The symbols you use all have exact meanings. In particular, avoid muddling up the following three things:

'equals': $A = B$ means that A and B are identical;
'implies': $A \implies B$ means that A implies B (that is, it means that whenever A is true, then B must be true; it does not require that A is true);
'therefore': $A \therefore B$ means A is true, therefore B is true.

It will help you to write accurate mathematics if you try and write in complete sentences – this may feel artificial at first but will make your own thinking more clear. One good way to check that your mathematics makes sense is to read it through afterwards.

1.1. Algebra.

Algebra begins with the idea that we can go beyond arithmetic by letting *letters* represent other things. This allows us to make statements in a very convenient form, and prove statements about infinitely many different things in one go. In this section, letters $a, b, c, \ldots$ will denote real numbers.

1.1.1. *Powers.* For $n \geq 1$,

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$$

Example: $7^3 = 7 \times 7 \times 7 = 343$

is a multiplied by itself n times or a raised to the nth power. In this expression, a is the *base* and n is the *exponent*. The most important properties of raising to powers (or *exponentiation*) are:

Example: $2^3 \times 2^2 = 2^5$

- $a^m \times a^n = a^{m+n}$;
- $a^1 = a$ for any value of a;
- for $a \neq 0$, $a^0 = 1$;
- $(a^m)^n = a^{mn}$;
- $a^{-n} = \frac{1}{a^n}$.

Example: $2^{-3} = \frac{1}{8}$

You cannot combine exponentials with different bases and exponents, but you can combine them if the base is the same (we have seen that $a^m \times a^n = a^{m+n}$) or if the exponent is the same:

$$a^m \times b^m = (a \times b)^m.$$

The inverse operation of 'raise to the nth power' is 'take the nth root', so

Example: $27^{1/3} = 3$

$$a^{1/n} = \sqrt[n]{a}.$$

Notice that this means

$$\left(\sqrt[n]{a}\right)^n = \left(a^{1/n}\right)^n = a^{n/n} = a,$$

which is reassuring. Combining all the rules for exponentiation gives some complicated expressions.

Example: $(\frac{64}{27})^{-1/3} = (\frac{27}{64})^{1/3} = \frac{27^{1/3}}{64^{1/3}} = \frac{3}{4}$

Expressions involving things that cannot be simplified, like $3+\sqrt{2}$, are usually best left in that form – though you should always try and write them in the form $a + b\sqrt{c}$. To do this, there is a specific trick: if $a + b\sqrt{c}$ appears in the *denominator*, then multiply by $\frac{a-b\sqrt{c}}{a-b\sqrt{c}}$. The reason this helps is that

$$(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - b^2c,$$

which does not involve any square roots, so the denominator has been simplified.

Example 1.1. First a simple example:

$$\frac{1}{1+2\sqrt{3}} = \frac{1}{1+2\sqrt{3}} \times \frac{1-2\sqrt{3}}{1-2\sqrt{3}} = \frac{1-2\sqrt{3}}{1-12} = -\tfrac{1}{11} + \tfrac{2}{11}\sqrt{3}.$$

Then a more involved simplification:

$$\begin{aligned}
\frac{3+2\sqrt{5}}{1-2\sqrt{7}} &= \frac{3+2\sqrt{5}}{1-2\sqrt{7}} \times \frac{1+2\sqrt{7}}{1+2\sqrt{7}} \\
&= \frac{(3+2\sqrt{5})(1+2\sqrt{7})}{(1-2\sqrt{7})(1+2\sqrt{7})} \\
&= \frac{3+6\sqrt{7}+2\sqrt{5}+4\sqrt{35}}{1-4\times 7} \\
&= -\tfrac{1}{9} - \tfrac{2}{9}\sqrt{7} - \tfrac{2}{27}\sqrt{5} - \tfrac{4}{27}\sqrt{35}
\end{aligned}$$

which does not simplify further.

Example 1.2. Simplify the expression $(\sqrt{3}-2)^{-2}$.
Solution: There are several ways to do this. Notice that

$$(\sqrt{3}-2)^{-2} = \frac{1}{(\sqrt{3}-2)^2}.$$

First simplify the denominator: we know that

$$(a+b)^2 = a^2 + 2ab + b^2$$

as shown in Figure 1.

So taking $a = \sqrt{3}$ and $b = -2$ we get

Notice the notation: $2\cdot 3$ *means* 2×3

$$(\sqrt{3}-2)^2 = (\sqrt{3})^2 - 2\cdot 2\cdot \sqrt{3} + 4 = 7 - 4\sqrt{3}.$$

Now deal with the reciprocal as usual:

$$\begin{aligned}\frac{1}{(\sqrt{3}-2)^2} &= \frac{1}{7-4\sqrt{3}} \\ &= \frac{7+4\sqrt{3}}{(7-4\sqrt{3})(7+4\sqrt{3})} \\ &= \frac{7+4\sqrt{3}}{49-16\times 3} \\ &= 7+4\sqrt{3}.\end{aligned}$$

State complete answer clearly

We conclude that

$$\frac{1}{(\sqrt{3}-2)^2} = 7+4\sqrt{3}.$$

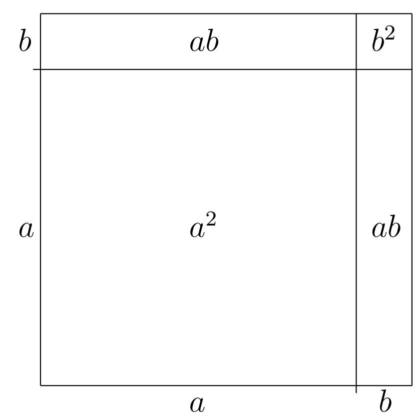


FIGURE 1. Picture proof that $(a+b)^2 = a^2 + 2ab + b^2$.

1.1.2. *Scientific notation.* One of the most important uses of the idea of exponentiation is to use powers of 10 in *scientific notation* for large or small numbers.

Example 1.3. Here are some large and small numbers:

- The Andromeda Galaxy contains at least

$$200,000,000,000$$

  stars.
- The mass of an alpha particle, which is emitted in the radioactive decay of Plutonium-239, is

$$0.000,000,000,000,000,000,000,000,006,645 \text{ kg}.$$

- In 2002 the United States GDP (Gross Domestic Product) is predicted to be

$$\$10,857,800,000,000.$$

All these expressions are very cumbersome.

The number of stars in the Adromeda Galaxy can be written as:

$$2.0 \times 100,000,000,000.$$

It is that large number, 100,000,000,000, which causes the problem. But this is just a multiple of ten:

$$100,000,000,000 = 10^{11}.$$

Notice that the exponent 11 is the number of zeros following the '1'. So we would write 200,000,000,000 in scientific notation as:

$$2.0 \times 10^{11}$$

This number is read 'two point zero times ten to the eleventh.'

The number 200,000,000,000 should be written as 2.0×10^{11}, with the first number chosen to be between 1 and 10. It could also be written as 20×10^{10}, but by convention the number is usually written as 2.0×10^{11} so that the lead number is less than 10, followed by as many decimal places as necessary. It is easy to see that all the variations above are just different ways to represent the same number:

$$200,000,000,000 = 20 \times 10^{10} = 2.0 \times 10^{11} = 0.2 \times 10^{12}.$$

This illustrates another way to think about Scientific Notation: the exponent will tell you how the decimal point moves; a positive exponent moves the decimal point to the right, and a negative one moves it to the left. So for example: $4.0 \times 10^2 = 400$ (2 places to the right of 4); $4.0 \times 10^{-2} = 0.04$ (2 places to the left of 4).

Negative exponents indicate negative powers of 10, which are expressed as fractions with 1 in the numerator and the power of 10 in the denominator (on the bottom). So:

$$10^{-1} = \tfrac{1}{10}; \quad 10^{-2} = \tfrac{1}{100}; \quad 10^{-3} = \tfrac{1}{1{,}000},$$

and so on.

The key to adding or subtracting numbers in Scientific Notation is to make sure the exponents are the same. For example,

$$(2.0 \times 10^2) + (3.0 \times 10^3)$$

can be rewritten as:

$$(0.2 \times 10^3) + (3.0 \times 10^3) = 3.2 \times 10^3.$$

When multiplying numbers expressed in scientific notation, the exponents can simply be added together. Consider for example:

$$(4.0 \times 10^5) \times (3.0 \times 10^{-1}).$$

The 4 and the 3 are *multiplied*, giving 12, and the exponents 5 and -1 are *added*, so the answer is:

$$12 \times 10^4 = 1.2 \times 10^5.$$

Division is similar:

$$(6.0 \times 10^8) \div (3.0 \times 10^5) = 2.0 \times 10^3$$

where we have divided the 6 by the 3 and *subtracted* the exponents.

Example 1.4. In 2002 the US Gross Domestic Product will be

$$\$10,857,800,000,000.$$

and the population will be about

$$300,000,000$$

Write these numbers in scientific notation, and use this to find the GDP *per capita* (that is, per person).
Solution: The population is written 3×10^8 in scientific notation. The GDP is

$$1.08578 \times 10^{13},$$

so we need to divide. First the inital terms:

$$\frac{1.08578}{3} = 0.361927$$

and then we *subtract* the exponents:

$$\frac{10^{13}}{10^8} = 10^5.$$

So our answer is

$$0.361927 \times 10^5 = 3.61927 \times 10^4$$

in scientific notation. Two final steps: First, the population figure was given as approximate, so it is doubtful that all those digits really mean anything, so round it to

$$3.6 \times 10^4.$$

Second – and very important – is a quick reality check. The answer we have arrived at is 36 thousand dollars per person GDP. That is plausible for the US, so we can be confident that our answer is right.

Scientific notation allowed us to deal with those two very large numbers easily. The next example is similar.

Example 1.5. Light travels at about $300,000$ kilometres per second in vacuum. How many metres does light travel in a vacuum in a century? Solution: This is an extreme problem – if you try and do it by writing out all the numbers in full you will fill a lot of paper and make a lot of mistakes. All we need to do is find out how many seconds there are in a century and multiply that by the number of metres light travels in a second.

In fact it travels at exactly $299,792,458$ metres per second in vacuum. This is exact because since 1983 the metre length has been defined to be $\frac{1}{299,792,458}$th of the distance light travels in a vacuum in one second...

First the seconds. There are 60 in a minute, of which there are 60 in an hour, of which there are 24 in a day... This is already a big number! There are

$$60 \times 60 \times 24 = (6 \times 10^1) \times (6 \times 10^1) \times (2.4 \times 10^1) = 86.4 \times 10^3 = 8.64 \times 10^4$$

seconds in a day. So there are

$$(3.65 \times 10^2) \times (8.64 \times 10^4) = 31.536 \times 10^6 = 3.1536 \times 10^7$$

seconds in a year. So there are

$$3.1536 \times 10^9$$

seconds in a century.

Now think about the speed of light: a kilometre is a thousand or 10^3 metres, so the speed of light is

$$3 \times 10^5 \text{ km/s} = 3 \times 10^8 \text{ m/s}.$$

Finally, we multiply the speed by the number of seconds: light travels

$$(3.1536 \times 10^9) \times (3 \times 10^8) = 9.4608 \times 10^{17}$$

metres in a century. Again, this was a slightly slapdash calculation – the speed of light was not precise, we ignored leap years and so on. So a more reasonable answer is: light travels about 9×10^{17} metres in a century.

The Galaxy STIS 123627+621755, referred to informally as "Sharon," was identified by NASA's Hubble Space Telescope. It is estimated that light from Sharon takes 10 billion years (that is, 10^{10} years) to reach the earth. How far away is Sharon in metres?

1.1.3. *Other powers.* It will become more clear when you do differentiation that exponentials with other bases are very important. The expression a^b can be defined for any real numbers a and b as long as $a \neq 0$. Usually the name 'exponential function' is saved for the function $x \mapsto e^x$, where $e = 2.71828\ldots$ is a special number, the *base of the natural logarithms*.

1.2. **Logarithms.** One way of thinking of the formula $a^{m+n} = a^m \times a^n$ is this: the exponential operation $m \mapsto a^m$ sends *addition* to *multiplication*. The reverse operation is called the 'logarithm base a', and takes *multiplication* to *addition*.

Definition 1.6. Let a be a strictly positive number. We say that $y = \log_a x$ (log base a of x) if and only if $a^y = x$.

It is not obvious that this really makes sense, but we'll see from the graphs later that $\log_a(x)$ exists for any $x > 0$.

Only the bases 10 and e are commonly used, giving the *common logarithm function* $\log = \log_{10}$ and the *natural logarithm function* $\ln = \log_e$.

Example 1.7. To see logarithms in action, check the following using the definition.

(1) $\log(1000) = 3$;
(2) $\log(1/100) = -2$;
(3) $\log(1000 \times 100) = \log(1000) + \log(100) = 5$;
(4) $\log(\sqrt{10}) = \frac{1}{2}$.

The general properties of any logarithm function $\log_a$ for $a > 0$ are as follows:

Example: $\log(10) + \log(100) = \log(1000) = 3$

- $\log_a(a) = 1$;
- $\log_a(x \times y) = \log_a(x) + \log_a(y)$;
- $\log_a(x^n) = n \log_a(x)$;
- $\log_a(\frac{1}{x}) = -\log_a(x)$;
- $\log_a(1) = 0$

for any $x, y > 0$.

1.3. **Binomial theorem.** The binomial (two-names) theorem gives a method for expanding expressions like $(a+b)^n$. We'll initially deal with whole numbers n. First we need the *binomial coefficients*,

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

Example: $\binom{5}{3} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 10$

where $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$. A convenient way to calculate these numbers is to notice that $\binom{n}{j}$ is the $(j+1)$th entry in the $(n+1)$th row in *Pascal's triangle*, obtained by adding the two numbers above each entry to get the next entry.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

The numbers in Pascal's triangle give the *coefficients* in binomial expansions:

$$
\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{aligned}
$$

Sometimes the expression we want to expand has other coefficients which need to be included in the a and b; this is illustrated in the next example.

Example 1.8. The x^3 coefficient in $(2+\frac{1}{2}x)^5$ is 5. To see this, we can use Pascal's triangle as follows. Set $a = 2x$ and $b = \frac{1}{2}x$; by Pascal's triangle the coefficient with a b^3 in the expansion of $(a+b)^5$ is $10a^2b^3$. So the expression with an x^3 in $(2+\frac{1}{2}x)^5$ is $10\cdot 4\cdot \frac{1}{8}x^3$, and the coefficient is 5.

What this example means is that

$$(2+\tfrac{1}{2}x)^5 = 5x^3 + \text{expressions not involving } x^3.$$

2. Trigonometry (4 lectures)

Trigonometry is a powerful set of methods for understanding all sorts of shapes and problems involving angles. Most such problems can be reduced to problems on *triangles*, and for these there are three useful results: Pythagoras' theorem, the sine rule and the cosine rule.

2.1. Angles. An angle between a pair of lines that cross expresses the proportion of a whole circle that they subtend. There are two common ways to do this:

- Degrees: here the whole circle is 360^o, so a right angle is 90^o;
- Radians: here the whole circle is 2π radians, so a right angle is $\frac{\pi}{2}$ radians.

Degrees are more familiar, but once you come on to calculus radians are much easier to work with. Fractions of a degree are measured in *minutes* and *seconds*; a minute is one sixtieth of a degree and a second is one sixtieth of a minute.

2.2. Triangles. A useful convention for triangles is to label the angles and the sides as in Figure 2, with angles associated with their *opposite* sides. Notice that each angle has exactly one *opposite side*.

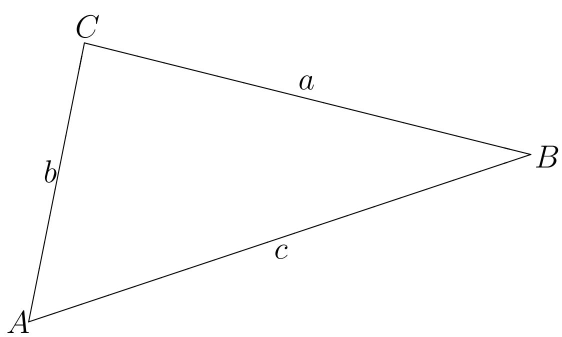


FIGURE 2. Notation for triangles.

Sum of angles in a triangle

The first property of triangles holds for any triangle: the angles sum to 180^o:

$$A + B + C = 180^o. \tag{2.1}$$

2.3. Right triangles. A triangle is a *right triangle* if one of its angles is a right angle. In a right triangle each angle other than the right angle has a unique *opposite* side and a unique *adjacent* side. Figure 3 shows this for the angle A. The long side of a right triangle is always called the *hypotenuse*.

Pythagorus' theorem

The oldest property of right triangles is Pythagorus' theorem:

$$b^2 = a^2 + c^2 \tag{2.2}$$

or

$$\text{hypotenuse}^2 = \text{opposite}^2 + \text{adjacent}^2$$

This is one of the oldest theorems in Mathematics, and you should see at least one proof of it.

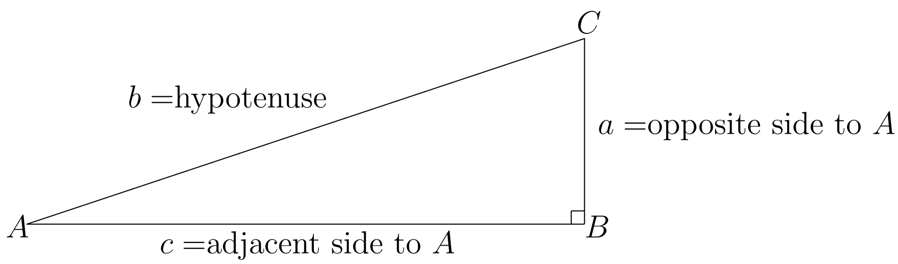


FIGURE 3. Notation for right triangles.

2.3.1. *Proof of Pythagorus' theorem.* There are many proofs of this theorem: the one presented here is *very* old – it was probably known to Hindu scholars in India hundreds of years before Pythagorus was alive.

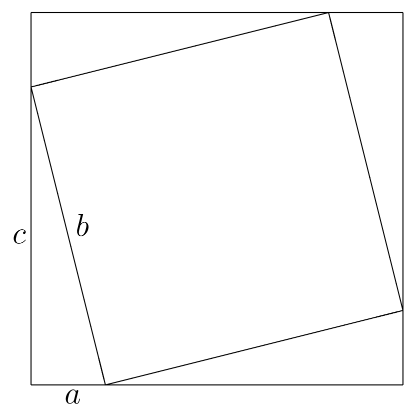


FIGURE 4. Picture proof of Pythagorus' Theorem.

In Figure 4, there are four triangles each with area $\frac{1}{2}ac$. There is one skew square in the middle with area b^2. It follows that the total area is

$$\text{total area } = 4(\tfrac{1}{2}ac) + b^2 = 2ac + b^2. \tag{2.3}$$

On the other hand, the whole figure is a square with side length $(a+c)$, so

$$\text{total area } = (a+c)^2 = a^2 + 2ac + c^2. \tag{2.4}$$

Comparing (2.3) and (2.4) shows that

$$a^2 + c^2 = b^2,$$

which is Pythagorus' Theorem.

2.4. **Trigonometric functions.** Associated with a right triangle are three *trigonometric* functions, which are defined by the properties:

Definition of trigonometric functions

- $\sin(A) = \frac{a}{b} = \frac{\text{opposite}}{\text{hypotenuse}}$;
- $\cos(A) = \frac{c}{b} = \frac{\text{adjacent}}{\text{hypotenuse}}$;
- $\tan(A) = \frac{\sin(A)}{\cos(A)} = \frac{a}{c} = \frac{\text{opposite}}{\text{adjacent}}$.

These relations are often used the other way round: if you know the angle A and the hypotenuse b, then

- opposite$= a = b\sin(A)$;
- adjacent$= c = b\cos(A)$.

Notice that we keep these definitions for large angles, by saying that in Figure 2 if C lies *below* B, then the length BC is negative. Similarly, if C is to the left of the vertical through A then AB is negative. Try to see why (for example) $\sin(330^o)$ is negative and $\cos(110^o)$ is negative.

Example 2.1. Using these relations, Pythagorus' theorem can be written in a different way:

$$b^2 = b^2\sin^2(A) + b^2\cos^2(A),$$

Pythagorus' theorem

so

$$1 = \sin^2(A) + \cos^2(A). \tag{2.5}$$

In fact for any three numbers a, b, c satisfying (2.2) there is a right triangle with those side lengths.

Example 2.2. Find the angles in a right triangle whose longest side is 5 and which has one side length 3.

Example of solving a right triangle

Solution: The longest side is always the hypotenuse, so the third side has length $\sqrt{5^2 - 3^2} = 4$. In Figure 3 we can take $b = 5$, $c = 4$ and $a = 3$. The angle B is 90^o, and so by (2.1) we must have $A + C = 90^o$. It is therefore enough to find A, and for this we can use

$$\sin(A) = \frac{a}{b} = \frac{3}{5} = 0.6.$$

Now from a calculator we see that $A = 36^o52'$, and therefore the third angle is $C = 53^o8'$.

2.5. **Cosine and sine rule.** For this section, refer again to Figure 2. Notice that each angle has *two* sides that are adjacent to it, so we cannot speak of the adjacent side anymore. However, each angle does have a unique opposite side. There are two main results valid for all triangles that we will use. The first is a generalization of Pythagorus called the *cosine rule*:

Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos(A). \tag{2.6}$$

Notice that this does generalize Pythagorus: if $A = 90^o$ then $\cos(A) = 0$, so the cosine rule reads $a^2 = b^2 + c^2$ for a right triangle.

Example 2.3. Find a triangle (that is, find the angles) with sides of length 10m, 20m and 12m. *Example using cosine rule*
Solution: First rearrange the cosine rule to express angles in terms of sides,

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}, \tag{2.7}$$

and notice that we can apply these at each corner in turn. Write $a = 10$, $b = 20$ and $c = 12$. Then

$$\cos(A) = \frac{400 + 144 - 100}{2 \cdot 20 \cdot 12} = \frac{444}{480} = 0.925.$$

Similarly

$$\cos(B) = \frac{a^2 + c^2 - b^2}{2ac} = -\frac{156}{240} = -0.65$$

and

$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab} = \frac{356}{400} = 0.89.$$

Therefore

$$A = 22^o 20', B = 130^o 32', \text{ and } C = 27^o 7'.$$

Notice that we should have $A + B + C = 180^o$ (we are out by one minute due to rounding errors).

There is one other general property of triangles, the *sine rule* which states that *Sine rule*

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}. \tag{2.8}$$

This can be used to solve a triangle given the length of one side, the opposite angle, and one further side length or angle.

Example 2.4. A tower T stands on the far side of a river. Two points A and B on this side of the river are chosen which are 200 metres apart. The angle between AB and AT is measured to be 40^o and the angle between AB and BT is measured to be 30^o. How far is the tower from A? *Example using sine rule*
Solution: The situation is shown in Figure 5. The third angle is given by (2.1):

$$T = 180^o - 40^o - 30^o = 110^o.$$

By the sine rule,

$$\frac{\overline{AB}}{\sin(T)} = \frac{\overline{AT}}{\sin(B)},$$

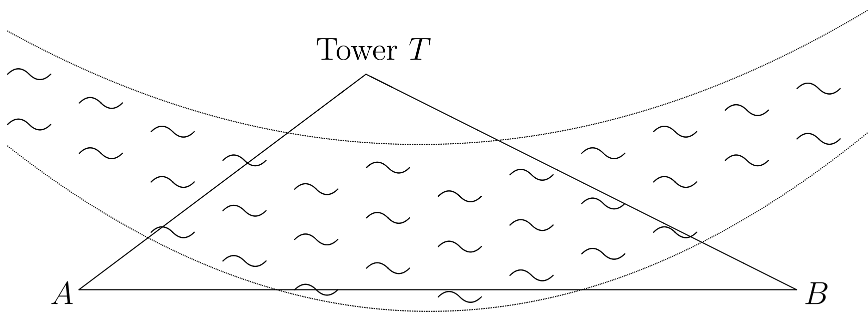


FIGURE 5. How far is the tower from A?

so

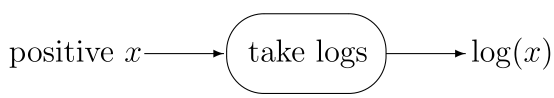
$$\overline{AT} = \sin(B)\frac{\overline{AB}}{\sin(T)} = \sin(30^{\circ})\frac{200}{\sin(110^{\circ})} = 106.4,$$

so the tower T is about 106.4 metres from A.

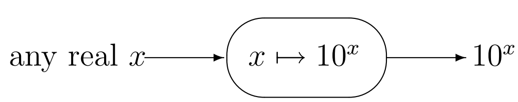
3. Functions and equations (3 lectures)

A *function* assigns a number to a given number chosen from a special set (the *domain*). You can think of a function as a box that does something to a number of the right type. There are many different ways to write functions, but we will usually write something like $y = f(x)$ or $f : x \mapsto f(x)$. You can think of a function as something that acts on a number of the right sort to give another number.

- The function log takes any positive number and returns a number:



- The exponential function $x \mapsto 10^x$ takes any real number and gives a positive number and is the inverse of the logarithm function:



In a function $y = f(x)$ the x and y are called *variables* because they are allowed to take on any suitable value.

3.1. **Solving equations.** There is no 'method' for solving equations, but many special kinds of equations can be solved using algebra. It may help to see a few equations first.

- $3x + 2 = 8$ is a *linear* equation, which is solved simply by rearranging:

$$\begin{aligned} 3x+2 &= 8 \\ \therefore 3x &= 6 \\ \therefore x &= 2. \end{aligned}$$

- $x^2 - 5x + 6 = 0$ is a *quadratic* equation that *factorizes* in a nice way:

$$x^2 - 5x + 6 = (x-2)(x-3).$$

  So we can argue as follows:

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ \therefore (x-2)(x-3) &= 0 \\ \therefore x = 2 \quad &\text{or} \quad x = 3. \end{aligned}$$

- $x^2 - 6x + 4 = 0$ is a *quadratic* equation that does not seem to factorize in a nice way. We'll develop a convenient method later for these, but I prefer to do it by *completing the square*: the only trick here is to notice that *Completing the square*

$$(x-3)^2 = x^2 - 6x + 9.$$

  So we try and make our equation contain $x^2 - 6x + 9$:

$$\begin{aligned} x^2 - 6x + 4 &= 0 \\ \therefore x^2 - 6x + 9 &= 5 \\ \therefore (x-3)^2 &= 5 \\ \therefore x - 3 &= \pm\sqrt{5} \\ \therefore x = 3 + \sqrt{5} \quad &\text{or} \quad x = 3 - \sqrt{5}. \end{aligned}$$

- $\cos(x) = 0.5$ is a *trigonometric* equation. One solution will be found easily using a calculator: $x = 60^o$. But if you look at the graph of cos you'll find there are infinitely many solutions: $x = 60^o + 360^o, 60^o + 2 \times 360^o, \ldots$ and so on, as well as $x = -60^o, -360^o + 60^o, -360^o - 60^o, \ldots$. In fact the solutions can be written

$$x = k \times 360^o \pm 60^o \text{ for all integers } k.$$

- $\cos(x) = x$ (here x is in radians) is an equation that mixes up trigonometric functions with polynomials, so in general we cannot hope to solve it with formulas. When all else fails, we can always draw a graph as in Figure 6 which will tell you (if you draw it *very very carefully*) that

$$x = 0.73908513321516064165531208767387 3\ldots$$

I am cheating here: this very accurate answer was obtained by Picard iteration not from a graph.

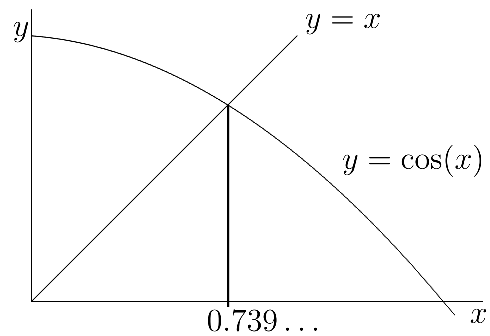


FIGURE 6. Graph to solve $\cos(x) = x$.

3.2. **Graphs of functions.** If you want to understand what a function is really doing, there is nothing better than a graph. Just as with solving equations, there is no single 'method' for sketching a graph: instead you need to use a mixture of common sense, calculation, and methods that will come later in the course from calculus. For now we will just learn the basic language of graphs. For a function $y = f(x)$ the *independent* variable x will run along the horizontal axis or *abscissa*, and the resulting y values – the *dependent* variable because it depends on x – run along the vertical axis or *ordinate*. The first and simplest approach to sketching a graph is to compute several points and join them up with a smooth curve. It is a matter of taste and experience to decide which range of values to compute.

It is a very good idea to become familiar with the graphs of functions you use a lot (like sin, cos, simple polynomials, the hyperbola $y = \frac{1}{x}$). Some of the most important functions are sketched in Figures 7 to 14.

A cubic curve.

Example 3.1. Sketch the curve $y = x^3 + x^2 - 2x + 3$.
Solution: For large positive values of x, y is very large and positive. Similarly, for large negative values of x, y is large and negative. So the interesting features will happen when x is quite small (close to 0). Figure 15 shows a few values (I have worked out more than you would need to).

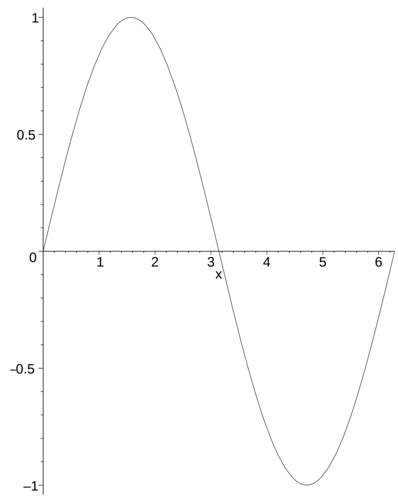


FIGURE 7. Graph of $y = \sin(x)$ for $0 \leq x \leq 2\pi$ radians.

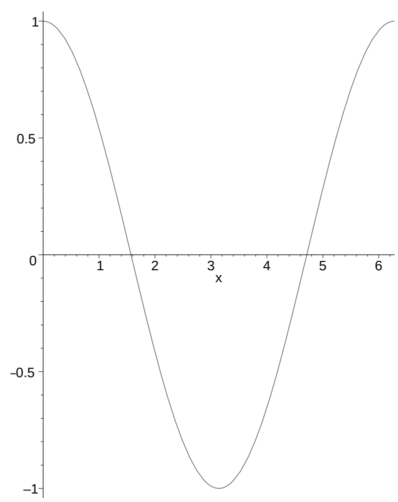


FIGURE 8. Graph of $y = \cos(x)$ for $0 \leq x \leq 2\pi$ radians.

Using these values, the graph can easily be sketched (we will do this in lectures).

3.3. **Quadratic formula.** Many problems end up with quadratic equations and it is useful to be able to solve them easily, and the *quadratic formula* does this (though I prefer completing the square!). The general quadratic equation has the form

General quadratic equation

$$ax^2 + bx + c = 0$$

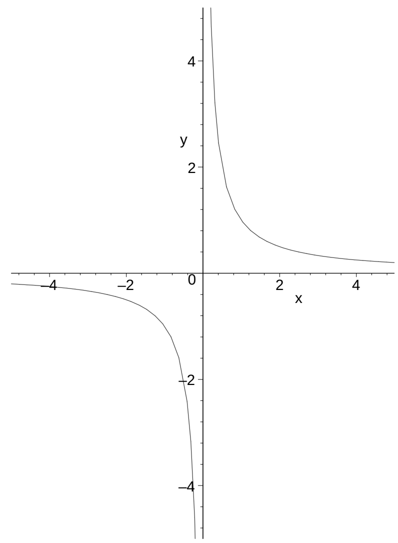


FIGURE 9. Graph of $y = 1/x$ for $-5 \le x \le 5$ radians.

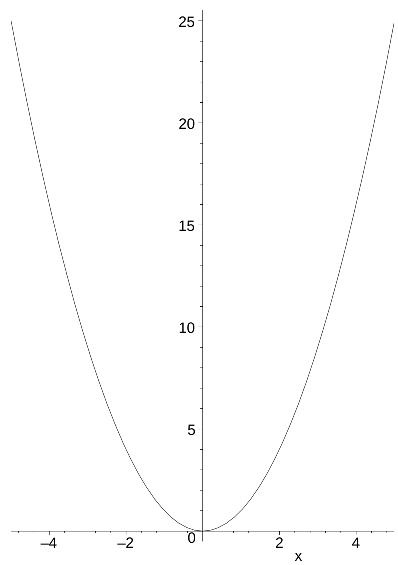


FIGURE 10. Graph of $y = x^2$ for $-5 \le x \le 5$.

with $a \neq 0$ (if $a = 0$ then it is a linear equation). After dividing by a, we can write this as

$$x^2 + \tfrac{b}{a}x + \tfrac{c}{a} = 0. \tag{3.1}$$

In order to complete the square, notice that

$$\left(x + \tfrac{b}{2a}\right)^2 = x^2 + \tfrac{b}{a}x + \tfrac{b^2}{4a^2}.$$

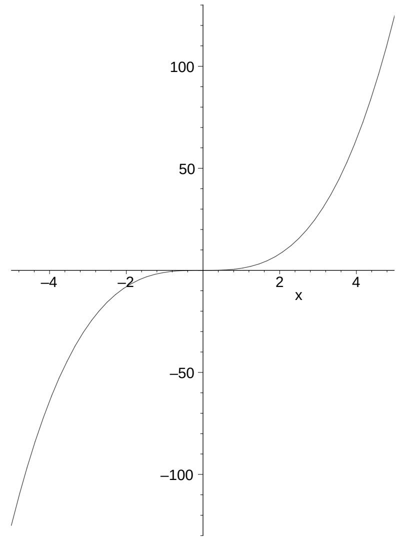


FIGURE 11. Graph of $y = x^3$ for $-5 \leq x \leq 5$.

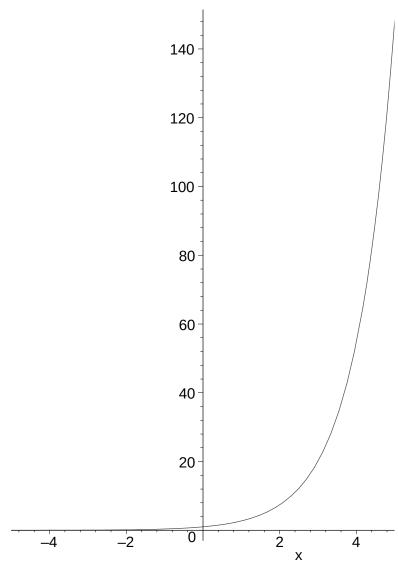


FIGURE 12. Graph of $y = e^x$ for $-5 \leq x \leq 5$.

So we can write (3.1) as

$$\left(x + \tfrac{b}{2a}\right)^2 = x^2 + \tfrac{b}{a}x + \tfrac{b^2}{4a^2} = \tfrac{b^2}{4a^2} - \tfrac{c}{a}. \tag{3.2}$$

It is now easy to find the solutions by taking square roots and subtracting $\frac{b}{2a}$:

$$x = -\tfrac{b}{2a} \pm \sqrt{\tfrac{b^2}{4a^2} - \tfrac{c}{a}}.$$

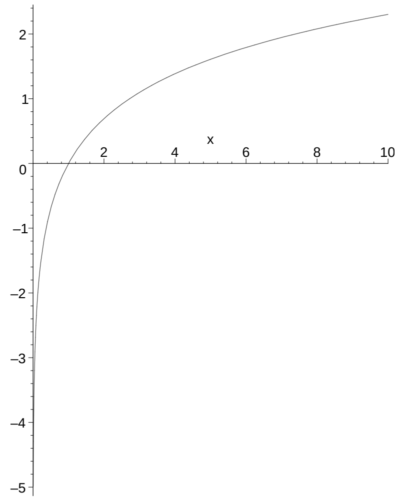


FIGURE 13. Graph of $y = \log(x)$ for $0 < x \leq 10$.

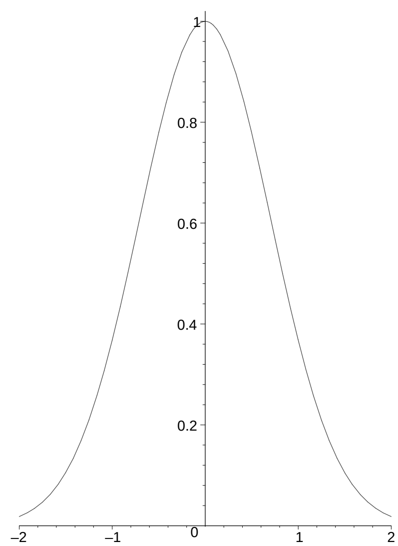


FIGURE 14. Graph of $y = e^{-x^2}$ for $-2 \leq x \leq 2$.

Quadratic formula

This can also be written in the more familiar way

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are three possibilities:

- If $b^2 - 4ac = 0$, then the two solutions are identical, so this is called a *double root* (cf. Figure 16).

x	y
-2.0	3.000
-1.8	4.008
-1.6	4.664
-1.4	5.016
-1.2	5.112
-1.0	5.000
-0.8	4.728
-0.6	4.344
-0.4	3.896
-0.2	3.432
0.0	3.000
0.2	2.648
0.4	2.424
0.6	2.376
0.8	2.552
1.0	3.000
1.2	3.768
1.4	4.904
1.6	6.456
1.8	8.472
2.0	11.00

FIGURE 15. Points on the curve $y = x^3 + x^2 - 2x + 3$.

- If $b^2 - 4ac > 0$, then there are two distinct real roots (cf. Figure 17).
- If $b^2 - 4ac < 0$, then there are no real roots (cf. Figure 18).

3.3.1. *Graphs of quadratics.* For simplicity we will think of the quadratic equation in the standard form (3.1) (this is to avoid having to think of a positive and a negative separately). Now solving the equation (3.1) is exactly the same as asking for the points where the graph of the function

$$y = x^2 + \tfrac{b}{a}x + \tfrac{c}{a}$$

crosses the x-axis. Now the three cases for the quadratic equation can be seen as follows.

Example 3.2. A rectangle has one side 2 metres longer than the other side. The area of the rectangle is 24 square metres. What are the lengths of the sides?

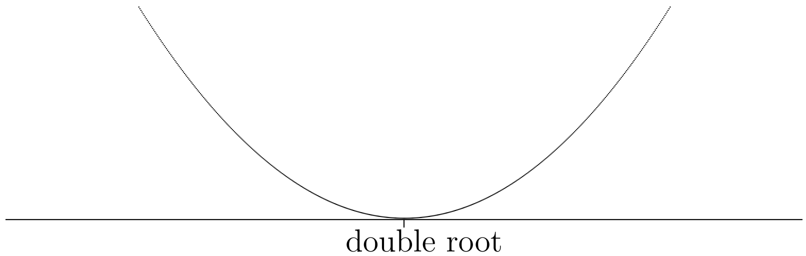


FIGURE 16. Quadratic with $b^2 - 4ac = 0$.

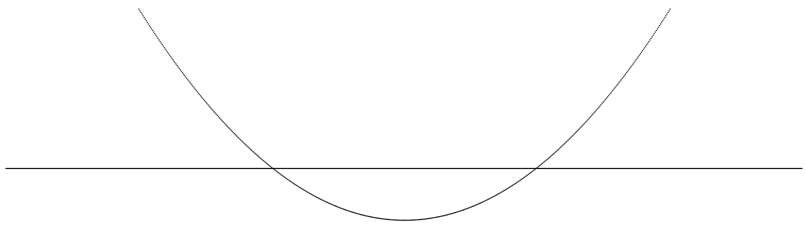


FIGURE 17. Quadratic with $b^2 - 4ac > 0$.

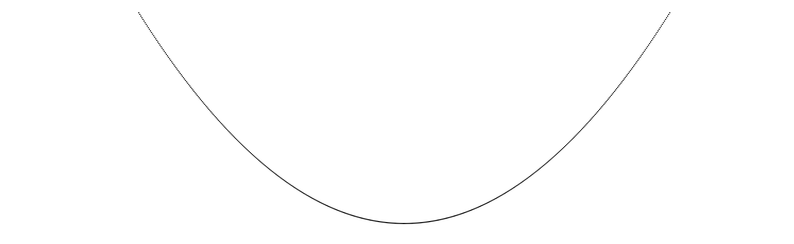


FIGURE 18. Quadratic with $b^2 - 4ac < 0$.

Solution: Let the rectangle have sides x and $(x + 2)$. Then the given information says that

$$x(x + 2) = 24,$$

which is a quadratic equation. Write this in standard form,

$$x^2 + 2x - 24 = 0.$$

Now you can either use the quadratic formula or notice a factorisation:

$$x^2 + 2x - 24 = (x - 4)(x + 6).$$

So

$$(x-4)(x+6)=0,$$

which shows that x is either 4 or -6. Since we are looking for a positive length, the answer must be $x=4$, so the rectangle has sides of length 4 metres and 6 metres.

4. Useful web sites

There are on-line mathematics tutorials at several sites, including
http://www.ping.be/math/mathindex.htm
http://www.netsrq.com/~hahn/calculus.html
http://sunsite.ubc.ca/LivingMathematics/
The figures for Section 1.1.2 come from
http://www.whitehouse.gov/omb/budget/
http://particleadventure.org/
http://nedwww.ipac.caltech.edu/
http://science.nasa.gov/
http://hubble.stsci.edu/
There are many beautiful proofs of Pythagorus' theorem at
http://www.mcn.net/~jimloy/pythag.html

E-mail address: `t.ward@uea.ac.uk`

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