DIGRAPHS WITH TREE-LIKE DESCENDANT SETS

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Abstract. We give certain properties which are satisfied by the descendant set of a vertex in a primitive distance-transitive digraph of finite out-valency and provide a strong structure theory for digraphs satisfying these properties. In particular, we show that there are only countably many possibilities for the isomorphism type of such a descendant set, thereby confirming a conjecture of the first Author.

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1. Introduction and Notation

The main purpose of this note is to prove a conjecture of the first Author from [2] stating that there are only countably many isomorphism types of digraphs which can arise as the descendant set of a vertex in a primitive highly arc transitive digraph of finite out-valency and infinite in-valency. Before describing this in detail, we fix the following notation and terminology.

A digraph D consists of a set VD of vertices, and a set $ED \subseteq VD \times VD$ of ordered pairs of vertices, the (directed) edges. Our digraphs will have no loops and no multiple edges. The *out-valency* of a vertex α is the size of the set $\{u \in VD : (\alpha, u) \in ED\}$; similarly, the *in-valency* of α is the size of the set $\{u \in VD \mid (u, \alpha) \in ED\}$. Let $s \geq 0$ be an integer. An *s-arc* from u to v in D is a sequence $u_0 u_1 \ldots u_s$ of $s+1$ vertices such that $u_0 = u$, $u_s = v$ and $(u_i, u_{i+1}) \in ED$ for $0 \leq i < s$ and $u_{i-1} \neq u_{i+1}$ for $0 < i < s$. We denote by $D^s(u)$ the set of vertices of D which are reachable by an s-arc from u. The *descendant set* $D(u)$ (or $\mathrm{desc}(u)$) of u is $\bigcup_{s \geq 0} D^s(u)$. Similarly the set $\mathrm{anc}(u)$ of ancestors of u is the set of vertices of which u is a descendant.

We think of any subset X of VD as a digraph in its own right by considering the full induced subdigraph on X and we will abuse terminology by referring to 'the digraph X' and writing $x \in X$ to indicate that x is a vertex of X. In particular, fix $\alpha \in D$, and let $\Gamma = D(\alpha)$. If $\mathrm{Aut}(D)$ is transitive on the set of vertices of D, then $D(u) \cong \Gamma$ for all vertices u, and we shall speak of the digraph Γ as *the descendant set* of D.

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Henceforth, we shall be interested in the structure of a descendant set $\Gamma = \Gamma(\alpha)$ in some transitive digraph with finite out-valency m. We refer to α as the *root* of Γ and write $\Gamma = \Gamma(\alpha)$ to indicate that any vertex of Γ is a descendant of α. Similarly, we write Γ^i instead of $\Gamma^i(\alpha)$ and if $\beta \in \Gamma(\alpha)$, then we let $\Gamma(\beta) = \mathrm{desc}(\beta) \subseteq \Gamma(\alpha)$.

We work with digraphs Γ having the following properties:

G0 $\Gamma = \Gamma(\alpha)$ is a rooted digraph with finite out-valency m and $\Gamma^s(\alpha) \cap \Gamma^t(\alpha) = \varnothing$ whenever $s \neq t$.
G1 $\Gamma(u) \cong \Gamma$ for all $u \in \Gamma$.
G2 For $n \in \mathbb{N}$ we have $|\Gamma^n(\alpha)| < |\Gamma^{n+1}(\alpha)|$.
G3 There is a number $k \in \mathbb{N}$ such that if $\ell \geq k$ and $x \in \Gamma^\ell(\alpha)$ and $z \in \Gamma(x)$, then $\mathrm{anc}(z) \cap \Gamma^1(\alpha) = \mathrm{anc}(x) \cap \Gamma^1(\alpha)$.

A priori there could be continuum-many isomorphism types of digraphs with these properties. Our main result (Theorem 3.2) is that there are only countably many isomorphism types of digraph which satisfy G0, G1 and G3. To establish this, we show in Section 2 that there is a natural equivalence relation ρ on Γ (refining the 'layering' of Γ given by G0) such that the quotient digraph Γ/ρ is a directed tree. If G2 holds then this is not a directed line and the size of the layers Γ^n grows exponentially.

In the rest of this introduction we explain how the above conditions arise from rather more natural hypotheses and in particular relate them to Amato's conjecture.

In [7], Neumann asked whether there exists a primitive permutation group having an infinite suborbit which is paired with a finite suborbit. This amounts to asking whether there is a digraph with infinite in-valency and finite out-valency whose automorphism group is transitive on edges and primitive on vertices. Countable digraphs of this sort were constructed in [6] using amalgamation methods developed in model theory. Similar methods we used in [5] to construct continuum-many non-isomorphic such countable digraphs and this strongly suggests that a classification of such digraphs is out of the question. Nevertheless, work in [2] suggested that a classification of the *descendant sets* in these digraphs might be possible, at least under stronger hypotheses on the automorphism group of the digraph.

The digraphs constructed in [6] and [5] are highly arc transitive: the automorphism group is transitive on s-arcs for all finite s. Primitive, highly arc transitive digraphs of finite out-valency are analysed in [2]. It is shown that the descendent set in such a digraph without directed cycles satisfies certain properties: P1 (giving our G0, G1) and P2, P3 (stronger forms of our G2, G3). A structure theory was developed in [2] for digraphs with these properties and it was conjectured that only countably many isomorphism types of digraph have them. We reproduce most of this structure theory in Section 2 here under weaker assumptions. The conjecture follows from Corollary 1.5 and Corollary 3.2 here.

We say that a digraph D is *directed-distance transitive* if for every $s \geq 0$ its automorphism group is transitive on pairs (u, v) for which there is an s-arc from u to v, but no t-arc for $t < s$. Note that this implies vertex and edge transitivity, but is weaker than being highly arc transitive.

Lemma 1.1. *Suppose D is of finite out-valency, has no directed cycles and is directed-distance transitive. Then any descendant set $\Gamma(\alpha)$ in D satisfies G0.*

Proof. This is the same as the proof of Proposition 3.10 in [4], so we omit the details.

□

Lemma 1.2. *Suppose D is a digraph of finite out-valency with a directed cycle and whose automorphism group is either primitive on vertices or transitive on edges. Then D has finite in-valency.*

Proof. First, suppose that D is edge-transitive. Then there is a K such that every edge of D is in a directed K-cycle. Let $\alpha \in D$. Then every in-vertex β of α is in $D^{K-1}(\alpha)$. But this set is finite, as D has finite out-valency.

Now suppose D is vertex-primitive. Consider the relation $\sim$ on D given by $u \sim v \Leftrightarrow u \in D(v)$ and $v \in D(u)$. This is an $\mathrm{Aut}(D)$-invariant equivalence relation on D and as D contains a directed cycle, its classes are not singletons. Thus, by primitivity $u \sim v$ for all $u, v \in D$. In particular, every edge of D is contained in a cycle. We can then argue as in the first case. □

Lemma 1.3. *Suppose Γ satisfies G0, G1 and that for each $i \in \mathbb{N}$ the automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on Γ^i. Then Γ satisfies G3.*

Proof. For $x \in \Gamma^i$, let $t_i = |\mathrm{anc}(x) \cap \Gamma^1|$. By the transitivity assumption, this depends only on i. As $\mathrm{anc}(x) \cap \Gamma^1 \subseteq \mathrm{anc}(z) \cap \Gamma^1$ when $z \in \Gamma(x)$, we have $t_1 \leq t_2 \leq t_3 \leq \ldots \leq m$. Choosing k so that t_k is as large as possible, the result follows. □

Remark 1.4. Note that in the above if G2 also holds, then $t_i < m$. Otherwise, for $\beta \in \Gamma^1$ we have $\Gamma^{i-1}(\beta) = \Gamma^i(\alpha)$ and so $|\Gamma^{i-1}| = |\Gamma^i|$ (by G1), contradicting G2.

Corollary 1.5. *Suppose D is a directed-distance transitive digraph with infinite in-valency and finite out-valency. Then the descendant set Γ in D satisfies G0, G1, G3. If the automorphism group of D is primitive on vertices, then Γ satisfies G2.*

Proof. By Lemma 1.2, D has no directed cycles, so by Lemma 1.1, Γ satisfies G0. As D has transitive automorphism group, G1 holds.

Directed-distance transitivity implies that $\mathrm{Aut}(\Gamma)$ is transitive on each Γ^i, so G3 holds. Finally if G2 does not hold for some n then for $\beta, \beta' \in \Gamma^1(\alpha)$ we have $\Gamma^n(\beta) = \Gamma^n(\beta')$. This gives a non-trivial equivalence relation on D which is clearly preserved by $\mathrm{Aut}(D)$. $\square$

2. Structure theory

Throughout this section we assume that Γ satisfies G0, G1, G3. We let k be an integer satisfying the condition in G3. The proofs in this section are all adapted from [2].

Lemma 2.1. *Suppose* $\beta \in \Gamma^n(\alpha)$, $\ell \geq k$, $x \in \Gamma^{n+\ell}(\alpha)$ *and* $z \in \Gamma(x) \cap \Gamma(\beta)$. *Then* $x \in \Gamma(\beta)$.

Proof. This is by induction on n. The case $n = 0$ is trivial as then $\beta = \alpha$. In general let $\gamma \in \Gamma^{n-1}(\alpha)$ be an ancestor of β. By induction hypothesis, $x \in \Gamma^{\ell+1}(\gamma)$. Now work with $\Gamma(\gamma) \cong \Gamma$ (by G1). As $\ell \geq k$ and $z \in \Gamma(x)$ we have $\mathrm{anc}(z) \cap \Gamma^1(\gamma) = \mathrm{anc}(x) \cap \Gamma^1(\gamma)$ (by G3 in $\Gamma(\gamma)$). So $\beta \in \mathrm{anc}(x)$, that is $x \in \Gamma(\beta)$, as required. $\square$

Definition 2.2. (1) Suppose $\beta \in \Gamma$, $x \in \Gamma^n(\beta)$ and $s \leq n$. Define

$$\Gamma_\beta^{-s}(x) = \{w \in \Gamma^{n-s}(\beta) : x \in \Gamma(w)\}.$$

(2) For $\ell \geq k$ and $x, y \in \Gamma^\ell(\alpha)$ write $\rho(x, y)$ iff

$$\Gamma_\alpha^{-k+1}(x) = \Gamma_\alpha^{-k+1}(y).$$

(Say that $\rho(x, y)$ does not hold in all other cases.)

So for $x, y \in \Gamma^\ell(\alpha)$ we have that $\rho(x, y)$ holds iff x, y have the same ancestors in $\Gamma^{\ell-k+1}(\alpha)$. Clearly ρ is an $\mathrm{Aut}(\Gamma)$-invariant equivalence relation on $\bigcup_{\ell \geq k} \Gamma^\ell$.

Lemma 2.3. *Suppose* $\ell \geq k$ *and* $x, y \in \Gamma^\ell(\alpha)$. *If* $\Gamma(x) \cap \Gamma(y) \neq \varnothing$, *then* $\rho(x, y)$ *holds.*

Proof. Note that the result holds for $\ell = k$ by G3.

Suppose $\ell = n+k$ with $n \geq 1$ and that $z \in \Gamma(x) \cap \Gamma(y)$. Let $B = \{\beta \in \Gamma^n(\alpha) : z \in \Gamma(\beta)\}$. If $\beta \in B$, then by Lemma 2.1, $x, y \in \Gamma^k(\beta)$. Thus (by the case $\ell = k$ in $\Gamma(\beta)$) we have $\mathrm{anc}(x) \cap \Gamma^1(\beta) = \mathrm{anc}(y) \cap \Gamma^1(\beta)$. But $\Gamma_\alpha^{-k+1}(x), \Gamma_\alpha^{-k+1}(y) \subseteq \bigcup_{\beta \in B} \Gamma^1(\beta)$. Thus $\Gamma_\alpha^{-k+1}(x) = \Gamma_\alpha^{-k+1}(y)$, so $\rho(x, y)$. $\square$

For $\ell \geq k$ and $x \in \Gamma^l(\alpha)$ we write $[x]_\rho$ for the ρ-equivalence class containing x. We use notation such as $\mathbf{v}$, $\mathbf{w}$ etc. for such classes and write $\Gamma(\mathbf{u}) = \bigcup_{x \in \mathbf{u}} \Gamma(x)$ and $\Gamma^s(\mathbf{u}) = \bigcup_{x \in \mathbf{u}} \Gamma^s(x)$.

Lemma 2.4. *Suppose* $\ell \geq k$ *and* $\mathbf{v} \subseteq \Gamma^\ell(\alpha)$ *is a* ρ*-class. Let* $w \in \Gamma(\mathbf{v})$. *Then* $[w]_\rho \subseteq \Gamma(\mathbf{v})$.

Proof. It suffices to prove this when $w \in \Gamma^{\ell+1}(\alpha)$. So suppose that $(v,w),(v',w')$ are directed edges and $\rho(w,w')$ holds. We need to show that $\rho(v,v')$ holds. Let $A=\Gamma_\alpha^{-1}(w)$ and $A'=\Gamma_\alpha^{-1}(w')$. By Lemma 2.3, $A\subseteq [v]_\rho$ and $A'\subseteq [v']_\rho$. By definition, $\Gamma_\alpha^{-k+1}(w)=\bigcup_{a\in A}\Gamma_\alpha^{-k+2}(a)$ and $\Gamma_\alpha^{-k+1}(w')=\bigcup_{a'\in A'}\Gamma_\alpha^{-k+2}(a')$. So $\bigcup_{a\in A}\Gamma_\alpha^{-k+2}(a)=\bigcup_{a'\in A'}\Gamma_\alpha^{-k+2}(a')$, as $\rho(w,w')$ holds. It follows (by taking ancestors one level back) that $\bigcup_{a\in A}\Gamma_\alpha^{-k+1}(a)=\bigcup_{a'\in A'}\Gamma_\alpha^{-k+1}(a')$. But as $A\subseteq[v]_\rho$, the left hand side is equal to $\Gamma_\alpha^{-k+1}(v)$ and similarly the right hand side is equal to $\Gamma_\alpha^{-k+1}(v')$. Thus $\rho(v,v')$ holds. □

Corollary 2.5. *Suppose $\ell\geq k$ and $v\in\Gamma^\ell(\alpha)$. Let $\mathbf{v}$ be the ρ-class containing v. Then the quotient digraph $\Gamma(\mathbf{v})/\rho$ is a rooted directed tree with finite out-valencies.*

Proof. The statement follows from Lemmas 2.3 and 2.4. □

Note that for $\beta\in\Gamma(\alpha)$ we can consider the equivalence relation ρ computed in both $\Gamma(\alpha)$ and $\Gamma(\beta)$, where in the latter we only consider ancestors in $\Gamma(\beta)$ when defining ρ: *a priori* this gives a coarser relation.

Lemma 2.6. *Suppose $\beta\in\Gamma^n(\alpha)$ and $x\in\Gamma^\ell(\beta)$ with $\ell\geq 2k-1$. Then the ρ-class containing x is the same whether it is computed in $\Gamma(\alpha)$ or $\Gamma(\beta)$.*

Proof. Note that $x\in\Gamma^{n+\ell}(\alpha)$. First observe that if $y\in[x]_\rho$ (computed in $\Gamma(\alpha)$) then x,y have the same ancestors in $\Gamma^{n+\ell-k+1}(\alpha)$ and so also in $\Gamma^n(\alpha)$: in particular $y\in\Gamma(\beta)$. So to prove the statement, it suffices to show that $\Gamma_\beta^{-k+1}(x)=\Gamma_\alpha^{-k+1}(x)$. It is clear from the definition that $\Gamma_\beta^{-k+1}(x)\subseteq\Gamma_\alpha^{-k+1}(x)$. Conversely, suppose $w\in\Gamma_\alpha^{-k+1}(x)$. Then $w\in\Gamma^{n+\ell-k+1}(\alpha)$ and by assumption $n+\ell-k+1\geq n+k$. So by Lemma 2.1 we have $w\in\Gamma(\beta)$ and therefore $w\in\Gamma_\beta^{-k+1}(x)$. □

Let $\ell\geq 2k-1$ and let $\mathbf{v}$ be a ρ-class in $\Gamma^\ell(\alpha)$. Let $T(\mathbf{v})$ be the structure consisting of the induced digraph on $\Gamma(\mathbf{v})$ together with the equivalence relation induced by ρ (coming from $\Gamma(\alpha)$). Recall that by Lemma 2.4, $T(\mathbf{v})$ is a union of ρ-classes in $\Gamma(\alpha)$. If $\mathbf{w}$ is another ρ-class (in $\bigcup_{\ell\geq 2k-1}\Gamma^l(\alpha)$) then by a *$\rho$-isomorphism* between $T(\mathbf{v})$ and $T(\mathbf{w})$ we mean a digraph isomorphism which respects ρ.

Corollary 2.7. *Suppose $\mathbf{v}$ is a ρ-class in $\Gamma^\ell(\alpha)$ with $l\geq 2k-1$. Then there is a ρ-class $\mathbf{w}$ in $\Gamma^{2k-1}(\alpha)$ and a ρ-isomorphism from $T(\mathbf{w})$ to $T(\mathbf{v})$.*

Proof. Let $v\in\mathbf{v}$ and let $\beta\in\Gamma^{\ell-2k+1}(\alpha)$ be an ancestor of v. So $v\in\Gamma^{2k-1}(\beta)$ and by Lemma 2.6 it follows that $\mathbf{v}\subseteq\Gamma(\beta)$. So $T(\mathbf{v})\subseteq\Gamma(\beta)$ and the ρ-structure on $T(\mathbf{v})$ is the same whether it is computed in $\Gamma(\alpha)$ or $\Gamma(\beta)$. By G1 there is a digraph isomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$, and this induces a ρ-isomorphism between $T(\mathbf{w})$, for some ρ-class $\mathbf{w}\subseteq\Gamma^{2k-1}(\alpha)$, and $T(\mathbf{v})\subseteq\Gamma^{2k-1}(\beta)$, as required. □

Thus to any digraph Γ satisfying G0, G1, G3, there are associated a finite number of ρ-isomorphism types of $T(\mathbf{v})$. In particular, we can refine Corollary 2.5 to:

Corollary 2.8. *Suppose $\ell \geq 2k-1$ and $\mathbf{v} \subseteq \Gamma^{\ell}(\alpha)$ is a ρ-class. Then the quotient digraph $T(\mathbf{v})/\rho$ is a rooted directed tree with a finite number of out-valencies.* □

3. Counting isomorphism types

We let $\mathcal{T}$ be the class of structures T with the following properties

- T is a digraph of finite out-valency and $T = T(\mathbf{u})$ for some finite set $\mathbf{u} \subseteq T$.
- $T^s(\mathbf{u}) \cap T^t(\mathbf{u}) = \varnothing$ whenever $s \neq t$.
- There is an equivalence relation ρ on T such that each ρ-class is contained in a layer $T^s(\mathbf{u})$.
- The quotient digraph T/ρ is a directed forest.
- For every ρ-class $\mathbf{w}$ there is a ρ-class $\mathbf{v} \subseteq \mathbf{u}$ and a ρ-isomorphism between $T(\mathbf{v})$ and $T(\mathbf{w})$.

We show:

Theorem 3.1. *There are only countably many ρ-isomorphism types of structures in $\mathcal{T}$.*

Corollary 3.2. *There are only countably many isomorphism types of digraph Γ which have properties G0, G1 and G3.*

Proof of Corollary. Fix such a Γ. Let T be the disjoint union of digraphs $T(\mathbf{v})$ with the equivalence relation ρ as in the previous section, taking $\mathbf{v}$ to be a ρ-class in Γ^{2k-1}. So in fact, $T = \bigcup_{\ell \geq 2k-1} \Gamma^{\ell}$. Then $T \in \mathcal{T}$, by Corollaries 2.8 and 2.7. Moreover we can recover Γ from T by looking at the descendant set of any vertex in T. Thus there are only countably many possibilities for Γ, by the above Theorem. □

We now prove Theorem 3.1. Let $T = T(\mathbf{u}) \in \mathcal{T}$ and let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be the ρ-classes in $T^0 = \mathbf{u}$. We colour a ρ-class $\mathbf{v}$ in T with colour C_i if i is (as small as possible) such that $T(\mathbf{v})$ is ρ-isomorphic to $T(\mathbf{v}_i)$. If $d \in \mathbb{N}$, then we denote by B^d_T the digraph on $\bigcup_{s \leq d} T^s$ together with the structure given by the ρ-classes and the colouring on this set. Similarly if $\mathbf{v}$ is a ρ-class we denote by $B^d_T(\mathbf{v})$ the corresponding structure on $\bigcup_{s \leq d} T^s(\mathbf{v})$.

In the following, by a *$\rho - C$-isomorphism* we mean a digraph isomorphism which preserves the relation ρ and the colouring.

Lemma 3.3. *For $T \in \mathcal{T}$ there is a natural number $N = N_T$ with the property that if $d \geq N$, $\mathbf{v}, \mathbf{v}'$ are ρ-classes in T and $\alpha' : B^d_T(\mathbf{v}) \to B^d_T(\mathbf{v}')$ is a $\rho - C$-isomorphism, then there is a $\rho - C$-isomorphism $\alpha : T(\mathbf{v}) \to T(\mathbf{v}')$ with $\alpha(x) = \alpha'(x)$ for all $x \in \mathbf{v}$.*

Proof. Let A_0 be the group of permutations induced on T^0 by $\rho - C$-automorphisms of T which fix each ρ-class in T^0. Similarly for $d \geq 1$ let A_d be the group of permutations induced on T^0 by $\rho{-}C$-automorphisms of B^d_T which fix each ρ-class in T^0. Then $A_d \geq A_{d+1}$ and $A_0 = \bigcap_d A_d$, so there is a smallest integer $N \geq 1$ with $A_N = A_0$. In particular, for any ρ-class $\mathbf{v}$ in T^0, and $d \geq N$, any permutation of $\mathbf{v}$ which extends to a $\rho - C$-automorphism of $B^d_T(\mathbf{v})$ extends to an automorphism of $T(\mathbf{v})$. The same is therefore true for any ρ-class in T.

We show that this N has the required property. So let $\mathbf{v}, \mathbf{v}'$ etc be as in the statement. As $\mathbf{v}$, $\mathbf{v}'$ have the same colour, there is some $\rho - C$-isomorphism $\beta : T(\mathbf{v}) \to T(\mathbf{v}')$. Let β' be its restriction to $B^d_T(\mathbf{v})$. Then α', β' both have image $B^d_T(\mathbf{v}')$ and $\gamma' = (\beta')^{-1} \circ \alpha'$ is a $\rho - C$-automorphism of $B^d_T(\mathbf{v})$. So as $d \geq N$ there is a $\rho - C$-automorphism γ of $T(\mathbf{v})$ which agrees with γ' on $\mathbf{v}$. It is easy to check that $\alpha = \beta \circ \gamma$ is a $\rho - C$-isomorphism with the required properties. □

Proposition 3.4. *Suppose $T, S \in \mathcal{T}$ and $d > N_S$. If there is a $\rho - C$-isomorphism from B^d_T to B^d_S, then there is a $\rho - C$-isomorphism from B^{d+1}_T to B^{d+1}_S.*

Proof. Let $\Phi : B^d_T \to B^d_S$ be a $\rho - C$-isomorphism. Note that $d \geq 1$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_s$ be the ρ-classes in T^1 and $\mathbf{w}_i = \Phi(\mathbf{v}_i)$. So these are the ρ-classes in S. For $i \in \{1, \ldots, s\}$ there is a ρ-class $\mathbf{u}_i$ in T^0 and a $\rho - C$-isomorphism $f_i : T(\mathbf{u}_i) \to T(\mathbf{v}_i)$. Let $\mathbf{z}_i = \Phi(\mathbf{u}_i)$ and $\alpha'_i : B^{d-1}_S(\mathbf{z}_i) \to B^{d-1}_S(\mathbf{w}_i)$ be given by

$$\alpha'_i(y) = \Phi(f_i(\Phi^{-1}(y))).$$

So α'_i is a $\rho - C$-isomorphism. As $d - 1 \geq N_S$ it follows by Lemma 3.3 that there is a $\rho - C$-isomorphism $\alpha_i : S(\mathbf{z}_i) \to S(\mathbf{w}_i)$ which agrees with α' on $\mathbf{z}_i$.

We define $\Psi : B^{d+1}_T \to B^{d+1}_S$ as follows. For $x \in T^0$ we let $\Psi(x) = \Phi(x)$. If $x \in B^{d+1}_T \setminus T^0$ then there is a unique $i \leq s$ with $x \in B^d_T(\mathbf{v}_i)$ and in this case we define

$$\Psi(x) = \alpha_i(\Phi(f_i^{-1}(x))).$$

It is easy to see that Ψ is a well-defined bijection between B^{d+1}_T and B^{d+1}_S. As f_i, Φ and α_i all preserve ρ-classes and the colouring, the same is true of Ψ. So it remains to show that Ψ preserves edges and non-edges.

First we show that if $x \in B^1_T$, then $\Psi(x) = \Phi(x)$. If $x \in T^0$ then this is by definition of Ψ. If $x \in T^1$ then $x \in \mathbf{v}_i$ for some unique $i \leq s$. So $f_i^{-1}(x) \in \mathbf{u}_i$ and $\Phi(f_i^{-1}(x)) \in \mathbf{z}_i$, whence

$$\Psi(x) = \alpha_i \Phi f_i^{-1}(x) = \alpha'_i \Phi f_i^{-1}(x) = \Phi(x).$$

Thus Ψ preserves edges and non-edges in B^1_T.

If $x, y \in B^{d+1}_T \setminus T^0$ and (x, y) is an edge, then $x, y \in B^d_T(\mathbf{v}_i)$ for some i. Then $\Psi(x) = \alpha_i \Phi f_i^{-1}(x)$ and $\Psi(y) = \alpha_i \Phi f_i^{-1}(y)$ and so, as α_i, Φ

and f_i preserve edges, $(\Psi(x), \Psi(y))$ is an edge in B_S^d. By the same argument, if $x, y \in B_T^d(\mathbf{v}_i)$ and (x, y) is a non-edge, then $(\Psi(x), \Psi(y))$ is a non-edge. Finally, if x, y lie in different $B_T^d(\mathbf{v}_i)$ then $\Psi(x), \Psi(y)$ lie in different $B_S^d(\mathbf{w}_i)$, so $(\Psi(x), \Psi(y))$ is a non-edge. $\square$

Corollary 3.5. *Suppose $T, S \in \mathcal{T}$ and B_T^d, B_S^d are $\rho - C$-isomorphic for some $d \geq N_S$. Then T and S are $\rho - C$-isomorphic.*

Proof. By assumption and Proposition 3.4, for $n \geq N_S$ the set I_n of $\rho - C$-isomorphisms $B_T^n \to B_S^n$ is non-empty. Restriction gives a map $I_{n+1} \to I_n$ and so, as each I_n is finite, König's Lemma implies that there is a $\rho - C$-isomorphism $T \to S$. $\square$

Proof of Theorem 3.1. Suppose $T \in \mathcal{T}$. As above, consider this with a colouring of the ρ-classes. Let $N = N_T$ be as in Lemma 3.3. Then by Corollary 3.5, the (coloured) ball B_T^{N+1} determines T within $\mathcal{T}$ up to isomorphism. There are only countably many possibilities for this finite structure, hence the result. $\square$

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