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PERIODIC POINTS OF ENDOMORPHISMS ON SOLENOIDS AND RELATED GROUPS

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Abstract

This paper investigates the problem of finding the possible sequences of periodic point counts for endomorphisms of solenoids. For an ergodic epimorphism of a solenoid, a closed formula is given which expresses the number of points of any given period in terms of sets of places of finitely many algebraic number fields and distinguished elements of those fields. The result extends to more general epimorphisms of compact abelian groups.

1. *Introduction*

Let X be a compact metrizable abelian group and α a continuous endomorphism of X. The set of points of period $j \in \mathbb{N}$ for α is

$$F_j(\alpha) = \{x \in X : \alpha^j(x) = x\}.$$

Denote the cardinality of $F_j(\alpha)$ by $|F_j(\alpha)|$. Assuming $F_j(\alpha)$ is finite for all $j \in \mathbb{N}$, an obvious question is: what are the possible sequences of periodic point counts arising for such dynamical systems? One route to providing an answer is to find a reasonable formula for $|F_j(\alpha)|$; this may also provide further information about the dynamical system (X, α). Such a situation is exemplified by toral endomorphisms [**2**], [**16**].

After toral endomorphisms, the next most obvious candidates for study are solenoidal endomorphisms (X is a *solenoid* if it is connected and has finite topological dimension, equivalently if the Pontryagin dual group $\widehat{X}$ is a finite rank torsion-free abelian group). When X is one-dimensional, $\widehat{X} \subset \mathbb{Q}$ and one may refer to the convenient classification [**3**] of subgroups of $\mathbb{Q}$ as a starting point. However, when the dimension of X is two or more there is no such description available. For example, Kechris [**9**] considers the classification problem in rank two and shows that the appropriate Borel equivalence relation is not treeable [**9**, Th. 5].

An interesting special case where a periodic point formula is known is that of the S-integer dynamical systems, introduced by Chothi, Everest and Ward [**4**]. These systems can exhibit behaviour not present in the toral case, such as irrational zeta functions and erratic growth. The authors also show how certain questions concerning the growth of periodic points are equivalent to deep problems in number theory. Unfortunately, a major drawback of the S-integer framework is that it far from describes all solenoidal systems. For example, the *circle doubling map* $x \mapsto 2x$ on $\mathbb{T}$ is an S-integer dynamical system. However, even in dimension one there are uncountably many algebraically non-conjugate systems (failing to be S-integer systems) whose sequences of periodic point counts are identical to that of the circle doubling map (*cf.* Example 1).

Despite the issues raised above, a closed formula for periodic point counting is obtained that applies to a wide class of dynamical systems (encompassing all ergodic solenoidal systems).

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Some obvious conditions such as ergodicity and finite entropy are necessary to ensure that $F_j(\alpha)$ is finite for all $j \in \mathbb{N}$. The proof is independent of the S-integer framework and uses an algebraic approach stemming from the ideas of Kitchens and Schmidt [**10**]. Notably, this overcomes some important obstacles such as the classification problem [**9**] and the appearance of non-Noetherian dual modules (see Section 2).

The main result shows that the sequences of periodic point counts that arise are products of those arising from S-integer dynamical systems. This is a little surprising, since the main decomposition employed is a twisted skew product whose factors may be far from S-integer systems. By 'far from', we mean that there may not even be a finite-to-one projection of such a factor onto any S-integer system. Typically, the factors are projective limits of toral endomorphisms or full shifts. Our result is phrased in terms of global fields K and their places $\mathcal{P}(K)$ (see Section 2). Each of these fields will be an algebraic number field if X is a solenoid or a function field of transcendence degree one over a finite field if X is zero-dimensional. If X is neither a solenoid nor zero-dimensional, a mixture of such fields will arise.

THEOREM 1.1. *Let α be an ergodic finite entropy epimorphism of a finite-dimensional compact abelian group. Then there exist global fields $K_1, \ldots, K_n$, sets of finite places $P_i \subset \mathcal{P}(K_i)$ and $\xi_i \in K_i$, $1 \leqslant i \leqslant n$, such that for any $j \in \mathbb{N}$,*

$$|F_j(\alpha)| = \prod_{i=1}^{n} \prod_{v \in P_i} |\xi_i^j - 1|_v^{-1}. \tag{1.1}$$

In Section 3, the case of one-dimensional solenoids is considered separately, as it illustrates some of the ideas needed for the proof of the main result and also highlights important families of non-conjugate systems sharing common sequences of periodic point counts.

2. *Algebraic Background*

Let X be a compact abelian group and α an endomorphism of X. Throughout, X is assumed to be metrizable and α continuous. The Pontryagin dual group $\widehat{X}$ is a countable discrete abelian group and the dual map $\widehat{\alpha}$ is a monomorphism whenever α is an epimorphism. Let $R = \mathbb{Z}[t]$. The group $M = \widehat{X}$ has the structure of an R-module by identifying $\widehat{\alpha}$ with the map $x \mapsto tx$ and extending in an obvious way to polynomials. Conversely, if M is a countable R-module, multiplication by t on M dualizes to an endomorphism α_M of the compact abelian group $X = \widehat{M}$.

If α is an ergodic finite entropy epimorphism of a finite-dimensional group, the dual module M satisfies the following conditions:

(1) the set of associated primes $\mathrm{Ass}(M)$ is finite and consists entirely of non-zero principal ideals,
(2) the map $x \mapsto (t^j - 1)x$ is a monomorphism of M for all $j \in \mathbb{N}$ (equivalently, $t^j - 1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$ and all $j \in \mathbb{N}$),
(3) for each $\mathfrak{p} \in \mathrm{Ass}(M)$,

$$m(\mathfrak{p}) = \dim_{\mathbb{K}(\mathfrak{p})} M_{\mathfrak{p}} < \infty, \tag{2.1}$$

where $\mathbb{K}(\mathfrak{p})$ denotes the field of fractions of $R/\mathfrak{p}$ and $M_{\mathfrak{p}}$ is the localization of M at $\mathfrak{p}$.

The dynamical systems (X, α) and (Y, β) are *algebraically conjugate* if there is a continuous isomorphism $\phi : X \to Y$ such that $\phi \cdot \alpha = \beta \cdot \phi$. In terms of dual modules, such a conjugacy is manifested by a module isomorphism. Unless otherwise stated, *conjugacy* always means algebraic conjugacy in this paper. Further background for algebraic dynamical systems can be found in Schmidt's monograph [**15**].

Throughout, a *global field* is either a finite extension of $\mathbb{Q}$ or a finite extension of a function field of the form $\mathbb{F}_p(t)$. If $\mathfrak{p} \subset R$ is a principal prime ideal, $\mathbb{K}(\mathfrak{p})$ is a global field. The *places* $\mathcal{P}(K)$ of a global field K are the equivalence classes of absolute values on K. When $\mathrm{char}(K) = 0$, the *infinite* places are the archimedean ones. When $\mathrm{char}(K) > 0$, all places are non-archimedean and the *infinite* places comprise the extensions of the infinite place of $\mathbb{F}_p(t)$, given by $|f/g|_\infty = p^{\deg(f)-\deg(g)}$. All other places are said to be *finite*. Given a finite place of K, there corresponds a unique discrete valuation v whose precise value group is $\mathbb{Z}$. The corresponding normalized absolute value is,

$$|\cdot|_v = |\mathfrak{K}_v|^{-v(\cdot)},$$

where $\mathfrak{K}_v$ is the (necessarily finite) residue class field of v. Further background and terminology concerning global fields and their places can be found in [**5**].

Finally, it should be noted that by a *fractional ideal* I of a domain D with fraction field K, we mean a fractional ideal in the sense of Eisenbud [**6**, Sec. 11.3]. That is, I is a D-submodule of K. In particular, I is not necessarily finitely generated over D.

3. *Dimension One Groups*

If X is a solenoid of dimension one, the dual group $M = \widehat{X}$ may be identified with a subgroup of $\mathbb{Q}$. Following [**1**], the subgroups of $\mathbb{Q}$ can be described as follows. Let Π be the set of rational primes, $p \in \Pi$ and let M be a subgroup of $\mathbb{Q}$. The *p-height*, $h_p^M(a) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ of an element $a \in M$, is the largest integer $B \geqslant 0$ such that the equation $p^B x = a$ has a solution $x \in M$. If no such B exists, set $h_p^M(a) = \infty$. The *height sequence* corresponding to a is $(h_p^M(a))$, where p runs through the primes Π in their natural order. Two height sequences are *equivalent* if they differ in only finitely many coordinates and whenever infinite p-height is witnessed in one sequence it is also witnessed in the other. A subgroup of $\mathbb{Q}$ has the property that the height sequences of all its non-zero elements lie in the same equivalence class. Furthermore, the isomorphism classes of these subgroups are in one-to-one correspondence with equivalence classes of height sequences, called *types*. Given the definition of types, it makes sense to say that a subgroup has *infinite p-height* if it contains an element with this property.

An ergodic epimorphism α of a one-dimensional solenoid dualizes to a monomorphism of a subgroup $M \subset \mathbb{Q}$. Without loss of generality, M may be considered as a fractional ideal of a domain $D = \mathbb{Z}[\xi]$, where $\xi \in \mathbb{Q}\backslash\{-1, 1, 0\}$ and $\widehat{\alpha}$ is simply multiplication by ξ on M. To see this, note that M is an R-module with a single associated prime generated by a linear polynomial f and $D \cong R/(f)$ is isomorphic to a subring of $\mathbb{Q}$. Consequently, there is an induced embedding $D \hookrightarrow M \hookrightarrow \mathbb{Q}$. To calculate the number of points of period j for α, a straightforward duality argument (*cf.* Lemma 4.1) shows,

$$|F_j(\alpha)| = |M/(\xi^j - 1)M|.$$

Let $\Pi(D)$ be the set of rational primes which generate prime ideals in D and let $p \in \Pi(D)$. Since $M/(\xi^j - 1)M$ is finite, it has a composition series over D. To obtain associated primes and multiplicities, localize at each prime ideal generated by $p \in \Pi(D)$ and consider the dimension of $M_{(p)}/(\xi^j - 1)M_{(p)}$ as a vector space over the field $D_{(p)}/(p) = \mathbb{F}_p$. If $h_p^M(1) = \infty$ then $M_{(p)} = \mathbb{Q}$ and this dimension is zero. Therefore, it is only necessary to consider those $p \in \Pi(D)$ with $h_p^M(1) < \infty$. Let the set of such primes be P. If $p \in P$ then $M_{(p)} \neq \mathbb{Q}$ and it follows that $M_{(p)} \cong D_{(p)}$. Hence, $\dim_{\mathbb{F}_p} M_{(p)}/(\xi^j - 1)M_{(p)}$ is simply $v_p(\xi^j - 1)$, where v_p is the p-adic valuation. Thus, the following holds.

Theorem 3.1. *Let α be an ergodic epimorphism of a one-dimensional solenoid X. Then $M = \widehat{X}$ may be regarded as a fractional ideal of a domain $\mathbb{Z}[\xi] \subset \mathbb{Q}$. The periodic point counts* *for α depend only on the element $\xi \in \mathbb{Q}$ and the set P consisting of those primes $p \in \mathbb{Z}[\xi]$ for which M does not have infinite p-height. Moreover,*

(i) *For any $j \in \mathbb{N}$,*

$$|F_j(\alpha)| = \prod_{p \in P} |\xi^j - 1|_p^{-1}.$$

(ii) *If $|\xi|_p \neq 1$ for all $p \in P$, then there is exactly one point of every period.*

(iii) *If P is infinite, then there are uncountably many non-conjugate epimorphisms of one-dimensional solenoids with identical periodic point data to that of α.*

Proof. The formula (i) is evident from the above discussion, noting that $P = \varnothing$ implies $M = \mathbb{Q}$; the resulting empty product reflecting the fact that 0_X is the only fixed point of α^j, $j \in \mathbb{N}$.

To prove (ii), for a non-trivial case assume $P \neq \varnothing$. Let $p \in P$. If $|\xi|_p \neq 1$, since $h_p^M(\xi) < \infty$ and since $\mathbb{Z}[\xi]$ is a ring, it follows that $|\xi|_p < 1$. By the ultrametric inequality, this means $|\xi^j - 1|_p = 1$ for all $j \in \mathbb{N}$.

For (iii), first note that since P is countably infinite there are uncountably many distinct sequences (a_p) with $0 \leqslant a_p < \infty$ for $p \in P$ and $a_p = \infty$ for $p \in \Pi \setminus P$. Let the collection of all such sequences be Λ. Without loss of generality, we may assume that the type to which $(a_p) \in \Lambda$ belongs corresponds to a fractional ideal of $\mathbb{Z}[\xi]$. Since each type in $\mathbb{Q}$ contains only countably many height sequences, it follows that Λ must split into uncountably many types $\widetilde{\Lambda}$. Finally, since the value of $|F_j(\alpha)|$ depends only on P and the divisibility properties of $\xi^j - 1$ with respect to P, $\widetilde{\Lambda}$ indexes a set of distinct fractional ideals of $\mathbb{Z}[\xi]$ and these generate non-conjugate dynamical systems, all of which have a common sequence of periodic point counts. □

Example 1. Let Π be the set of rational primes and M a subgroup of $\mathbb{Q}$ which contains $\mathbb{Z}$, is not a ring and has no elements of infinite p-height, $p \in \Pi$. There is an uncountable family $\{M_\lambda\}_{\lambda \in \Lambda}$ of non-isomorphic subgroups of this form. Set $X_\lambda = \widehat{M_\lambda}$ and let α_λ be the doubling map $x \mapsto 2x$ on X_λ. Then, $(X_\lambda, \alpha_\lambda)_{\lambda \in \Lambda}$ is an uncountable family of non-conjugate dynamical systems. By Theorem 3.1, for $j \in \mathbb{N}$,

$$|F_j(\alpha_\lambda)| = \prod_{p \in \Pi} |2^j - 1|_p^{-1} = |2^j - 1|_\infty = 2^j - 1,$$

where $|\cdot|_\infty$ is the infinite place of $\mathbb{Q}$. Hence, each α_λ has a sequence of periodic point counts identical to that of the circle doubling map.

4. *Proof of Theorem 1.1*

We will repeatedly apply the following straightforward consequence of Pontryagin duality to count periodic points.

Lemma 4.1. *Let N be an R-module. If either $|F_j(\alpha_N)|$ or $|N/(t^j - 1)N|$ is finite, then these quantities are equal.*

Proof. See for example [**11**, Lem. 7.2]. □

The following result is also needed.

Lemma 4.2. *Let $L \subset N$ be R-modules and $g \in R$.*

(i)

$$\left|\frac{N}{gN}\right| = \left|\frac{N/L}{g(N/L)}\right|\left|\frac{L}{L\cap gN}\right|$$

(ii) *If N/L is finite and the map $x\mapsto gx$ is a monomorphism of N then $|N/gN| = |L/gL|$.*

Proof. The first part simply uses the isomorphism theorems; for example, see the proof of [**14**, Th. 3.2]. For the second part, first note that since N/L is finite all its associated primes are maximal. Furthermore, there is a prime filtration of N/L which lifts to a finite chain of R-modules,

$$L = L_0 \subset L_1 \subset \cdots \subset L_n = N,$$

with the property that $L_i/L_{i-1} \cong R/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i \subset R$, $1 \leqslant i \leqslant n$. Hence, it is sufficient to prove the claim assuming $N/L \cong R/\mathfrak{m}$, for a maximal ideal $\mathfrak{m} \subset R$; the more general case follows by induction.

Since $x \mapsto gx$ is injective, a straightforward argument (see for example [**14**, Lem. 3.4]) shows

$$g \notin \mathfrak{m} \implies gN\cap L = gL \text{ and } N/L \cong g(N/L),$$

and

$$g \in \mathfrak{m} \implies gN\cap L = gN \text{ and } N/L \cong gN/gL.$$

By substitution into the equation established in the first part of the lemma, the required result is immediate for the case $g \notin \mathfrak{m}$. If $g \in \mathfrak{m}$, upon noting $g(N/L) = \{0\}$, the first part of the lemma gives,

$$|N/gN| = |N/L|\left|\frac{L/gL}{gN/gL}\right| = |L/gL|.$$

□

Our first application of Lemma 4.2 is to prove the following. Note that as a consequence, $|M/(t^j-1)M| < \infty$, for all $j \in \mathbb{N}$.

LEMMA 4.3. *Let N be an R-module for which $\mathrm{Ass}(N)$ consists of finitely many non-trivial principal ideals and suppose $m(\mathfrak{p}) < \infty$ for each $\mathfrak{p} \in \mathrm{Ass}(N)$, where $m(\mathfrak{p})$ is given by (2.1). If $g \in R$ is such that the map $x \mapsto gx$ is a monomorphism of N, then N/gN is finite.*

Proof. The conditions on N mean there is a Noetherian submodule $L \subset N$ such that $\mathrm{Ass}(N/L)$ consists entirely of maximal ideals and $\mathrm{Ass}(L) = \mathrm{Ass}(N)$. A straightforward induction using Lemma 4.2 shows that L/gL is finite. Furthermore, there is a chain of R-modules

$$L = L_0 \subset L_1 \subset L_2 \subset \cdots$$

such that for all $i \geqslant 1$, L_i/L_{i-1} is finite and

$$\varinjlim L_i/(gN\cap L_i) \cong N/gN.$$

However, by Lemma 4.2(ii),

$$\left|\frac{L_i}{gN\cap L_i}\right| \leqslant |L_i/gL_i| = |L/gL|,$$

for all $i \geqslant 1$. Therefore, N/gN must be finite. □

The multiplicative set

$$U = \bigcap_{\mathfrak{p}\in\mathrm{Ass}(M)} R\setminus\mathfrak{p}$$

has $U \cap \mathrm{ann}(a) = \varnothing$ for all non-zero $a \in M$, so the natural map $M \to U^{-1}M$ is a monomorphism. Identifying localizations of R with subrings of $\mathbb{Q}(t)$, the domain

$$\mathfrak{R} = U^{-1}R = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(M)} R_{\mathfrak{p}}$$

is a finite intersection of discrete valuation rings and is therefore a principal ideal domain by [**12**, Th. 12.2]. The assumptions of finite entropy and finite topological dimension force $U^{-1}M$ to be a Noetherian $\mathfrak{R}$-module (see for example [**13**, Exs. 3.1 and 3.2]). Hence, there is a prime filtration

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = U^{-1}M, \tag{4.1}$$

in which $M_i/M_{i-1} \cong \mathfrak{R}/\mathfrak{q}_i$ for non-trivial primes $\mathfrak{q}_i \subset \mathfrak{R}$, $1 \leqslant i \leqslant n$. Moreover, $\mathfrak{p}_i = \mathfrak{q}_i \cap R \in \mathrm{Ass}(M)$ for all $1 \leqslant i \leqslant n$. Identifying M with its image in $U^{-1}M$ and intersecting the chain (4.1) with M gives a chain

$$\{0\} = L_0 \subset L_1 \subset \cdots \subset L_n = M. \tag{4.2}$$

Considering (4.2) as a chain of R-modules, for each $1 \leqslant i \leqslant n$ there is an induced inclusion

$$\frac{L_i}{L_{i-1}} \hookrightarrow \frac{M_i}{M_{i-1}} \cong \frac{\mathfrak{R}}{\mathfrak{q}_i} \cong K_i = \mathbb{K}(\mathfrak{p}_i) \tag{4.3}$$

and $N_i = L_i/L_{i-1}$ may be considered as a fractional ideal of $E_i = R/\mathfrak{p}_i$.

Using Lemma 4.2(i),

$$\left|\frac{L_i}{gL_i}\right| = \left|\frac{N_i}{gN_i}\right| \left|\frac{L_{i-1}}{L_{i-1} \cap gL_i}\right|,$$

where $g = t^j - 1$ and $1 \leqslant i \leqslant n$. Let $y \in L_i$, let η denote the image of y in N_i and let ξ_i denote the image of t in E_i. If $gy \in L_{i-1}$ then $(\xi_i^j - 1)\eta = 0$. The ergodicity assumption implies $g \notin \mathfrak{p}_i$, so $\xi_i^j - 1 \neq 0$. Therefore, $\eta = 0$ and $y \in L_{i-1}$. It follows that $L_{i-1} \cap gL_i = gL_{i-1}$ and hence,

$$\left|\frac{L_i}{gL_i}\right| = \left|\frac{N_i}{gN_i}\right| \left|\frac{L_{i-1}}{gL_{i-1}}\right|. \tag{4.4}$$

Successively applying (4.4) to each of the modules L_i, $1 \leqslant i \leqslant n$, gives,

$$|M/gM| = \prod_{i=1}^{n} |N_i/gN_i|.$$

We now turn our attention to an individual term $|N_i/gN_i|$. For ease of notation fix i; let $N = N_i$, $E = E_i$, $K = K_i$ and $\xi = \xi_i$. When $\mathrm{char}(E) > 0$, $E \cong \mathbb{F}_p[t]$ for some rational prime p, so E is a Dedekind domain; this need not be the case if $\mathrm{char}(E) = 0$. Since E is a finitely generated domain, [**6**, Th. 4.14] shows that the integral closure, D say, of E in K is a finitely generated Dedekind domain. Therefore, D is a finitely generated integral E-algebra and hence is also finitely generated as an E-module. Using standard identifications, we may consider $I = D \otimes_E N$ as a fractional ideal of D. Since D is finitely generated as a module over E, by $a_1, \ldots, a_r \in D$ say, $I = \sum_{k=1}^{r} a_k N$. Since $E \subset D \subset K$, there exists a non-zero $z \in D$ such that $za_k \in E$, for all $k = 1, \ldots, r$. Hence $zI \subset N$. By Lemma 4.3, since multiplication by z is a monomorphism of I, I/zI is finite and this forces I/N to be finite as it is a homomorphic image of I/zI. Therefore, Lemma 4.2(ii) shows that

$$|N/(\xi^j - 1)N| = |I/(\xi^j - 1)I|.$$

By considering $I/(\xi^j - 1)I$ as a D-module, finding a composition series for this module and successively localizing at each of its associated primes to obtain multiplicities, it follows that

$$|I/(\xi^j - 1)I| = \prod_{\mathfrak{m} \in \mathrm{Ass}(I/(\xi^j - 1)I)} q_{\mathfrak{m}}^{\delta_{\mathfrak{m}}(\xi, I)}, \tag{4.5}$$

where $q_{\mathfrak{m}} = |D/\mathfrak{m}|$ and

$$\delta_{\mathfrak{m}}(\xi, I) = \dim_{D/\mathfrak{m}}(I/(\xi^j - 1)I)_{\mathfrak{m}}.$$

Let $P = \{\mathfrak{m} \in \operatorname{Spec}(D) : I_{\mathfrak{m}} \neq K\}$. If $\mathfrak{m}$ is a prime of D such that $I_{\mathfrak{m}} = K$,

$$(I/(\xi^j - 1)I)_{\mathfrak{m}} \cong I_{\mathfrak{m}}/(\xi^j - 1)I_{\mathfrak{m}} = \{0\}.$$

Since D has Krull dimension 1, [**12**, Th. 6.5] shows that the set of prime ideals $\mathfrak{m} \subset R$ for which $(I/(\xi^j - 1)I)_{\mathfrak{m}} \neq \{0\}$ coincides with the elements of $\operatorname{Ass}(I/(\xi^j - 1)I)$, which is therefore a subset of P. This also means that the product (4.5) may be taken over all $\mathfrak{m} \in P$ to yield the same result. Note that each localization $D_{\mathfrak{m}}$ is a distinct valuation ring of K and P may be identified with a set of finite places of the global field K. Furthermore, every element of K may be written in the form $u\pi^k$, where u is a unit of $D_{\mathfrak{m}}$, $k \in \mathbb{Z}$ and π is a uniformizer for $D_{\mathfrak{m}}$. Hence, if $I_{\mathfrak{m}} \neq K$ then there exists $B \in \mathbb{N}$ such that $\pi^{-B} \in I_{\mathfrak{m}}$ and $v_{\mathfrak{m}}(x) \geqslant -B$ for all $x \in I_{\mathfrak{m}}$. But then $I_{\mathfrak{m}} = D_{\mathfrak{m}}\pi^{-B}$ which is isomorphic to $D_{\mathfrak{m}}$ as a fractional ideal of D. Hence,

$$|I/(\xi^j - 1)I| = \prod_{\mathfrak{m} \in P} q_{\mathfrak{m}}^{\delta_{\mathfrak{m}}(\xi, D)}. \tag{4.6}$$

Finally, since $\delta_{\mathfrak{m}}(\xi, D) = v_{\mathfrak{m}}(\xi^j - 1)$, each factor in the product (4.6) may be replaced by $|\xi^j - 1|_{\mathfrak{m}}^{-1}$, where $|\cdot|_{\mathfrak{m}}$ is the normalized absolute value arising from $D_{\mathfrak{m}}$. This concludes the proof.

Remark 1. Continuing with the notation used above, the set of places $P \subset \mathcal{P}(K)$ may also be obtained directly from the fractional ideal N of the domain E, which is not necessarily a Dedekind domain (the above proof would be simpler if E was always Dedekind). In fact P is identical to

$$Q = \{v \in \mathcal{P}(K) : |N|_v \text{ is a bounded subset of } \mathbb{R}\}.$$

Let T be the set of places which define the integral closure D of E in K. Since $E \subset N$, $Q \subset T$ and $|D|_v \subset [0, 1]$ for all $v \in Q$.

Let $\mathfrak{m}(v) = \{x \in K : |x|_v < 1\}$ be the maximal ideal of D which corresponds to a place $v \in T$. It will follow that $Q = P$ if $I_{\mathfrak{m}(v)} \neq K$ precisely when $|N|_v$ is a bounded subset of $\mathbb{R}$. If $I_{\mathfrak{m}(v)} \neq K$ then $|I|_v$ is bounded and hence so is $|N|_v$. Conversely, if $|N|_v$ is bounded, since $|D|_v \subset [0, 1]$ and since every element of I may be written in the form $\sum_{k=1}^{r} a_k x_k$, where $x_k \in N$ and $a_1, \ldots, a_r$ are as in the above proof, by the ultrametric inequality, $|I|_v$ is also bounded. Hence $I_{\mathfrak{m}(v)} \neq K$.

5. *Concluding Remarks and Examples*

To retrieve the periodic point counting formula for S-integer dynamical systems given by [**4**, Lem. 5.2] from the formula (1.1), one simply applies the product formula for global fields. A significant point of note is that in [**4**], the periodic point formula is obtained using an adelic description of X and Haar measures. The method used here is necessarily different as there is no such convenient description of X. It is also important to note that only the final paragraph of the proof of Theorem 1.1 is required for the dynamical systems covered in [**4**].

As Example 1 illustrates, even for one-dimensional solenoids there can be many dynamical systems which are not conjugate to any S-integer dynamical system, but which have a common sequence of periodic point counts given by one. An even more fundamental example is the following.

Example 2. Let α be the ergodic epimorphism of the torus $X = \mathbb{T}^2$ given by the matrix,

$$A = \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}.$$

The system (X, α) fails to be an S-integer system because it fails to satisfy [**4**, Lem. 2.1] (dually, $M \cong R/(t^2 - 5)$ and this is not a Dedekind domain). However, in the language of Theorem 1.1,

$$|F_j(\alpha)| = \prod_{v \in P} |(\sqrt{5})^j - 1|_v^{-1},$$

where P is the set of all finite places of $\mathbb{Q}(\sqrt{5})$. Hence, via the product formula for global fields,

$$|F_j(\alpha)| = |(\sqrt{5})^j - 1||(-\sqrt{5})^j - 1|;$$

this being the more familiar form for counting periodic points of toral endomorphisms, using the eigenvalues of A.

The following example illustrates a two-dimensional system which cannot be expressed as a direct product of two one-dimensional systems, yet its sequence of periodic point counts decomposes via a short exact sequence involving two such systems.

Example 3. The following construction is introduced in [**8**, Sec. 88, Ex. 2] as an example of an indecomposable group. Consider the rank two subgroup of $\mathbb{Q}^2$ given by

$$M = D + E + \langle (\tfrac{1}{5}, \tfrac{1}{5}) \rangle,$$

where $D = \langle (2^{-k}, 0) : k \in \mathbb{N} \rangle \cong \mathbb{Z}[\frac{1}{2}]$ and $E = \langle (0, 3^{-k}) : k \in \mathbb{N} \rangle \cong \mathbb{Z}[\frac{1}{3}]$. Then, M cannot be expressed as a direct product of two rank one groups. Consider the doubling map $\alpha : x \mapsto 2x$ on $X = \widehat{M}$. Following the method of Theorem 1.1, the R-module chain (4.2) comprises the first three terms in the short exact sequence of R-modules

$$\{0\} \to D \to M \to M/D \to \{0\}.$$

Subsequently, the periodic point formula for (X, α) is established by finding the rational primes p for which the rank one groups D and M/D have infinite p-height. These primes are 2 and 3 respectively and, applying the product formula for global fields to (1.1), we obtain

$$|F_j(\alpha)| = |2^j - 1|_\infty^2 |2^j - 1|_2 |2^j - 1|_3 = |2^j - 1|_\infty^2 |2^j - 1|_3. \tag{5.1}$$

By duality, the submodule $D \subset M$ corresponds to the invertible extension of the circle doubling map and the submodule $E \subset M$ corresponds to the isometric extension studied in [**7**]. Furthermore, (5.1) also counts the periodic points of $\beta : x \mapsto 2x$ on $Y = \widehat{D} \times \widehat{E}$. However, there can be no conjugacy between (Y, β) and (X, α), as M is indecomposable. In fact, at best there is a finite-to-one projection of (X, α) onto (Y, β).

Finally, it is important to note that the assumptions of ergodicity, finite-dimension and finite entropy are essential in Theorem 1.1. Without these, there are numerous possibilities including systems with infinitely many points of every period. More subtly, one can produce examples with prescribed growth rates by forming infinite products of carefully chosen constituent systems, such as in [**17**].

References

1. D. M. Arnold. *Finite rank torsion free abelian groups and rings*, volume 931 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982.
2. M. Artin and B. Mazur. On periodic points. *Ann. of Math. (2)*, 81:82–99, 1965.
3. R. A. Beaumont and H. S. Zuckerman. A characterization of the subgroups of the additive rationals. *Pacific J. Math.*, 1:169–177, 1951.

4. V. Chothi, G. Everest, and T. Ward. S-integer dynamical systems: periodic points. *J. Reine Angew. Math.*, 489:99–132, 1997.
5. P. M. Cohn. *Algebraic numbers and algebraic functions*. Chapman and Hall Mathematics Series. Chapman & Hall, London, 1991.
6. D. Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.
7. G. Everest, V. Stangoe, and T. Ward. Orbit counting with an isometric direction. In *Algebraic and topological dynamics*, volume 385 of *Contemp. Math.*, pages 293–302. Amer. Math. Soc., Providence, RI, 2005.
8. L. Fuchs. *Infinite abelian groups. Vol. II*. Academic Press, New York, 1973. Pure and Applied Mathematics. Vol. 36-II.
9. A. S. Kechris. On the classification problem for rank 2 torsion-free abelian groups. *J. London Math. Soc. (2)*, 62(2):437–450, 2000.
10. B. Kitchens and K. Schmidt. Automorphisms of compact groups. *Ergodic Theory Dynam. Systems*, 9(4):691–735, 1989.
11. D. Lind, K. Schmidt, and T. Ward. Mahler measure and entropy for commuting automorphisms of compact groups. *Invent. Math.*, 101(3):593–629, 1990.
12. H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.
13. R. Miles. Homoclinic points of non-expansive automorphisms. *Aequationes Math.* To appear.
14. R. Miles. Zeta functions for elements of entropy rank one actions. *Ergodic Theory Dynam. Systems*, 27(2): 567–582, 2007.
15. K. Schmidt. *Dynamical systems of algebraic origin*, volume 128 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1995.
16. S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73:747–817, 1967.
17. T. Ward. Group automorphisms with few and with many periodic points. *Proc. Amer. Math. Soc.*, 133(1):91–96 (electronic), 2005.

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