Homoclinic points of non-expansive automorphisms

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Abstract

We study homoclinic points of non-expansive automorphisms of compact abelian groups. Connections between the existence of non-trivial homoclinic points, expansiveness, entropy and adjoint automorphisms (in the sense of Einsiedler and Schmidt) are explored. Some implications for countable abelian group actions by automorphisms of compact abelian groups are also considered and it is shown that if every element of such an action has finite entropy, there can be no non-trivial common homoclinic points, unless the action is generated by a single automorphism.

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1 Introduction

Let α be an automorphism of a compact abelian group X. Throughout, all groups are assumed to be metrizable and all automorphisms are continuous. The dynamical system (X, α) is *expansive* if there exists a neighbourhood U of 0_X such that

$$\bigcap_{n\in\mathbb{Z}} \alpha^n(U) = \{0_X\}.$$

A point $x \in X$ is a *homoclinic point* if

$$\alpha^n(x) \to 0_X \text{ as } |n| \to \infty.$$

The set $\Delta(X, \alpha)$ of all homoclinic points forms a subgroup of X. When the underlying space is clear from the context, the notation will be abbreviated to $\Delta(\alpha)$.

It is natural to extend the definition of a homoclinic point to actions of $\mathbb{Z}^d$ by automorphisms of compact abelian groups. Lind and Schmidt [10] have shown that for expansive $\mathbb{Z}^d$-actions, properties of the resulting homoclinic group are closely related to the topological entropy of the action; for example, the homoclinic group is non-trivial if and only if the action has positive entropy. Homoclinic points of expansive $\mathbb{Z}^d$-actions receive further treatment in [13], [6], [7] and [5].

Homoclinic points have also proven useful in finding symbolic representations of expansive automorphisms [9], [19]. However, some of the most fundamental non-expansive examples do not have non-trivial homoclinic points and the symbolic representation problem is much more involved in this setting [12].

The homoclinic groups of non-expansive algebraic $\mathbb{Z}^d$-actions can be more complicated than those of expansive actions, as the examples of [10, Sec. 7] illustrate. However, when $d = 1$ some of the difficulties arising in the higher rank setting do not appear. In Section 3, we consider the homoclinic groups of such automorphisms. Several results involve finite entropy automorphisms and the well-known problems associated with identifying such automorphisms (in particular Lehmer's problem [18, Sec. 19]) are avoided as a consequence of a simple algebraic constraint.

The Pontryagin dual group $\widehat{X}$ can be considered as a countable module M over the ring of Laurent polynomials $R = \mathbb{Z}[u^{\pm 1}]$ by identifying multiplication by u with the application of the dual automorphism $\widehat{\alpha}$. Conversely, a countable R-module M induces an automorphism $\alpha = \alpha_M$ of the compact abelian group $X = \widehat{M}$. The case of interest here is that of automorphisms α_M *with unfaithful modules*, that is ones for which

$$\{f \in R : fa = 0 \text{ for all } a \in M\} = \operatorname{ann}(M) \neq \{0\}.$$

Notably, automorphisms of solenoids and similar important examples fall into this class (see Examples 3.1 and 3.2). Hence, M need not be a Noetherian module.

Einsiedler and Schmidt [6] have introduced the concept of an adjoint automorphism for an expansive automorphism and the definition works equally well for any automorphism of a compact abelian group for which $\Delta(X, \alpha)$ is countable. In this case, the α-invariant subgroup $\Delta(X, \alpha)$ becomes an R-module M^*. Adopting the discrete topology on M^*, via duality a new automorphism $\alpha^* = \alpha_{M^*}$ of the compact abelian group $X^* = \widehat{M^*}$ is obtained, called *the adjoint* of α. If $\Delta(X^*, \alpha^*)$ is countable, the process can be repeated to obtain a second adjoint automorphism α^{**}. The automorphism α is said to be *reflexive* if (X, α) and (X^{**}, α^{**}) are algebraically conjugate dynamical systems. Einsiedler and Schmidt show that ergodic expansive automorphisms are reflexive and

$$\mathsf{h}(\alpha^*) = \mathsf{h}(\alpha),$$

where $\mathsf{h}(\cdot)$ denotes the topological entropy. It is assumed throughout that *ergodic* means ergodic with respect to the normalized Haar measure on X.

We summarize the results of Section 3 in the following theorem. In our setting, note that $\Delta(X, \alpha)$ is potentially uncountable without the assumption of finite entropy.

Theorem 1.1. *Suppose α_M is an automorphism of an infinite compact abelian group with unfaithful module.*

1. *There exists $f \in R$ such that*
$$\Delta(\alpha_M) \cong \operatorname{Hom}_R(M/fM, R/(f)). \tag{1}$$
2. *$\Delta(\alpha_M) \neq \{0\}$ if and only if there is an infinite index submodule $L \subset M$ such that $\alpha_{M/L}$ is expansive.*

Suppose also that $\mathsf{h}(\alpha_M) < \infty$. Then

3. *The restriction of α_M to the closure of $\Delta(\alpha_M)$ is expansive and hence $\Delta(\alpha_M)$ is countable.*
4. *The automorphism α_M is expansive if and only if for every infinite index submodule $L \subset M$, $\Delta(\alpha_{M/L}) \neq \{0\}$.*
5. *If α_M is non-expansive, α_M is not reflexive. Moreover, if α_M is also ergodic,*
$$\mathsf{h}(\alpha_M^*) < \mathsf{h}(\alpha_M). \tag{2}$$

In Section 4, some consequences for actions of countable abelian groups by automorphisms of compact abelian groups are considered. For ease of notation, α will denote the action of such a group G by automorphisms α_g, $g \in G$. By identifying certain degenerate actions of rational rank one groups (Theorem 4.3) and using the results of Section 3, the following is established. This extends a similar result for $\mathbb{Z}^2$-actions obtained by Manning and Schmidt [13].

Theorem 1.2. *Let α be an action of a countable torsion-free abelian group G by automorphisms of a compact abelian group X such that $\mathsf{h}(\alpha_g) < \infty$ for all $g \in G$. If G is not isomorphic to $\mathbb{Z}$, there are no non-trivial homoclinic points common to every non-trivial element of the action.*

2 Algebraic background

Let α be an action of a countable abelian group G by automorphisms α_g of a compact abelian group X. Let R_G denote the group ring $\mathbb{Z}[G]$. When $G = \mathbb{Z}$ and where no confusion will arise, simply write $R = R_G$ and use α to denote the automorphism that generates the action.

The Pontryagin dual group $M = \widehat{X}$ is countable, discrete and becomes an R_G-module by defining
$$fa = \sum_{g \in G} c_g \widehat{\alpha_g}(a)$$

where $a \in M$, $c_g \in \mathbb{Z}$ and $f = \sum_{g \in G} c_g g \in R_G$ has $c_g = 0$ for all but finitely many g. Conversely, for any countable abelian group G, a countable R_G-module induces a natural action of G on $X = \widehat{M}$ by setting α_g to be the automorphism dual to multiplication by g on M. This perspective translates dynamical properties into algebraic ones and is explained in detail for actions of $\mathbb{Z}^d$ in [18]. For more general acting groups, the underlying ring R_G need not be Noetherian; this can make algebraic interpretations more difficult, see for example [16], [17].

Let $\mathfrak{R}$ be a commutative ring. The *height* $\mathrm{ht}(\mathfrak{p})$ of a prime ideal $\mathfrak{p} \subset \mathfrak{R}$ is the maximal length r of a strictly decreasing chain of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r.$$

The *Krull dimension* of $\mathfrak{R}$, denoted $\mathrm{kdim}(\mathfrak{R})$, is the supremum of the lengths r taken over all such chains of prime ideals in $\mathfrak{R}$. For example, $R = \mathbb{Z}[u^{\pm 1}]$ has $\mathrm{kdim}(R) = 2$. The height one primes are generated by the irreducible (possibly constant) polynomials of R. Any height two prime $\mathfrak{p}$ is maximal, non-principal, contains a rational prime and $R/\mathfrak{p}$ is a finite field.

Now suppose $\mathfrak{R}$ is Noetherian and let M be an $\mathfrak{R}$-module. A prime ideal $\mathfrak{p} \subset \mathfrak{R}$ is *associated* with M if it is the annihilator of an element of M. That is, there exists $a \in M$ such that

$$\{f \in \mathfrak{R} : fa = 0\} = \mathrm{ann}(a) = \mathfrak{p}.$$

Denote the set of associated primes by $\mathrm{Ass}(M)$ and the subset of height one primes of $\mathrm{Ass}(M)$ by $\mathrm{Ass}^1(M)$.

A Noetherian $\mathfrak{R}$-module M has a finite set of associated primes and admits a *prime filtration*, that is a chain of submodules

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_r = M, \tag{3}$$

where for each $1 \leq i \leq r$, we have $M_i/M_{i-1} \cong \mathfrak{R}/\mathfrak{p}_i$ for a prime ideal $\mathfrak{p}_i \subset \mathfrak{R}$ which contains an associated prime or is equal to an associated prime of M. All elements of $\mathrm{Ass}(M)$ always appear in such a filtration. Prime filtrations need not be unique and this can sometimes be useful.

Lemma 2.1. *Let $R = \mathbb{Z}[u^{\pm 1}]$, let M be an unfaithful Noetherian R-module and suppose $P \subset \mathrm{Ass}^1(M)$ is non-empty. Then there is a prime filtration of the form (3) and $1 \leq j \leq r$ such that $\{\mathfrak{p}_1, \ldots \mathfrak{p}_j\} = \mathrm{Ass}^1(M_j) = P$ and $\mathrm{Ass}^1(M/M_j) = \mathrm{Ass}^1(M) \setminus P$.*

Proof. If $L \subset M$ is a submodule, $\mathrm{Ass}^1(M/L) \subset \mathrm{Ass}^1(M)$. For, if $\mathrm{ann}(a+L) = \mathfrak{p}$ has $\mathrm{ht}(\mathfrak{p}) = 1$, $\mathfrak{p}$ is a minimal prime containing $\mathrm{ann}(a) \neq \{0\}$ and so is contained in $\mathrm{Ass}^1(Ra) \subset \mathrm{Ass}^1(M)$ (see [14, Th. 6.5]).

Choose a submodule $M_1 \subset M$ such that $M_1 \cong R/\mathfrak{p}_1$ for some $\mathfrak{p}_1 \in P$. If $\mathrm{Ass}^1(M/M_1) \cap P \neq \varnothing$, there is a submodule $M_2 \subset M$ containing M_1 such that $M_2/M_1 \cong R/\mathfrak{p}_2$ for some $\mathfrak{p}_2 \in P$. If there is no such $\mathfrak{p}_2$ then $\mathrm{Ass}^1(M/L) \cap P = \varnothing$ for all submodules $L \subset M$ containing M_1 and we follow the usual algorithm for obtaining a prime filtration. By choosing primes from P for as long as possible, the desired form of filtration is obtained, j being minimal with respect to the property, $\mathrm{Ass}^1(M/M_j) \cap P = \varnothing$. The process terminates because M is Noetherian. ☐

For automorphisms of compact abelian groups, there is a useful entropy addition formula due to Yuzvinskiĭ [20]. For convenience, the formula may be phrased in terms of countable R-modules $L \subset M$, as follows.

$$\mathsf{h}(\alpha_M) = \mathsf{h}(\alpha_L) + \mathsf{h}(\alpha_{M/L}). \tag{4}$$

Let α and β be actions of a countable abelian group G by automorphisms of compact abelian groups X and Y respectively. Then β is an *algebraic factor* of α if there is a continuous surjective homomorphism $\phi : X \to Y$ such that

$$\phi \cdot \alpha_g = \beta_g \cdot \phi,$$

for all $g \in G$. In terms of R_G-modules $M = \widehat{X}$ and $L = \widehat{Y}$, via duality, this means L may be regarded as a submodule of M, $\widehat{\beta}$ identifying with the appropriate restriction of $\widehat{\alpha}$. Conversely, given R_G-modules $L \subset M$, α_L is an algebraic factor of α_M. The actions α and β are *algebraically conjugate* if ϕ is an isomorphism.

Finally, the following algebraic interpretation of ergodicity will be helpful.

Lemma 2.2. *Let α be an automorphism of a compact abelian group and let M be the dual R-module. Then the following conditions are equivalent.*

1. *α is ergodic.*
2. *For each non-zero $a \in M$, $u^n - 1 \notin \mathrm{ann}(a)$ for all $n \in \mathbb{Z}^\times$.*
3. *For each $\mathfrak{p} \in \mathrm{Ass}(M)$, $u^n - 1 \notin \mathfrak{p}$, for all $n \in \mathbb{Z}^\times$.*

Proof. See [18, Prop. 6.6]. ☐

3 Homoclinic groups of single automorphisms

We will be interested in automorphisms arising from unfaithful R-modules. This algebraic constraint is satisfied by a range of well known dynamical systems, such as the following.

Example 3.1 (*Solenoidal automorphisms*)**.** Suppose that α is an automorphism of a solenoid X; that is, a connected group of finite topological dimension. Since X has finite topological dimension, the dual module $M = \widehat{X}$ has finite rational rank and since X is connected, M is a torsion-free abelian group. Therefore, there is an embedding of M into the localization $M' = \mathcal{U}^{-1}M$ of M at $\mathcal{U} = \mathbb{Z}^\times$ and M' is a finite-dimensional vector space over $\mathbb{Q}$. Furthermore, M' naturally becomes a Noetherian module over the principal ideal domain $\mathfrak{R} = \mathcal{U}^{-1}R = \mathbb{Q}[u^{\pm 1}]$. Hence, $\mathrm{Ass}(M)$ may be obtained from $\mathrm{Ass}_{\mathfrak{R}}(M')$ by contracting these primes to R. Since M' is a Noetherian $\mathfrak{R}$-module, $\mathrm{Ass}_{\mathfrak{R}}(M') = \{(f_1), \dots, (f_r)\}$ is finite and does not contain (0) as $r = \dim_{\mathbb{Q}}(M') < \infty$. Moreover, there is a non-zero integer k such that $k(f_1 f_2 \cdots f_r)^r \in \mathrm{ann}(M)$ and so M is an unfaithful R-module.

Example 3.2 (*Ergodic, finite entropy automorphisms of groups having finite topological dimension*)**.** Suppose X is a compact abelian group of finite (possibly zero) topological dimension and let α be an automorphism of X with $\mathsf{h}(\alpha) < \infty$. Let $M = \widehat{X}$ be the dual module and $L \subset M$ the submodule of elements $a \in M$ with $\mathrm{ann}(a) \cap \mathbb{Z}^{\times} \neq \varnothing$. Then M/L is a torsion-free abelian group and $\widehat{M/L}$ identifies with the connected component of 0 in X. Therefore, the previous example shows that there exists a non-zero $g \in R$ such that $gM \subset L$.

Now consider the torsion abelian group L. As α is ergodic, Lemma 2.2 implies $\mathrm{Ass}(L) = \mathrm{Ass}^1(L)$, each prime ideal of which must be generated by a rational prime. The set $\mathrm{Ass}(L)$ is finite, otherwise L contains a chain of submodules

$$L_1 \subset L_2 \subset \cdots, \tag{5}$$

with $\mathsf{h}(\alpha_{L_{i+1}/L_i}) \geq \log 2$, $i \geq 1$, contradicting the finite entropy assumption by (4). If $\mathcal{U} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(L)} R \setminus \mathfrak{p}$, $\mathrm{ann}(a) \cap \mathcal{U} = \varnothing$ for all $a \in L$ and the natural map $L \to L' = \mathcal{U}^{-1}L$ is therefore injective. Furthermore, L' can be considered as a module over

$$\mathfrak{R} = \mathcal{U}^{-1}R = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(L)} R_{\mathfrak{p}} \subset \mathbb{Q}(u),$$

which is a finite intersection of discrete valuation rings and is hence a principal ideal domain [14, Th. 12.2]. As an $\mathfrak{R}$-module, L' is Noetherian. For, $(0) \notin \mathrm{Ass}_{\mathfrak{R}}(L')$ and L' may be written as a limit $\lim_{\to} L'_i$ of Noetherian submodules

$$L'_1 \subset L'_2 \subset \cdots, \tag{6}$$

where $L'_{i+1}/L'_i \cong \mathfrak{R}/\mathfrak{q}_i$ for $\mathfrak{q}_i \in \mathrm{Ass}_{\mathfrak{R}}(L')$, $i \geq 1$. Intersecting (6) with L, one obtains a chain of the form (5), where $L_i = L'_i \cap L$, $i \geq 1$. Subsequently, for each $i \geq 1$, there are embeddings

$$R/\mathfrak{p}_i \hookrightarrow L_{i+1}/L_i \hookrightarrow L'_{i+1}/L'_i,$$

where $\mathfrak{p}_i = \mathfrak{q}_i \cap R \in \mathrm{Ass}_R(L)$. As above, the finite entropy assumption means (5) must terminate; hence so too must (6). As L' is a Noetherian $\mathfrak{R}$-module and $(0) \notin \mathrm{Ass}_{\mathfrak{R}}(L')$, there exists a non-zero $h' \in \mathfrak{R}$ with $h'L' = \{0\}$. Since $\mathfrak{R} \subset \mathbb{Q}(u)$, there is a non-zero $h \in R$ with $f = hh' \in R$. Thus, $fg \in \mathrm{ann}(M)$ and M is an unfaithful R-module.

For a simple example of a non-expansive automorphism with a non-trivial homoclinic group, consider the following.

Example 3.3. Let σ be the shift map $(x_m) \mapsto (x_{m+1})$ on $\mathbb{T}^{\mathbb{Z}}$ and let

$$X = \{(x_m) \in \mathbb{T}^{\mathbb{Z}} : 2x_{m+1} - 2x_m = 0_{\mathbb{T}}\},$$

which is a σ-invariant subgroup of $\mathbb{T}^{\mathbb{Z}}$. Then any point of the form

$$(\dots, 0, 0, \tfrac{1}{2}, 0, 0, \dots) \in X$$

is a homoclinic point. Finite sums of translates of points of this form are also homoclinic. Note that $\sigma|_X$ is non-expansive because for any $\varepsilon \in \mathbb{T}$, X contains a point of the form (y_m), where $y_m = \varepsilon$ for all $m \in \mathbb{Z}$.

To study homoclinic groups of more general non-expansive automorphisms, the following results will be useful.

Lemma 3.4. *Let M be a countable R-module and $L \subset M$ a submodule.*

1. *If $\Delta(\alpha_L) = \{0\}$ then $\Delta(\alpha_M) = \Delta(\alpha_{M/L})$.*
2. *If $\Delta(\alpha_{M/L}) = \{0\}$, the natural projection $\widehat{M} \to \widehat{L}$ is injective on $\Delta(\alpha_M)$.*
3. *If $\Delta(\alpha_L) = \{0\}$ for all cyclic submodules L then $\Delta(\alpha_M) = \{0\}$.*

Proof. Let $X = \widehat{M}$, $Y = L^{\perp}$ and note $\widehat{L} \cong X/Y$ and $\widehat{M/L} \cong Y$.

To obtain (1), simply note that if $w \in X \setminus Y$ is a homoclinic point then the image of w in X/Y is a non-trivial homoclinic point of the induced factor α_L, since the natural projection $X \to X/Y$ is continuous.

To obtain (2), suppose $v, w \in \Delta(\alpha_M) \subset X$ and let $\overline{v}, \overline{w}$ denote the respective images in X/Y. If $\overline{v} = \overline{w}$ then there exists $y \in Y$ such that $v - w = y$. Hence, $y \in \Delta(\alpha_{M/L})$ which implies $y = 0$. Therefore, the natural projection $X \to X/Y$ is injective on $\Delta(\alpha_M)$.

For (3), consider $N = \bigoplus_{a \in M} Ra$. The natural projection $N \to M$ given by $\oplus_{a \in M} f_a a \mapsto \sum_{a \in M} f_a a$, where $f_a \in R$ is non-zero for only finitely many $a \in M$, dualizes so that $\widehat{M}$ may be regarded as a closed α_N-invariant subgroup of $\widehat{N}$ and α_M identifies with the restriction of α_N. Furthermore, $\Delta(\alpha_N) = \{0\}$ and so $\Delta(\alpha_M) = \{0\}$. $\square$

The property of expansiveness is now considered in more detail.

Definition 3.5. An automorphism α of an infinite compact abelian group X is *totally non-expansive* if the only expansive algebraic factors of (X, α) are finite.

Expansiveness and total non-expansiveness are reflected algebraically as follows. Let $\mathbb{S} \subset \mathbb{C}$ denote the unit circle and for an ideal $\mathfrak{a} \subset R$, set

$$V_{\mathbb{C}}(\mathfrak{a}) = \{z \in \mathbb{C}^{\times} : f(z) = 0 \text{ for all } f \in \mathfrak{a}\}.$$

Note that a Noetherian R-module M with a filtration of the form (3) is finite if and only if for $1 \leq i \leq r$, each prime $\mathfrak{p}_i$ is maximal.

Lemma 3.6. *Let α_M be an automorphism of an infinite compact abelian group. Then*

1. *α_M is expansive if and only if M is a Noetherian R-module satisfying $V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{S} = \varnothing$ for all $\mathfrak{p} \in \mathrm{Ass}^1(M)$.*

2. α_M is totally non-expansive if and only if $V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{S} \neq \varnothing$ for all $\mathfrak{p} \in \operatorname{Ass}^1(M)$.

Proof. Upon noting that $V_{\mathbb{C}}(\mathfrak{p}) = \varnothing$ when $\operatorname{ht}(\mathfrak{p}) = 2$, (1) follows from [18, Th. 6.5].

For (2), first suppose α_M is totally non-expansive. For any $\mathfrak{p} \in \operatorname{Ass}^1(M)$ there is a submodule of M isomorphic to $R/\mathfrak{p}$ and this is infinite as $\operatorname{ht}(\mathfrak{p}) = 1$. By duality, $\alpha_{R/\mathfrak{p}}$ is a non-expansive factor of α_M and hence $V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{S} \neq \varnothing$ by part (1). Now suppose $V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{S} \neq \varnothing$ for all $\mathfrak{p} \in \operatorname{Ass}^1(M)$. If α_L is a factor of α_M, α_L is induced by a submodule $L \subset M$. Since $\operatorname{Ass}^1(L) \subset \operatorname{Ass}^1(M)$, the conditions for expansiveness in part (1) will be violated unless L is Noetherian and $\operatorname{Ass}^1(L) = \varnothing$. But then both L and $\widehat{L}$ are finite. □

Remarks 3.7.

1. Lemma 19.1 of [18] shows that the only principal prime ideals $\mathfrak{p} \subset R$ with $\mathsf{h}(\alpha_{R/\mathfrak{p}}) = 0$ are those generated by cyclotomic polynomials. Hence, if $\mathfrak{p} \subset R$ is a principal prime ideal such that $\alpha_{R/\mathfrak{p}}$ is expansive, $\mathsf{h}(\alpha_{R/\mathfrak{p}}) > 0$.

2. If an automorphism α_M of an infinite group has zero entropy then every $\mathfrak{p} \in \operatorname{Ass}^1(M)$ satisfies $\mathsf{h}(\alpha_{R/\mathfrak{p}}) = 0$ and hence α_M is totally non-expansive.

3. An expansive automorphism α_M has an unfaithful module M, because Lemma 3.6(1) implies $(0) \notin \operatorname{Ass}(M)$. So, $\mathfrak{a} = \cap_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$ is non-trivial and suitable powers of elements of $\mathfrak{a}$ lie in $\operatorname{ann}(M)$.

In view of Lemma 3.6, expansiveness and total non-expansiveness have natural interpretations for elements $f \in R$ according to the behaviour of $\alpha_{R/(f)}$. For example, a non-unit f is expansive if and only if all the complex roots of f do not lie on $\mathbb{S}$. At the other extreme, a non-unit f is totally non-expansive if and only if every irreducible factor of f in R has a root on $\mathbb{S}$, as the elements of $\operatorname{Ass}(R/(f))$ are generated by the irreducible factors of f. More generally, if α_M is an automorphism of an infinite compact abelian group with unfaithful module and if $f \in \operatorname{ann}(M)$ is totally non-expansive, so too is α_M because $f \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}^1(M)$.

Lemma 3.8. *Let α_M be an automorphism of a compact abelian group with unfaithful module. If α_M is totally non-expansive, $\Delta(\alpha_M) = \{0\}$.*

Proof. First assume that $M = R/\mathfrak{p}$ for a prime ideal $\mathfrak{p} \subset R$. If $\operatorname{ht}(\mathfrak{p}) = 2$ then M is finite and $\Delta(\alpha_M) = \{0\}$. If $\operatorname{ht}(\mathfrak{p}) = 1$ and α_M is non-expansive, adaptions of either [15, Lem. 5.2.3] or [12, Cor. 4.2] show that $\Delta(\alpha_M) = \{0\}$. The former uses the explicit description of automorphisms of the form $\alpha_{R/\mathfrak{p}}$ using adeles [4, Sec. 6] and the argument is essentially that of Example 3.11.

Now assume that M is Noetherian. Since $(0) \notin \operatorname{Ass}(M)$, by the above, there is a prime filtration of M of the form (3) such that $\Delta(\alpha_{M_{i+1}/M_i}) = \{0\}$, for all $0 \leq i \leq r$. As $\Delta(\alpha_{M_1}) = \{0\}$, Lemma 3.4(1) implies

$$\Delta(\alpha_{M_2}) = \Delta(\alpha_{M_2/M_1}) = \{0\}.$$

By induction, it follows that $\Delta(\alpha_M) = \{0\}$.

Finally, suppose M is not Noetherian. For an infinite cyclic submodule L, α_L is totally non-expansive because $\mathrm{Ass}(L) \subset \mathrm{Ass}(M)$. So, $\Delta(\alpha_L) = \{0\}$ as the above arguments apply to the Noetherian module L. Hence, Lemma 3.4(3) gives the required result. □

A straightforward consequence of Lemma 3.8 and Remark 3.7(2) is that

$$\mathsf{h}(\alpha_M) = 0 \implies \Delta(\alpha_M) = \{0\},$$

which may also be seen from [10, Prop. 3.5].

Lemma 3.9. *Let $f \in R$ be non-zero and $\mathfrak{R} = R/(f)$. If f is a non-unit and is expansive then there exists a non-trivial fundamental homoclinic point $w^\Delta \in \Delta(\alpha_\mathfrak{R})$ such that the map $R \to \Delta(\alpha_\mathfrak{R})$ given by*

$$g \mapsto \sum_{n\in\mathbb{Z}} c_n \alpha_\mathfrak{R}^n(w^\Delta),$$

where $g = \sum_{n\in\mathbb{Z}} c_n u^n$ has $c_n = 0$ for all but finitely many $n \in \mathbb{Z}$, is a surjective ring homomorphism with kernel (f). Furthermore, $\Delta(\alpha_\mathfrak{R})$ is dense in $\widehat{\mathfrak{R}}$.

Proof. Upon noting that $\alpha_\mathfrak{R}$ is ergodic, this is given by [10, Lem. 4.5]. □

Proof of Theorem 1.1(1). When α_M is totally non-expansive, the result holds trivially with $f = 1$. Hence, assume α_M has an infinite expansive factor. Then there is at least one $\mathfrak{p} \in \mathrm{Ass}^1(M)$ such that $\alpha_{R/\mathfrak{p}}$ is expansive. Consequently, for a non-zero $h \in \mathrm{ann}(M)$ we can write $h = fg$, where $f \neq 0$ is expansive and g is a unit or is totally non-expansive. Assuming the latter case, since $g(fM) = \{0\}$, if fM is infinite, α_{fM} is totally non-expansive. Hence,

$$\Delta(\alpha_{fM}) = \{0\} \text{ and } \Delta(\alpha_M) = \Delta(\alpha_{M/fM}), \tag{7}$$

by Lemma 3.4(1). Note that (7) also holds if g is a unit or if fM is finite.

Set $N = M/fM$ and note $fN = \{0\}$, so N may be considered as a module over $\mathfrak{R} = R/(f)$. The rest of the proof closely follows that of [6, Prop. 4.3], the main difference being that in our situation N need not be a Noetherian R-module. However, this does not significantly alter the argument, the key points of which are as follows.

Since $\alpha_\mathfrak{R}$ is expansive, Lemma 3.9 can be used to obtain a fundamental homoclinic point w^Δ for $\Delta(\alpha_\mathfrak{R})$ and this group is dense in $\widehat{\mathfrak{R}}$. Define a map $\chi : \mathrm{Hom}_R(N, \mathfrak{R}) \to \widehat{N}$ by setting

$$\langle \chi(\phi), a\rangle = \langle w^\Delta, \phi(a)\rangle,$$

where $\phi \in \mathrm{Hom}_R(N, \mathfrak{R})$ and $a \in N$. The topology of pointwise convergence on $\widehat{N}$ can be used to show that $\chi(\mathrm{Hom}_R(N, \mathfrak{R})) \subset \Delta(\alpha_N)$. Restricting the range to $\Delta(\alpha_N)$, χ can be shown to be an R-module homomorphism. To see that χ is surjective, proceed as follows. For each $a \in N$, there is an R-module homomorphism $\psi_a : \mathfrak{R} \to N$ given by $\psi_a(k) = ka$. The continuous dual map $\widehat{\psi_a} : \widehat{N} \to \widehat{\mathfrak{R}}$ sends homoclinic points of α_N to homoclinic points of $\alpha_{\mathfrak{R}}$ and if $w \in \Delta(\alpha_N)$, by Lemma 3.9, there is a unique $h_a \in \mathfrak{R}$ such that $\widehat{\psi_a}(w) = h_a w^{\Delta}$. Define a module homomorphism $\phi : N \to \mathfrak{R}$ by $\phi(a) = h_a$, which delivers $\chi(\phi) = w$. Injectivity of χ follows from the density of $\Delta(\alpha_{\mathfrak{R}})$ in $\widehat{\mathfrak{R}}$ and the required isomorphism is thus obtained. $\square$

Lemma 3.10. *Let α_M be an expansive automorphism of an infinite compact abelian group. Then there is a finite index submodule $L \subset M$ such that α_L is ergodic.*

Proof. Firstly, M is necessarily Noetherian with non-zero associated primes by Lemma 3.6. As M is infinite, $\mathrm{Ass}^1(M) \neq \varnothing$. By Lemma 2.1, a prime filtration of M may be obtained via an algorithm which selects the elements of $\mathrm{Ass}^1(M)$ first, thus giving a submodule $L \subset M$ with $\mathrm{Ass}(L) = \mathrm{Ass}^1(M)$ and $\mathrm{Ass}^1(M/L) = \varnothing$. By Lemma 3.6, the elements of $\mathrm{Ass}(L)$ do not have roots on $\mathbb{S}$ and α_L is therefore ergodic by Lemma 2.2. Since $\mathrm{Ass}^1(M/L)$ is empty, M/L is finite. $\square$

Proof of Theorem 1.1(2). Let $L \subset M$ be an infinite index submodule and suppose $\alpha_{M/L}$ is expansive. Then by Lemma 3.10 there is an ergodic factor β of $\alpha_{M/L}$ which is also expansive. Hence,

$$\mathsf{h}(\alpha_{M/L}) \geq \mathsf{h}(\beta) > 0$$

and [10, Th. 4.1] shows that $\Delta(\alpha_{M/L}) \neq \{0\}$, so $\Delta(\alpha_M) \neq \{0\}$.

Conversely, if $\Delta(\alpha_M) \neq \{0\}$, α_M cannot be totally non-expansive. By the proof of Theorem 1.1(1), there exists an expansive polynomial $f \in R$ and a non-trivial homomorphism $\phi : M/fM \to R/(f)$. The image $\phi(M/fM)$ is an ideal $\mathfrak{a} \subset R/(f)$ which must be infinite. Hence, the kernel of the composition of surjective R-module homomorphisms

$$M \to M/fM \to \mathfrak{a},$$

has infinite index in M. Finally, since $\mathfrak{a}$ is a submodule of $R/(f)$, $\alpha_{\mathfrak{a}}$ is expansive by [18, Cor. 6.15]. $\square$

The following well known example illustrates how non-trivial homomorphisms of the form (1) are essential for the existence of non-trivial homoclinic points.

Example 3.11. Consider $M = \mathbb{Z}[\frac{1}{6}]$ as an R-module via the substitution map $u \mapsto 2$. This module corresponds to the doubling map $x \mapsto 2x$ on a one-dimensional solenoid $X = \widehat{M}$. Setting $W = \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$, there is an isomorphism $X \cong W/\iota(M)$, where ι is the natural diagonal embedding of M as a discrete co-compact subgroup of W (see [4, Sec. 6]). Under this isomorphism, α_M identifies with the doubling map β on $W/\iota(M)$, which lifts naturally to W. Furthermore, $\Delta(W/\iota(M), \beta)$ is given by the intersection of the projections of

$$\{(x, 0, 0) : x \in \mathbb{R}\} \text{ and } \{(0, y, 0) : y \in \mathbb{Q}_2\}$$

to $W/\iota(M)$, which is trivial.

An alternative perspective is offered by Theorem 1.1(1). Firstly, α_M is not totally non-expansive because $\mathrm{Ass}(M) = \{(u-2)\}$. However, $(u-2)M = \{0\}$ and (1) implies

$$\Delta(\alpha_M) \cong \mathrm{Hom}_R(\mathbb{Z}[\tfrac{1}{6}], \mathbb{Z}[\tfrac{1}{2}]) = \{0\}.$$

A similar argument may be applied to [10, Ex. 3.7]. The dual R-module of the automorphism given there identifies with the domain $\mathbb{Z}[\frac{1}{2}, \sqrt{5}]$ via the substitution map $u \mapsto \xi$, where $\xi = (1+\sqrt{5})/2$. The polynomial $f = u^2 - u - 1$ is an expansive polynomial, obtained as in the proof of Theorem 1.1(1), which annihilates the module. Using (1), the homoclinic group is isomorphic to

$$\mathrm{Hom}_R(\mathbb{Z}[\tfrac{1}{2}, \sqrt{5}], \mathbb{Z}[\xi^{\pm 1}]) = \{0\}.$$

Example 3.12. Consider again Example 3.3. The dual module is

$$M = R/(2u-2),$$

which is annihilated by $2(u-1)$. So, $f = 2$ is a suitable expansive polynomial for use in (1). Set $\mathfrak{R} = R/(2) \cong \mathbb{F}_2[u^{\pm 1}]$, then

$$\Delta(\alpha_M) \cong \mathrm{Hom}_R\left(M/fM, R/(f)\right) \cong \mathrm{Hom}_R\left(\mathfrak{R}, \mathfrak{R}\right) \cong \mathfrak{R}.$$

Since Theorem 1.1(1) permits the appearance of uncountable homoclinic groups, a further constraint is needed in order for the module $M^* = \Delta(\alpha_M)$ and the associated adjoint automorphism $\alpha_M^* = \alpha_{M^*}$ to be defined, as discussed in Section 1. This is made possible by insisting that $\mathsf{h}(\alpha_M) < \infty$.

Lemma 3.13. *Let α_M be an automorphism of a compact abelian group with unfaithful module and finite entropy. Then there exists a Noetherian submodule $L \subset M$ such that $\mathsf{h}(\alpha_{M/L}) = 0$ and $\mathsf{h}(\alpha_M) = \mathsf{h}(\alpha_L)$.*

Proof. Since $\mathrm{Ass}^1(M)$ is finite, the set of primes,

$$P = \{\mathfrak{p} \in \mathrm{Ass}(M) : \mathsf{h}(\alpha_{R/\mathfrak{p}}) > 0\} \subset \mathrm{Ass}^1(M),$$

is also finite. For a non-trivial case, assume $P \neq \varnothing$. The module M can be expressed as a limit $M = \lim_{\to} M_i$ of Noetherian submodules

$$M_1 \subset M_2 \subset \cdots$$

and

$$\mathrm{Ass}^1(M_{i+1}/M_i) \subset \mathrm{Ass}^1(M_{i+1}) \subset \mathrm{Ass}^1(M),$$

for all $i \geq 1$. Hence, if $\mathfrak{p} \in \mathrm{Ass}(M_{i+1}/M_i)$ has $\mathsf{h}(\alpha_{R/\mathfrak{p}}) > 0$ then $\mathfrak{p} \in P$. Therefore,

$$\mathsf{h}(\alpha_{M_{i+1}/M_i}) > 0 \implies \mathsf{h}(\alpha_{M_{i+1}/M_i}) \geq \min\{\mathsf{h}(\alpha_{R/\mathfrak{p}}) : \mathfrak{p} \in P\}.$$

This shows that there exists $j \in \mathbb{N}$ such that $\mathsf{h}(\alpha_{M_{i+1}/M_i}) = 0$ for every $i \geq j$, otherwise by (4), $\mathsf{h}(\alpha_M) = \infty$. Again using (4), $\mathsf{h}(\alpha_{M/M_j}) = 0$ and $\mathsf{h}(\alpha_M) = \mathsf{h}(\alpha_{M_j})$. $\square$

Proof of Theorem 1.1(3). Let Y denote the closure of $\Delta(\alpha_M)$ in M which is an α-invariant subgroup. Since $\Delta(\alpha_M)$ is dense in Y, the dual of $\Delta(\alpha_M)$ coincides with $N = \widehat{Y}$. Furthermore, Lemma 3.13 implies there is a Noetherian submodule $L \subset M$ such that $\mathsf{h}(\alpha_{M/L}) = 0$ and $\mathsf{h}(\alpha_M) = \mathsf{h}(\alpha_L)$. So, $\Delta(\alpha_{M/L}) = \{0\}$ and by Lemma 3.4(2) there is a topological isomorphism between $\Delta(\alpha_M)$ and an α_L-invariant subgroup of $\Delta(\alpha_L)$. Furthermore, the appropriate restrictions of α_M and α_L are conjugate via this isomorphism. Thus, N is a Noetherian R-module. If $\mathfrak{p} \in \mathrm{Ass}^1(N)$, since $R/\mathfrak{p}$ is isomorphic to an infinite submodule of N, $\Delta(\alpha_{R/\mathfrak{p}}) \neq \{0\}$ by the density of $\Delta(\alpha_N)$ in $\widehat{N}$. Hence, by Lemma 3.8, $\alpha_{R/\mathfrak{p}}$ is expansive for all $\mathfrak{p} \in \mathrm{Ass}^1(N)$ and α_N is expansive by Lemma 3.6. The final part of the result is given by [10, Lem. 3.2]. □

Proof of Theorem 1.1(4). Suppose α_M is not expansive. If M is a Noetherian module, by Lemma 3.6 there exists a non-expansive prime $\mathfrak{p} \in \mathrm{Ass}^1(M)$. Since $\mathfrak{p}$ is a minimal prime over $\mathrm{ann}(M)$, the $\mathfrak{p}$-primary component $L = \ker(M \to M_{\mathfrak{p}})$ satisfies $\mathrm{Ass}(M/L) = \{\mathfrak{p}\}$ and M/L is infinite as $\mathrm{ht}(\mathfrak{p}) = 1$. Furthermore, $\alpha_{M/L}$ is totally non-expansive, so $\Delta(\alpha_{M/L}) = \{0\}$.

If M is not Noetherian, Lemma 3.13 shows there is an infinite index submodule $L \subset M$ such that $\mathsf{h}(\alpha_{M/L}) = 0$. Hence, $\Delta(\alpha_{M/L}) = \{0\}$.

Now suppose α_M is expansive and $L \subset M$ is an infinite index submodule. The automorphism $\alpha_{M/L}$ is again expansive and since M/L is infinite, $\mathsf{h}(\alpha) > 0$ by Remark 3.7. The result now follows from [10, Th. 4.1]. □

Proof of Theorem 1.1(5). Since Theorem 1.1(3) shows that the restriction of α_M to the closure of $\Delta(\alpha_M)$ is expansive, [6, Prop. 4.3] shows that α_M^{**} is expansive and so α_M is not reflexive.

Suppose α_M is ergodic. Lemma 3.13 implies there is a Noetherian submodule $L \subset M$ such that $\mathsf{h}(\alpha_{M/L}) = 0$ and $\mathsf{h}(\alpha_M) = \mathsf{h}(\alpha_L)$. So, $\Delta(\alpha_{M/L}) = \{0\}$ and there is an inclusion $\Delta(\alpha_M) \hookrightarrow \Delta(\alpha_L)$ by Lemma 3.4(2).

First assume $\alpha_{R/\mathfrak{p}}$ is non-expansive for some $\mathfrak{p} \in \mathrm{Ass}(L)$. Let $P \subset \mathrm{Ass}(L)$ be the set of all such non-expansive primes. If $P = \mathrm{Ass}(L)$ then α_L is totally non-expansive and by Theorem 1.1(1), $\Delta(\alpha_L) = \{0\}$, giving $\mathsf{h}(\alpha_M^*) = \mathsf{h}(\alpha_L^*) = 0$. Hence, assume $P \neq \mathrm{Ass}(L)$ and use Lemma 2.1 to obtain a submodule $N \subset L$ such that $\mathrm{Ass}(N) = P$ and $\mathrm{Ass}^1(L/N) = \mathrm{Ass}(L) \setminus P$. Then α_N is totally non-expansive and $\alpha_{L/N}$ is expansive, so $\Delta(\alpha_N) = \{0\}$ and $\Delta(\alpha_L) = \Delta(\alpha_{L/N})$ by Lemma 3.4(1). As α_N is ergodic, $\mathsf{h}(\alpha_N) > 0$ and $\mathsf{h}(\alpha_{L/N}) < \mathsf{h}(\alpha_L)$ by (4). Since $\mathrm{Ass}^1(L/N) \neq \varnothing$, L/N is infinite and Lemma 3.10 shows there is a finite index submodule $J \subset L/N$ such that α_J is ergodic. By Lemma 3.4(2), $\Delta(\alpha_{L/N}) \subset \Delta(\alpha_J)$ and hence

$$\mathsf{h}(\alpha_M^*) \leq \mathsf{h}(\alpha_L^*) = \mathsf{h}(\alpha_{L/N}^*) \leq \mathsf{h}(\alpha_J^*). \tag{8}$$

Additionally,

$$\mathsf{h}(\alpha_J) = \mathsf{h}(\alpha_{L/N}) < \mathsf{h}(\alpha_L) = \mathsf{h}(\alpha_M). \tag{9}$$

Since α_J is expansive and ergodic, [6, Th. 4.6] shows that $\mathsf{h}(\alpha_J^*) = \mathsf{h}(\alpha_J)$ and the required result follows from (8) and (9).

Now assume that $\alpha_{R/\mathfrak{p}}$ is expansive for all $\mathfrak{p} \in \operatorname{Ass}(L) = \operatorname{Ass}(M)$, so that α_L is expansive and M is non-Noetherian by Lemma 3.6. Since α_L is ergodic, [6, Th. 4.6] shows that $\mathsf{h}(\alpha_L^*) = \mathsf{h}(\alpha_L)$ and [6, Th. 4.7] implies α_L^* is expansive. As $\operatorname{Ass}(M/L)$ consists entirely of maximal ideals, M can be written as a limit $\lim_{\rightarrow} L_i$ of Noetherian modules

$$L = L_1 \subset L_2 \subset \cdots ,$$

where $|L_{i+1}/L_i| = \lambda(i)$, for positive integers $\lambda(i) > 1$, $i \geq 1$. Dually, $X = \widehat{M}$ can be expressed as a projective limit $X = \lim_{\leftarrow}(X_i, \pi_i)$, where $X_i = \widehat{L_i}$ and π_i is the projection dual to the inclusion $L_i \to L_{i+1}$. For each $i \geq 1$, let π_i' denote the natural composite map $\pi_i' : X_{i+1} \to X_1$ and note $|\ker(\pi_i')| = \prod_{j=1}^{i} \lambda(j)$. Since $\mathsf{h}(\alpha_{M/L_i}) = 0$, Lemma 3.8 and Lemma 3.4(2) shows there is an inclusion

$$\Delta(\alpha_M) \hookrightarrow \Delta(\alpha_{L_i})$$

via the projection of X to X_i. Furthermore, [6, Lem. 5.5] shows that the restriction of π_i' to $\Delta(\alpha_{L_{i+1}})$ is also injective and $\pi_i'(\Delta(\alpha_{L_{i+1}}))$ has index $|\ker(\pi_i')|$ in $\Delta(\alpha_L)$. Since $|\ker(\pi_i')| \to \infty$ as $i \to \infty$, it follows that $M^* = \Delta(\alpha_M)$ embeds as an infinite index submodule of $L^* = \Delta(\alpha_L)$.

Now L^*/M^* is infinite and L^* is Noetherian, so $\operatorname{Ass}^1(L^*/M^*) \neq \varnothing$. Furthermore, $\operatorname{Ass}^1(L^*/M^*) \subset \operatorname{Ass}^1(L^*)$ and since α_L^* is expansive, $\operatorname{Ass}^1(L^*/M^*)$ contains an expansive prime ideal. Hence, $\mathsf{h}(\alpha_{L^*/M^*}) > 0$ by Remark 3.7(1) and so by (4),

$$\mathsf{h}(\alpha_M^*) = \mathsf{h}(\alpha_L^*) - \mathsf{h}(\alpha_{L^*/M^*}) < \mathsf{h}(\alpha_L^*) = \mathsf{h}(\alpha_L) = \mathsf{h}(\alpha_M).$$

□

Remark 3.14. Theorem 1.1(5) offers yet another perspective on examples such as Example 3.11. Using the notation of Theorem 1.1(1), the module M in Example 3.11 is annihilated by an expansive irreducible polynomial f. Setting $\mathfrak{R} = R/(f)$, if the homoclinic module $M^* = \operatorname{Hom}_R(M, \mathfrak{R})$ is non-trivial, $\operatorname{Ass}(M^*) = \{(f)\}$. Therefore, $\mathsf{h}(\alpha_M^*) \geq \mathsf{h}(\alpha_{\mathfrak{R}})$. However, in this example, upon identifying $\mathfrak{R}$ with a submodule of M, $\mathsf{h}(\alpha_{M/\mathfrak{R}}) = 0$. Hence,

$$\mathsf{h}(\alpha_M) = \mathsf{h}(\alpha_{\mathfrak{R}}) + \mathsf{h}(\alpha_{M/\mathfrak{R}}) = \mathsf{h}(\alpha_{\mathfrak{R}}),$$

giving a contradiction to Theorem 1.1(5). Thus, M^* must be trivial.

The following example shows that ergodicity is a necessary assumption for (2).

Example 3.15. Consider the product of R-modules $M = L \times N$, where $L = R/(f)$ for an expansive polynomial $f \in R$ and $N = R/(g)$ for a cyclotomic polynomial $g \in R$. Then α_N and α_M are non-ergodic by Lemma 2.2. Furthermore, $\mathsf{h}(\alpha_N) = 0$ and so $\mathsf{h}(\alpha_M) = \mathsf{h}(\alpha_L)$. As α_N is totally non-expansive, $\Delta(\alpha_N) = \{0\}$ and

$$\Delta(\alpha_M) \cong \Delta(\alpha_L) \times \Delta(\alpha_N) \cong \Delta(\alpha_L) \cong L,$$

by Lemma 3.9. Hence $\mathsf{h}(\alpha_M^*) = \mathsf{h}(\alpha_L) = \mathsf{h}(\alpha_M)$.

4 Entropy rank one actions

Throughout this section, G is a countable abelian group and R_G denotes the group ring $\mathbb{Z}[G]$. We begin with an extension of a definition from [4].

Definition 4.1. Let α be an action of a countable abelian group G by automorphisms α_g of a compact abelian group. Then α has *entropy rank one* if $\mathsf{h}(\alpha_g) < \infty$ for all $g \in G$.

Example 4.2 (*An infinitely generated action of entropy rank one*)**.** Let $p_1, p_2, \ldots$ be the sequence of rational primes and

$$R_\infty = \mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}, \ldots],$$

where $u_1, u_2, \ldots$ are independent indeterminates. The map $u_i \mapsto p_i$, $i \geq 1$, induces a ring endomorphism $R_\infty \to \mathbb{Q}$ under which $\mathbb{Q}$ becomes an R_∞-module. This induces an action α of the group $G = \bigoplus_{\mathbb{N}} \mathbb{Z}$ of eventually zero sequences of integers on the solenoid $X = \widehat{\mathbb{Q}}$ and represents a flow in the sense of Berend [1]. For any $g \in G$, the application of a single automorphism $\widehat{\alpha_g}$ corresponds to multiplication by an element $\xi_g \in \mathbb{Q}$ on $\mathbb{Q}$ and therefore each α_g has finite entropy by [11]. Hence, α is an entropy rank one action. Furthermore, if $g \neq 0$, $\xi_g \notin \{-1, 1\}$ and for any $n \in \mathbb{Z}$, $\xi_g^n = 1 \Leftrightarrow n = 0$. So every element of the action is ergodic by Lemma 2.2.

The following result is needed for the proof of Theorem 1.2.

Theorem 4.3. *Let α be an action of a torsion-free abelian group G of rational rank one on a compact abelian group X and suppose G is not isomorphic to $\mathbb{Z}$. For any $g \in G$, if $\mathsf{h}(\alpha_g) < \infty$ then every algebraic factor of α_g is non-ergodic.*

Proof. Suppose G is not isomorphic to $\mathbb{Z}$ and let $g \in G$. Since G is a torsion-free group of rational rank one, it may be considered as a subgroup of the additive group of $\mathbb{Q}$. Therefore, by [3, Prop. 1.6.7], as G is not isomorphic to $\mathbb{Z}$, G has a subgroup isomorphic to

$$H = \langle g/r_1, g/r_2, \ldots \rangle,$$

where (r_i) is a strictly increasing sequence of positive integers. The group ring R_H can be identified with the domain $\mathbb{Z}[u^{\pm 1/r_i} : i \geq 1]$, where u is an indeterminate. By [8, Prop. 9.2], $\operatorname{kdim}(R_H) = 2$. Without loss of generality, assume $r_1 = 1$.

Let L be a cyclic R_H-submodule of $M = \widehat{X}$ and let $R = \mathbb{Z}[u^{\pm 1}]$. Then $L \cong R_H/\mathfrak{a}$ for some ideal $\mathfrak{a} \subset R_H$ and L can be considered as an algebra over $\mathfrak{R} = R/(\mathfrak{a} \cap R)$. Write $\beta = \alpha_g$. The aim is to show that for each $\mathfrak{p} \in \operatorname{Ass}_{\mathfrak{R}}(L)$, $\beta_{\mathfrak{R}/\mathfrak{p}}$ is non-ergodic. Suppose, for a contradiction, that this is not the case for some $\mathfrak{p} \in \operatorname{Ass}_{\mathfrak{R}}(L)$.

Since L is integral over $\mathfrak{R}$, by [8, Prop. 9.2], there is a prime ideal $\mathfrak{q} \subset L$ with $\mathfrak{q} \cap \mathfrak{R} = \mathfrak{p}$. For each $j \geq 1$, let D_j denote the image of the ring

$$\mathbb{Z}[u^{\pm 1/r_i} : 1 \leq i \leq j] \subset R_H,$$

in $L/\mathfrak{q}$, noting $D_1 \cong \mathfrak{R}/\mathfrak{p}$. For any $j \geq 1$, if $\mathrm{kdim}(D_j) = 2$, $D_1 \cong \mathbb{Z}[u^{\pm 1}]$ and $\beta_{L/\mathfrak{q}}$ has a factor equivalent to a full-shift on $\mathbb{T}^{\mathbb{Z}}$ which has infinite entropy, giving a contradiction to $\mathsf{h}(\beta) < \infty$. Since $\mathrm{kdim}(D_j) = 0$ implies D_1 is finite, this is prohibited by the ergodicity assumption. Therefore, $\mathrm{kdim}(D_j) = 1$ for all $j \geq 1$.

Denote the field of fractions of $L/\mathfrak{q}$ by K and that of D_j by K_j. Suppose, for a contradiction, that $K|K_1$ is a finite extension; so K is either a finite extension of $\mathbb{Q}$ or a finitely generated extension of transcendence degree one over a finite field. Let $\overline{u}$ denote the image of u in K. If $\mathrm{char}(K) > 0$, $\overline{u}$ is transcendental over the prime subfield of K and there is a valuation of K with $v(\overline{u}) \neq 0$. If $\mathrm{char}(K) = 0$, denote the unit group of the ring of integers in K by $\mathcal{U}$ and note there is also such a valuation if $\overline{u} \notin \mathcal{U}$. However, all valuations on K should be discrete and a contradiction is obtained by attempting to apply v to the image of H in K. If $\mathrm{char}(K) = 0$ and $\overline{u} \in \mathcal{U}$, the image $\overline{H}$ of H in K is a subgroup of $\mathcal{U}$, which is finitely generated by the Unit Theorem [3, Th. 3.3.3]. However, the ergodicity assumption means the natural map $H \to \overline{H}$ is injective and again a contradiction is obtained. Hence, $K|K_1$ is not a finite extension.

A standard application of (4) shows

$$\mathsf{h}(\beta_{D_j}) = \mathsf{h}(\beta_{K_j}) = \dim_{K_1}(K_j) \cdot \mathsf{h}(\beta_{K_1}),$$

for all $j \geq 1$. As β_{D_1} is ergodic, $\mathsf{h}(\beta_{D_1}) > 0$ and by the above $\dim_{K_1}(K_j) \to \infty$ as $j \to \infty$. Moreover, for every $j \geq 1$,

$$\mathsf{h}(\beta_M) \geq \mathsf{h}(\beta_L) \geq \mathsf{h}(\beta_{L/\mathfrak{q}}) \geq \mathsf{h}(\beta_{D_j}),$$

which means $\mathsf{h}(\beta_M)$ cannot be finite, giving a contradiction. So, $\beta_{\mathfrak{R}/\mathfrak{p}}$ is non-ergodic for every $\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}}(L)$. Equivalently, $\beta_{R/\mathfrak{p}}$ is non-ergodic for every $\mathfrak{p} \in \mathrm{Ass}_R(L)$.

Finally, given an arbitrary R-submodule $L \subset M$, since

$$\mathrm{Ass}_R(L) = \bigcup_{a \in L} \mathrm{Ass}_R(R_H a),$$

the above shows that $\beta_{R/\mathfrak{p}}$ is non-ergodic for all $\mathfrak{p} \in \mathrm{Ass}(L)$ and hence β_L is non-ergodic by Lemma 2.2. $\square$

Theorem 4.3 highlights the degeneracy of entropy rank one actions of rational rank one groups other than $\mathbb{Z}$. The following is an example of such an action.

Example 4.4. Let p be a rational prime and $G = \mathbb{Z}[1/p]$. Let K be the field obtained by adjoining all p^j-th roots of unity, $j \geq 1$, to $\mathbb{Q}$ and let $X = \widehat{K}$. The group ring R_G may be identified with $\mathbb{Z}[u^{\pm 1/p^j} : j \geq 1]$, where u is an indeterminate. The map

$$u^{1/p^j} \mapsto \exp(2\pi i/p^j),$$

where $j \geq 0$, induces a ring homomorphism $R_G \to K$. Hence, K becomes an R_G-module and via duality there is a resulting G-action α on X. Furthermore, for every $g \in G$ the automorphism α_g is non-ergodic by Lemma 2.2 and has zero entropy by Remark 3.7(1).

Theorem 1.2 of [13] implies that certain expansive entropy rank one $\mathbb{Z}^2$-actions have no non-zero homoclinic points common to every non-trivial element of the action. Theorem 1.2 shows this is true in greater generality. Note that a point which is homoclinic for every non-trivial element of the action is not necessarily homoclinic in the sense of [10] and we refer to [13] instead of [10] in the proof below. The following consequence of the results of [4, Sec. 7] will also be useful.

Lemma 4.5. *Let α_M be an action of $G = \mathbb{Z}^d$ by automorphisms of a compact abelian group. If α_M has entropy rank one,* $\operatorname{kdim}(R_G/\mathfrak{p}) \leq 1$ *for every prime ideal $\mathfrak{p} \in \operatorname{Ass}(M)$.*

Proof of Theorem 1.2. Suppose G has rational rank one but is not isomorphic to $\mathbb{Z}$. Let β be any non-trivial automorphism in the action and let $M = \widehat{X}$ be the dual $\mathbb{Z}[u^{\pm 1}]$-module. If $L \subset M$ is any cyclic submodule, the entropy rank one assumption means $\operatorname{ann}(L)$ is non-trivial. Since every algebraic factor of β_L is non-ergodic by Theorem 4.3, it follows that L is either finite or β_L is totally non-expansive. Therefore, $\Delta(\beta_L) = \{0\}$ by Lemma 3.8 and Lemma 3.4(3) implies $\Delta(\beta_M) = \{0\}$.

The only remaining possibility is that G has rational rank at least two. Hence, without loss of generality, assume $G \cong \mathbb{Z}^2$, as a lack of non-trivial common homoclinic points for a sub-action implies the same result for the whole action. Let $M = \widehat{X}$ be the dual R_G-module and identify R_G with the ring of Laurent polynomials $\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]$, where u_1 and u_2 are independent indeterminates. Denote the set of homoclinic points common to every non-trivial element of the action by $\Delta(\alpha_M)$. Consider the product action generated by the R_G-module $N = \bigoplus_{a \in M} R_G a$. The natural projection of N onto M dualizes so that X can be considered as an α_N-invariant subgroup of $\widehat{N}$. Hence, $\Delta(\alpha_M) \subset \Delta(\alpha_N)$, and it is therefore sufficient to assume M is an arbitrary cyclic module with α_M of entropy rank one and show that $\Delta(\alpha_M) = \{0\}$.

Consider a prime filtration of the Noetherian R_G-module M of the form (3), let $\mathfrak{q}$ be any prime appearing in this filtration and set $\mathfrak{R} = R/\mathfrak{q}$. For a non-trivial case, assume $\mathfrak{R}$ is not finite. By Lemma 4.5, $\operatorname{kdim}(\mathfrak{R}) = 1$. If $\alpha_{\mathfrak{R}}$ is expansive, [13, Th. 1.2] shows that $\Delta(\alpha_{\mathfrak{R}}) = \{0\}$. Hence, suppose $\alpha_{\mathfrak{R}}$ is non-expansive. Let β denote the automorphism dual to multiplication by u_1 on $\mathfrak{R}$ and consider $\mathfrak{R}$ as a module over $R = \mathbb{Z}[u_1^{\pm 1}] \subset R_G$. Every element of $\mathfrak{R}$ is annihilated by the prime $\mathfrak{p} = \mathfrak{q} \cap R$ which is non-zero. If $\mathfrak{p}$ is generated by a non-expansive polynomial or is maximal then $\beta_{\mathfrak{R}}$ is totally non-expansive and Lemma 3.8 shows that $\Delta(\alpha_{\mathfrak{R}}) \subset \Delta(\beta_{\mathfrak{R}}) = \{0\}$. Otherwise, $\mathfrak{q}$ is generated by an expansive polynomial and Remark 3.14 shows that $\Delta(\beta_{\mathfrak{R}}) = \{0\}$. An induction argument similar to that used in the proof of Lemma 3.8 now shows that $\Delta(\alpha_M) = \{0\}$. □

Excluding $\mathbb{Z}$-actions, the case for non-expansive actions that do not have entropy rank one has not been considered here. In this setting, there can be non-trivial homoclinic points common to every element of the action. Section 7 of [10] gives some examples of such actions that show the situation is more involved than that considered here.

For expansive entropy rank one $\mathbb{Z}^d$-actions with $d > 1$, there can be no non-trivial homoclinic points common to every non-trivial element of the action, but common homoclinic points do occur for elements lying in expansive components, in the sense of [5]. If α is an ergodic non-expansive entropy rank one $\mathbb{Z}^d$-action on a finite-dimensional group, Theorem 1.1(3) shows α acts expansively on the closure of $\Delta(\alpha_{\mathbf{n}})$, for any $\mathbf{n} \in \mathbb{Z}^d$. Therefore, the methods of [5, Sec. 9] and [7] can be applied to the appropriate restriction of α, exposing some structure for elements sharing common homoclinic points in the non-expansive entropy rank one setting.

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