FINITELY REPRESENTED CLOSED ORBIT SUBDYNAMICS FOR COMMUTING AUTOMORPHISMS

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Abstract. The purpose of this paper is to exhibit highly structured subdynamics for a class of non-expansive algebraic $\mathbb{Z}^d$-actions based on the closed orbits of elements of an action. This is done using dynamical Dirichlet series to encode orbit counts. It is shown that there is a distinguished group homomorphism from $\mathbb{Z}^d$ onto a finite abelian group that controls the form of the Dirichlet series of elements of an action and that these series have common analytic properties. Corresponding orbit growth asymptotics are subsequently investigated.

1. INTRODUCTION

Let X be a compact metric space and $T : X \to X$ a continuous map. Let $\mathsf{F}_k(T)$ denote the cardinality of the set of points of period $k \in \mathbb{N}$,

$$\{x \in X : T^k(x) = x\}.$$

A *closed orbit* of length $k \in \mathbb{N}$ is a set of the form

$$\{x, T(x), T^2(x), \ldots, T^k(x) = x\}.$$

Denote the cardinality of the collection of closed orbits of length k by $\mathsf{O}_k(T)$.

One way to encode the sequence of periodic point data $(\mathsf{F}_k(T))$ is by using the well-known *dynamical zeta function* introduced in [1],

$$\zeta_T(z) = \exp \sum_{k=1}^{\infty} \frac{\mathsf{F}_k(T)}{k} z^k.$$

In many settings, for example if T is a hyperbolic toral automorphism, ζ_T turns out to be rational [15] and, coupled with the arithmetic relation between periodic points and orbits, this provides a route to obtaining a reasonable asymptotic expression for the *orbit growth function*

$$\pi_T(N) = \sum_{k \leqslant N} \mathsf{O}_k(T).$$

The thesis of Margulis with accompanying survey by Sharp [10] and the papers of Parry and Pollicott [13], [14] contain fundamental results in this

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area. An case in point is that of a hyperbolic toral automorphism T, for which

$$\pi_T(N) \sim C\frac{e^{hN}}{N},$$

where $C > 0$ is a constant and h is the topological entropy.

If (X, T) is a dynamical system where the supply of closed orbits is more limited, instead of using ζ_T it can be useful to handle orbit counts directly via the *dynamical Dirichlet series* [6], which is the formal series

$$\delta_T(z) = \sum_{k=1}^{\infty} \frac{\mathsf{O}_k(T)}{k^z}.$$

For an automorphism T of a finite-dimensional compact abelian group X, in [6] it is shown that $\delta_T(z)$ is a *rational function of exponential variables* (that is a rational function of a set of variables $\{c^{-z} : c \in \mathcal{C}\}$, where $\mathcal{C} \subset \mathbb{N}$ is finite) whenever $(\mathsf{F}_k(T))$ has finite rank in a canonical poset of sequences of periodic point counts. This condition is satisfied if T is ergodic, X is connected (such a group is called a *solenoid*) and the set of rational primes

$$\mathcal{P}(T) = \{p : p \text{ is prime and } p|\mathsf{F}_k(T) \text{ for some } k \in \mathbb{N}\} \tag{1}$$

is finite. The series δ_T is subsequently used in [6] to find an asymptotic expression for π_T in terms of elementary functions.

An accessible setting, avoiding the need for the combinatorial framework of [6], comes from adopting finiteness of $\mathcal{P}(T)$ as a hypothesis. Notably, this hypothesis is sufficiently broad to include all the connected examples in [6]. The simplest of these [6, Ex. 4.1] is given by the automorphism T dual to the map $x \mapsto 2x$ on the localization $\mathbb{Z}_{(3)}$ of the integers at the prime 3. It is straightforward to show that this system has $\mathsf{F}_k(T) = |2^k - 1|_3^{-1}$,

$$\delta_T(z) = 1 + \frac{1}{2^z}\left(\frac{1}{1-3^{-z}}\right) \quad \text{and} \quad \pi_T(N) = \frac{1}{\log 3}\log(N) + \mathrm{O}(1).$$

Suppose now that X is a compact metrizable abelian group and α is a $\mathbb{Z}^d$-action, $d \geqslant 2$, by continuous automorphisms $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$, of X. Following Schmidt [16], α is called an *algebraic* $\mathbb{Z}^d$*-action.* When an underlying action α is clear, write $\delta_{\mathbf{n}} = \delta_{\alpha^{\mathbf{n}}}$, $\zeta_{\mathbf{n}} = \zeta_{\alpha^{\mathbf{n}}}$ and $\pi_{\mathbf{n}} = \pi_{\alpha^{\mathbf{n}}}$. If one wishes to study the subdynamics of individual elements of such an action α, it is natural to focus on examples where every automorphism $\alpha^{\mathbf{n}}$ has finite entropy; these are *entropy rank one* actions in the sense of Einsiedler and Lind [4]. For example, $\mathbb{Z}^d$-actions on solenoids are entropy rank one actions. If α is an expansive entropy rank one action (see Section 2 for the definition of expansiveness), the form of $\zeta_{\mathbf{n}}$ is closely related to expansive subdynamics [11], [2] and there is a subsequent link with $\pi_{\mathbf{n}}$. A snapshot of these connections is provided by the standard ×2 ×3 example (Example 3.1). If α is non-expansive, subdynamical patterns can be harder to discern. A notable exception is the portrait of directional entropy for algebraic $\mathbb{Z}^d$-actions on solenoids which one may easily obtain by extending a result of

Einsiedler and Lind [4, Prop. 8.5] to non-Noetherian actions. In contrast, dynamical zeta functions are often ill-behaved and difficult to describe in the non-expansive case (see Example 3.2).

Using dynamical Dirichlet series instead of dynamical zeta functions, the goal here is to reveal a class of non-expansive $\mathbb{Z}^d$-actions that exhibit surprisingly structured subdynamical behaviour based on the closed orbits of elements of an action. Wishing to capitalise on the good behaviour of dynamical Dirichlet series explained above, the focus here is on $\mathbb{Z}^d$-actions on solenoids. To avoid encountering automorphisms with infinitely many closed orbits of a given length, we consider only

$$\mathcal{E}(\alpha) = \{\mathbf{n} \in \mathbb{Z}^d : \alpha^{\mathbf{n}} \text{ is ergodic}\} \subset \mathbb{Z}^d.$$

It will be shown (Lemma 4.2) that

$$|\mathcal{P}(\alpha^{\mathbf{m}})| < \infty \text{ for some } \mathbf{m} \in \mathcal{E}(\alpha) \Leftrightarrow |\mathcal{P}(\alpha^{\mathbf{n}})| < \infty \text{ for all } \mathbf{n} \in \mathcal{E}(\alpha). \quad (2)$$

For an action that satisfies either of these equivalent conditions, each $\delta_{\mathbf{n}}$, $\mathbf{n} \in \mathcal{E}(\alpha)$, is a rational function of exponential variables. Moreover, the following theorem shows that these dynamical Dirichlet series have a form which is controlled by a single homomorphism from $\mathbb{Z}^d$ onto a finite abelian group.

Theorem 1.1. *Let α be an algebraic $\mathbb{Z}^d$-action on a solenoid X such that $\mathcal{P}(\alpha^{\mathbf{m}})$ is finite for some $\mathbf{m} \in \mathcal{E}(\alpha)$. Then there exists a distinguished surjective homomorphism ϑ from $\mathbb{Z}^d$ onto a finite abelian group Γ and for each $\gamma \in \Gamma$ there exists a finite list of rational functions of exponential variables $\mathcal{G}(\gamma)$, such that*

$$\delta_{\mathbf{n}}(z) = \sum_{g \in \mathcal{G}(\vartheta(\mathbf{n}))} a_g(\mathbf{n}) g(z), \quad (3)$$

where $\mathbf{n} \in \mathcal{E}(\alpha)$ and $a_g(\mathbf{n}) \in \mathbb{N}$.

Explicit expressions for g and a_g are given in Section 4 (see (23) and (22)). In particular, a_g has the form

$$a_g(\mathbf{n}) = \prod_{v \in \mathcal{S}} |\xi_v(\mathbf{n}) - 1|_v^{-1},$$

where $\mathcal{S}$ is a finite list of finite places (see Section 2) and ξ_v, $v \in \mathcal{S}$, are homomorphisms into the group of units in the valuation ring corresponding to v. Using these expressions, a striking uniformity in analytic properties is revealed.

Theorem 1.2. *Let α be an algebraic $\mathbb{Z}^d$-action on a solenoid X such that $\mathcal{P}(\alpha^{\mathbf{m}})$ is finite for some $\mathbf{m} \in \mathcal{E}(\alpha)$. Then there exist $\sigma(\alpha) \geqslant 0$ and $K(\alpha) \in \mathbb{N}$ such that for every $\mathbf{n} \in \mathcal{E}(\alpha)$, $\delta_{\mathbf{n}}$ has abscissa of convergence $\sigma(\alpha)$ and this is a pole of order $K(\alpha)$.*

Example 4.3 of [6] shows that when the abscissa of convergence of $\delta_{\mathbf{n}}$ is a simple pole away from the origin, $\alpha^{\mathbf{n}}$ can have rather erratic orbit growth which is difficult to describe. Omitting this situation, results from [6] and a fundamental diophantine result of Yu [19] are applied to obtain the following orbit growth asymptotics. As discussed in [6], Tauberian theorems have limited applicability in our setting due to the possibility of arbitrarily close singularities of $\delta_{\mathbf{n}}$ on the line $z = \sigma(\alpha)$, so the methods used are more fundamental.

Theorem 1.3. If $\sigma(\alpha)$ is not a simple pole away from the origin, there exist finite lists of non-negative constants $\{C_g\}_{g\in\mathcal{G}(\gamma)}$, $\gamma \in \Gamma$, such that for any $\mathbf{n} \in \mathcal{E}(\alpha)$

$$\pi_{\mathbf{n}}(N) \sim \left(\sum_{g\in\mathcal{G}(\vartheta(\mathbf{n}))} C_g a_g(\mathbf{n})\right) N^{\sigma(\alpha)}(\log N)^{K(\alpha)-1},$$

where

$$C \leqslant \sum_{g\in\mathcal{G}(\vartheta(\mathbf{n}))} C_g a_g(\mathbf{n}) \leqslant D \max_{1\leqslant i\leqslant d}\{n_i\}^E,$$

for positive constants C, D, E independent of $\mathbf{n}$ and N.

The standard ×2 ×3 example (Example 3.1) is, in a very natural sense, the simplest example from the family of dynamical systems $\{(X,\alpha)\}$, where X is any solenoid on which the maps $x \mapsto 2x$ and $x \mapsto 3x$ are invertible and $\alpha^{\mathbf{n}}(x) = 2^{n_1}3^{n_2}x$, $x \in X$, $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$. This family will be referred to as the ×2 ×3 *family.* To contrast with the situation for the standard ×2 ×3 example, Theorems 1.1, 1.2 and 1.3 will be illustrated using other examples from the ×2 ×3 family.

2. Algebraic background

Let X be a compact metrizable abelian group, α an algebraic $\mathbb{Z}^d$-action on X and $M = \widehat{X}$ the Pontryagin dual of X. Then M becomes a module over the ring of Laurent polynomials $R_d = \mathbb{Z}[u_1^{\pm1}, \ldots, u_d^{\pm1}]$ by identifying the dual automorphism $\widehat{\alpha}^{\mathbf{n}}$ with multiplication by $u^{\mathbf{n}} = u_1^{n_1}\cdots u_d^{n_d}$ and extending this in a natural way to polynomials. Conversely, for any countable R_d-module M, there is an associated $\mathbb{Z}^d$-action α_M on the compact group $X_M = \widehat{M}$ obtained by dualizing the action induced by multiplying by monomials on M. Note that $X = X_M$ is a solenoid if and only if M is a subgroup of $\mathbb{Q}^r$ for some $r \in \mathbb{N}$. The action α is *irreducible* if every closed α-invariant subgroup $Y \subset X$ is finite. If β is another algebraic $\mathbb{Z}^d$-action on a compact abelian group Y, then β and α are *algebraically conjugate* if there exists an isomorphism $\phi : X \to Y$ such that $\phi \cdot \alpha^{\mathbf{n}} = \beta^{\mathbf{n}} \cdot \phi$, for all $\mathbf{n} \in \mathbb{Z}^d$. The action α is *expansive* if there exists a neighbourhood U of 0 in X such that $\bigcap_{\mathbf{n}\in\mathbb{Z}^d} \alpha^{\mathbf{n}}(U) = \{0\}$. A full introduction to algebraic $\mathbb{Z}^d$-actions is given in Schmidt's monograph [16].

Proposition 2.1. Let α be an algebraic $\mathbb{Z}^d$-action on a solenoid X and let M be the dual module.

(i) The topological entropy $h(\alpha^{\mathbf{n}})$ is finite for all $\mathbf{n} \in \mathbb{Z}^d$ (so α is an entropy rank one action in the sense of Einsiedler and Lind [4]).
(ii) The set of associated primes $\mathrm{Ass}(M)$ is finite and comprises prime ideals $\mathfrak{p} \subset R_d$ such that $R_d/\mathfrak{p}$ has krull dimension $\mathrm{kdim}(R_d/\mathfrak{p}) = 1$.
(iii) The action α is expansive if and only if M is Noetherian and for each $\mathfrak{p} \in \mathrm{Ass}(M)$ the complex variety

$$V_{\mathbb{C}}(\mathfrak{p}) = \{z \in (\mathbb{C}^{\times})^d : f(z) = 0 \text{ for all } f \in \mathfrak{p}\}$$

does not intersect $\mathbb{S}^d$, where $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$.
(iv) The automorphism $\alpha^{\mathbf{n}}$ is ergodic if and only if $(u^{k\mathbf{n}} - 1)x \neq 0$ for all $k \in \mathbb{N}$ and all non-zero $x \in M$. Equivalently, no $\mathfrak{p} \in \mathrm{Ass}(M)$ contains a polynomial of the form $u^{k\mathbf{n}} - 1$, with $k \in \mathbb{N}$.

Proof. It is well known that any automorphism of a solenoid has finite entropy. For (ii), a standard dimension argument using duality shows that $\mathrm{Ass}(M)$ must be finite and the methods of [4, Sec. 6] show $\mathrm{kdim}(R_d/\mathfrak{p}) \leqslant 1$ for every $\mathfrak{p} \in \mathrm{Ass}(M)$. Since $R_d/\mathfrak{p}$ embeds in M when $\mathfrak{p} \in \mathrm{Ass}(M)$ and since $R_d/\mathfrak{p}$ is a finite field when $\mathrm{kdim}(R_d/\mathfrak{p}) = 0$, we must have $\mathrm{kdim}(R_d/\mathfrak{p}) = 1$, as M is a torsion-free abelian group. The proofs of statements (iii) and (iv) may be found in [16, Th. 6.5]. □

This paper deals with non-expansive actions. Excepting Example 3.1, all the actions considered will be non-expansive because of a violation of the Noetherian condition in (iii). Lemma 4.2 provides further details.

The following notation and terminology are fixed throughout. The *finite places* of an algebraic number field $\mathbb{K}$ are the equivalence classes of non-archimedean absolute values on $\mathbb{K}$. To each such place v there corresponds a unique discrete valuation ring $\mathfrak{R}_v \subset \mathbb{K}$ with unique maximal ideal $\mathfrak{m}_v$ and finite residue class field $\mathfrak{K}_v = \mathfrak{R}_v/\mathfrak{m}_v$. For any $x \in \mathfrak{K}_v^{\times}$, let $\omega(x)$ denote the (multiplicative) order of x. By slight abuse of notation, $v(\cdot)$ is also used for the normalized valuation and the associated normalized absolute value is $|\cdot|_v = |\mathfrak{K}_v|^{-v(\cdot)}$. A homomorphism $\chi : \mathbb{Z}^d \to \mathbb{K}^{\times}$ is *unitary* at v if $\{|\chi(\mathbf{n})|_v : \mathbf{n} \in \mathbb{Z}^d\} = \{1\}$. If $x \in \mathfrak{R}_v^{\times}$, write $\overline{x}$ for the image of x in $\mathfrak{K}_v^{\times}$ under the natural map. If χ is unitary at v, there is an induced homomorphism $\overline{\chi} : \mathbb{Z}^d \to \mathfrak{K}_v^{\times}$ given by $\overline{\chi}(\mathbf{n}) = \overline{\chi(\mathbf{n})}$. For further relevant background in algebraic number theory, see [3].

3. The ×2 ×3 family

One of the most fundamental algebraic $\mathbb{Z}^2$-actions is given by the standard ×2 ×3 example, described below.

Example 3.1 (the standard ×2 ×3 example)**.** Let X be the one-dimensional solenoid dual to the ring $\mathbb{Z}[\frac{1}{6}]$. Then X is locally isomorphic to $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$. By duality, it is readily seen that the maps $x \mapsto 2x$ and $x \mapsto 3x$ on X are invertible and hence generate an action α of $\mathbb{Z}^2$. Furthermore, α is expansive, mixing and the entropies $h(\alpha^{\mathbf{n}})$, $\mathbf{n} \in \mathbb{Z}^d$, may be calculated using [4, Prop. 8.5].

According to the expansive subdynamics theory for algebraic $\mathbb{Z}^d$-actions [5, Ex. 2.11], there are 6 expansive components for this action, delineated by the intersecting lines $|2|_v^{n_1}|3|_v^{n_2} = 1$ in $\mathbb{R}^2$, where $|\cdot|_v$ runs through the 2-adic, 3-adic and infinite absolute values of $\mathbb{Q}$ (note the components do not contain points on the bounding lines). Starting in the first quadrant and working anticlockwise, label these $C_1, \dots, C_6$. Example 4.6 of [11] gives a parameterization of dynamical zeta functions $\zeta_{\mathbf{n}}$ corresponding to these components, as shown in Table 1. Using the dynamical zeta functions or the periodic point formulae from [11, Ex. 4.6] directly (see for example [7, Lem. 2.1]), one may calculate asymptotic expressions for the orbit counting functions $\pi_{\mathbf{n}}(N)$. These are also shown in Table 1.

$\mathbf{n} = (n_1, n_2)$	$h(\alpha^{\mathbf{n}})$	$\zeta_{\mathbf{n}}(z)$	$\pi_{\mathbf{n}}(N) \sim$
$\mathbf{n} \in C_1 \cup C_4$	$\log 2^{\lvert n_1\rvert}3^{\lvert n_2\rvert}$	$(1-z)/(1-2^{\lvert n_1\rvert}3^{\lvert n_2\rvert}z)$	$\dfrac{(2^{\lvert n_1\rvert}3^{\lvert n_2\rvert})^{N+1}}{(2^{\lvert n_1\rvert}3^{\lvert n_2\rvert}-1)N}$
$\mathbf{n} \in C_2 \cup C_5$	$\log 3^{\lvert n_2\rvert}$	$(1-2^{\lvert n_1\rvert}z)/(1-3^{\lvert n_2\rvert}z)$	$\dfrac{3^{\lvert n_2\rvert(N+1)}}{(3^{\lvert n_2\rvert}-1)N}$
$\mathbf{n} \in C_3 \cup C_6$	$\log 2^{\lvert n_1\rvert}$	$(1-3^{\lvert n_2\rvert}z)/(1-2^{\lvert n_1\rvert}z)$	$\dfrac{2^{\lvert n_1\rvert(N+1)}}{(2^{\lvert n_1\rvert}-1)N}$

TABLE 1. The functions $\zeta_{\mathbf{n}}$ and $\pi_{\mathbf{n}}$ for Example 3.1.

There are uncountably many solenoids on which the commuting maps $x \mapsto 2x$ and $x \mapsto 3x$ are invertible. The subsequent family of $\mathbb{Z}^2$-actions that are generated comprise the ×2 ×3 family described in the introduction. Up to algebraic conjugacy, the above example is the unique irreducible action in this family. Irreducibility of Example 3.1 is an instance of [17, Th. 2.6]. To prove uniqueness, suppose M is an R_2-module dual to an irreducible system in the ×2 ×3 family. Then $\mathfrak{p} = (u_1 - 2, u_2 - 3)$ is the only associated prime of M and by irreducibility and duality, $L = R_2/\mathfrak{p}$ is a finite index submodule of M. Identifying L with the ring $\mathbb{Z}[1/6]$ in an obvious way, it follows that M can be considered both as a subgroup of $\mathbb{Q}$ and as a finitely generated L-module, by $a_1, \cdots, a_r \in \mathbb{Q}$ say. If c denotes the product of the denominators of $a_1, \dots, a_r$, we may write $M = \frac{b}{c}L$, where $b = \gcd(ca_1, \dots, ca_r)$ and the map from M to L defined by $\frac{b}{c}x \mapsto x$ gives an R_2-module isomorphism. That is, (X_M, α_M) and (X_L, α_L) are algebraically conjugate.

Recall that we will be interested in situations where the set of rational primes $\mathcal{P}(\alpha^{\mathbf{n}})$ given by (1) is finite for some (equivalently all) $\mathbf{n} \in \mathcal{E}(\alpha)$. Unlike the standard ×2 ×3 example, such actions are non-expansive (see Lemma 4.2).

Example 3.2 (an action in the ×2 ×3 family with $\mathcal{P}(\alpha^{\mathbf{n}}) = \{5\}$, $\mathbf{n} \neq 0$)**.** Let X be the one-dimensional solenoid dual to the ring $\mathbb{Z}_{(5)}$, the 5-adic valuation ring in $\mathbb{Q}$. Again by duality, the maps $x \mapsto 2x$ and $x \mapsto 3x$ on X are invertible and generate an action α of $\mathbb{Z}^2$. This time α is mixing, but the action is non-expansive.

Suppose $\mathbf{n} \neq 0$ and $F \subset X$ is the closed subgroup of points fixed by $\alpha^{j\mathbf{n}}$, $j \in \mathbb{N}$. Then $F \cong \widehat{F} \cong \mathbb{Z}_{(5)}/\mathfrak{a}_F$ for some ideal $\mathfrak{a}_F \neq (0)$. If $\mathfrak{a}_F \neq \mathbb{Z}_{(5)}$ then $\mathbb{Z}_{(5)}/\mathfrak{a}_F$ is a finite-dimensional vector space over $\mathbb{F}_5$ and so $|F| = |\mathbb{Z}_{(5)}/\mathfrak{a}_F|$ is a power of 5. Therefore, $\mathcal{P}(\alpha^{\mathbf{n}}) = \{5\}$. Let $\vartheta : \mathbb{Z}^2 \to (\mathbb{Z}_{(5)}/(5))^\times \cong \mathbb{F}_5^\times$ be given by $\vartheta(\mathbf{n}) = 2^{n_1}3^{n_2} \mod 5$. The structure of the resulting partition of $\mathcal{E}(\alpha) = \mathbb{Z}^2 \setminus \{0\}$ into cosets is shown below. This turns out to subsequently describe the pattern of dynamical Dirichlet series.

$$\begin{array}{ccccccccc}
1 & 2 & 4 & 3 & 1 & 2 & 4 & 3 & 1 \\
2 & 4 & 3 & 1 & 2 & 4 & 3 & 1 & 2 \\
4 & 3 & 1 & 2 & 4 & 3 & 1 & 2 & 4 \\
3 & 1 & 2 & 4 & 3 & 1 & 2 & 4 & 3 \\
1 & 2 & 4 & 3 & \square & 2 & 4 & 3 & 1 \\
2 & 4 & 3 & 1 & 2 & 4 & 3 & 1 & 2 \\
4 & 3 & 1 & 2 & 4 & 3 & 1 & 2 & 4 \\
3 & 1 & 2 & 4 & 3 & 1 & 2 & 4 & 3 \\
1 & 2 & 4 & 3 & 1 & 2 & 4 & 3 & 1
\end{array}$$

A precise calculation of vector space dimensions gives,

$$\begin{aligned}
\mathsf{F}_k(\alpha^{\mathbf{n}}) &= |(2^{n_1}3^{n_2})^k - 1|_5^{-1} \\
&= \begin{cases} |k|_5^{-1}|(2^{n_1}3^{n_2})^{\omega(\vartheta(\mathbf{n}))} - 1|_5^{-1} & \text{if } \omega(\vartheta(\mathbf{n}))|k, \\ 1 & \text{otherwise,} \end{cases}
\end{aligned} \tag{4}$$

where $\omega(x)$ is the order of $x \in \mathbb{F}_5^\times$.

It can be shown that $\zeta_{\mathbf{n}}(z)$ is irrational with natural boundary on the circle of convergence $|z| = 1$. An appropriate method and details of typical expressions that arise may be found in [8]. The Dirichlet series, $\delta_{\mathbf{n}}(z)$ is much better behaved. Using formal Dirichlet convolution [18, Sec. 3.7], we have

$$\delta_T(z) = \frac{1}{\zeta(z+1)} \sum_{k=1}^{\infty} \frac{\mathsf{F}_k(T)/k}{k^z}, \tag{5}$$

where $\zeta(z+1) = \sum_{k=1}^{\infty}(1/k)k^{-z}$. The Dirichlet series $\delta_{\mathbf{n}}$ may be calculated by substituting the periodic point formula (4) into (5); an essential observation is that $\zeta(z+1)$ may subsequently be extracted as a factor of the series on the right hand side of (5) and cancelled with its reciprocal. Exploiting techniques from [6], a general method is explained in the next section.

Table 2 shows all the dynamical Dirichlet series and corresponding orbit growth estimates for this example. The directional entropies are the same as in the previous example and have no significant relation to the orbit growth structure, so these are omitted.

$\vartheta(\mathbf{n})$	$\delta_{\mathbf{n}}(z)$	$\pi_{\mathbf{n}}(N) \sim$
1	$\lvert 2^{n_1}3^{n_2}-1\rvert_5^{-1}\dfrac{1-5^{-z-1}}{1-5^{-z}}$	$\dfrac{\lvert 2^{n_1}3^{n_2}-1\rvert_5^{-1}}{\log 5}\log N$
2,3	$1-\dfrac{1}{4^{z+1}}\left(1-\lvert(2^{n_1}3^{n_2})^4-1\rvert_5^{-1}\dfrac{1-5^{-z-1}}{1-5^{-z}}\right)$	$\dfrac{\lvert(2^{n_1}3^{n_2})^4-1\rvert_5^{-1}}{\log 5}\log N$
4	$1-\dfrac{1}{2^{z+1}}\left(1-\lvert(2^{n_1}3^{n_2})^2-1\rvert_5^{-1}\dfrac{1-5^{-z-1}}{1-5^{-z}}\right)$	$\dfrac{\lvert(2^{n_1}3^{n_2})^2-1\rvert_5^{-1}}{\log 5}\log N$

TABLE 2. The functions $\delta_{\mathbf{n}}$ and $\pi_{\mathbf{n}}$ for Example 3.2.

4. PROOFS OF MAIN RESULTS

To begin, a periodic point counting formula is required. This is obtained by extending the method of [12] to cover $\mathbb{Z}^d$-actions. The full proof is quite involved and only the necessary modifications of [12] are included here.

Proposition 4.1. Let α be an algebraic $\mathbb{Z}^d$-action on a solenoid X. Then there exist algebraic number fields $\mathbb{K}_1,\dots,\mathbb{K}_r$, sets of finite places $\mathcal{T}_1,\dots,\mathcal{T}_r$ and homomorphisms $\chi_i:\mathbb{Z}^d\to\mathbb{K}_i^\times$, such that χ_i is unitary at every $v\in\mathcal{T}_i$, $1\leqslant i\leqslant r$, and

$$\mathsf{F}_k(\alpha^{\mathbf{n}})=\prod_{i=1}^{r}\prod_{v\in\mathcal{T}_i}\lvert\chi_i(\mathbf{n})^k-1\rvert_v^{-1}, \tag{6}$$

for all $\mathbf{n}\in\mathcal{E}(\alpha)$ and $k\in\mathbb{N}$.

Proof. Let $M=\widehat{X}$ be the dual R_d-module. By Proposition 2.1(ii), $\mathrm{Ass}(M)$ is finite and consists of primes $\mathfrak{p}\subset R_d$ with $\mathrm{kdim}(R_d/\mathfrak{p})=1$. Since M is a torsion-free group, for each $\mathfrak{p}\in\mathrm{Ass}(M)$, $\mathfrak{p}\cap\mathbb{Z}=\{0\}$ and it follows that the field of fractions $\mathbb{K}(\mathfrak{p})$ of $R_d/\mathfrak{p}$ is an algebraic number field.

Let $U=\mathbb{Z}\backslash\{0\}$. Localizing at U, the natural map $M\to U^{-1}M$ is injective (since M is a torsion-free group) and $U^{-1}M$ is a Noetherian module over $Q_d=\mathbb{Q}[u_1^{\pm1},\dots,u_d^{\pm1}]$ (since M has finite rational rank). Furthermore, there is a prime filtration

$$\{0\}=M_0\subset M_1\subset\cdots\subset M_r=U^{-1}M, \tag{7}$$

in which $M_i/M_{i-1}\cong Q_d/\mathfrak{q}_i$, for a list of maximal ideals $\mathfrak{q}_1,\dots,\mathfrak{q}_r\subset Q_d$ with $\mathfrak{p}_i=\mathfrak{q}_i\cap R_d\in\mathrm{Ass}(M)$ for all $1\leqslant i\leqslant r$. Identifying M with its image in $U^{-1}M$ and intersecting the chain (7) with M gives a chain

$$\{0\}=L_0\subset L_1\subset\cdots\subset L_r=M. \tag{8}$$

Considering (8) as a chain of R_d-modules, for each $1\leqslant i\leqslant r$, there is an induced inclusion

$$\frac{L_i}{L_{i-1}}\hookrightarrow\frac{M_i}{M_{i-1}}\cong\frac{Q_d}{\mathfrak{q}_i}\cong\mathbb{K}_i=\mathbb{K}(\mathfrak{p}_i),$$

via which $N_i = L_i/L_{i-1}$ naturally becomes a sub-$R_d/\mathfrak{p}_i$-module of $\mathbb{K}(\mathfrak{p}_i)$. Define a homomorphism $\chi_i : \mathbb{Z}^d \to \mathbb{K}_i^\times$ by $\chi_i(\mathbf{n}) = \overline{u}_1^{n_1} \cdots \overline{u}_d^{n_d}$, where $\overline{u}_j$ is the image of u_j in $R_d/\mathfrak{p}_i$, $1 \leqslant j \leqslant d$. Note that since $\mathbf{n} \in \mathcal{E}(\alpha)$, Proposition 2.1(iv) implies that

$$\chi_i(\mathbf{n})^k \neq 1 \tag{9}$$

for each $i = 1, \ldots, r$ and every $k \in \mathbb{N}$. Proceeding exactly as in [12, Sec. 4], we obtain (6), where $\mathcal{T}_i$ is the set of normalized finite places v of $\mathbb{K}_i$ such that $|N_i|_v$ is a bounded subset of $\mathbb{R}$ (see [12, Rem. 1]).

Finally, to see that each χ_i is unitary at every $v \in \mathcal{T}_i$, notice if $|\chi_i(\mathbf{n})|_v \neq 1$ then either $|\chi_i(\mathbf{n})|_v > 1$ or $|\chi_i(\mathbf{n})^{-1}|_v > 1$, so $|\mathbb{Z}[\chi_i(\mathbf{n})^{\pm 1}]|_v$ is an unbounded subset of $\mathbb{R}$. But $\mathbb{Z}[\chi_i(\mathbf{n})^{\pm 1}]$ is embedded in N_i, so this is a contradiction. □

Lemma 4.2. If α is an algebraic $\mathbb{Z}^d$-action on a solenoid X and $\mathcal{P}(\alpha^{\mathbf{m}})$ is finite for some $\mathbf{m} \in \mathcal{E}(\alpha)$ then the sets of places $\mathcal{T}_1, \ldots, \mathcal{T}_r$ appearing in (6) are all finite, $\mathcal{P}(\alpha^{\mathbf{n}})$ is finite for all $\mathbf{n} \in \mathcal{E}(\alpha)$ and α is non-expansive.

Proof. The notation here is the same as that used in the proof of Proposition 4.1.

First notice that all the places appearing in (6) are normalized, so for any $\mathbf{n} \in \mathcal{E}(\alpha)$, each factor appearing in the product is a positive integer. For a contradiction suppose $\mathcal{T}_i$ is infinite for some $1 \leqslant i \leqslant r$. For each $v \in \mathcal{T}_i$, let p_v be the characteristic of $\mathfrak{K}_v$ and for any finite subset $\mathcal{S} \subset \mathcal{T}_i$ let

$$k_{\mathcal{S}} = \prod_{v \in \mathcal{S}} \omega(\overline{\chi}_i(\mathbf{m})).$$

If $x \in \mathfrak{R}_v^\times$ then p_v divides $|x^k - 1|_v^{-1}$ whenever $\omega(\overline{x})$ divides k. Therefore, $\prod_{v \in \mathcal{S}} p_v$ divides $\prod_{v \in \mathcal{S}} |\chi_i(\mathbf{m})^{k_{\mathcal{S}}} - 1|_v^{-1}$ which in turn divides $\mathsf{F}_{k_{\mathcal{S}}}(\alpha^{\mathbf{m}})$ by (6). Since in any algebraic number field there are only finitely many places extending a given place of $\mathbb{Q}$, by choosing arbitrarily large $\mathcal{S}$, this shows $\mathcal{P}(\alpha^{\mathbf{m}})$ is infinite which is a contradiction. Consequently, (2) holds.

The action α is non-expansive because of the failure of the Noetherian condition in Proposition 2.1(iii). To see this, consider (8) and note that if M is a Noetherian R_d-module then so is each $N_i = L_i/L_{i-1}$. Therefore, the group $N_i \subset \mathbb{K}_i$ is a finitely generated $R_d/\mathfrak{p}_i$-module and the finite places v of $\mathbb{K}_i$ for which $|N_i|_v$ is an unbounded subset of $\mathbb{R}$ coincide with those for which $|R_d/\mathfrak{p}_i|_v$ is an unbounded subset of $\mathbb{R}$. Since $R_d/\mathfrak{p}_i$ is finitely generated, there can be only finitely many such v. Therefore, the sets $\mathcal{T}_1, \ldots, \mathcal{T}_r$ appearing in (6) are all infinite which is a contradiction. □

We now construct the fundamental homomorphism ϑ appearing in the statement of Theorem 1.1. For ease of notation, from now on denote the complete list of places appearing in the product (6) simply by $\mathcal{T}$ and for all $v \in \mathcal{T}_i$, write $\chi_v = \chi_i$, $1 \leqslant i \leqslant r$. By Lemma 4.2, the hypothesis of the theorem implies that $\mathcal{T}$ is finite and formula (6) becomes

$$\mathsf{F}_k(\alpha^{\mathbf{n}}) = \prod_{v \in \mathcal{T}} |\chi_v(\mathbf{n})^k - 1|_v^{-1}. \tag{10}$$

Set

$$\Gamma = \prod_{v \in \mathcal{T}} \overline{\chi}_v(\mathbb{Z}^d) \subset \prod_{v \in \mathcal{T}} \mathfrak{K}_v^\times \tag{11}$$

and define $\vartheta : \mathbb{Z}^d \to \Gamma$ by

$$\vartheta(\mathbf{n})_v = \overline{\chi}_v(\mathbf{n}). \tag{12}$$

In order to proceed with the proof of Theorem 1.1, it is necessary to obtain a more useful expression for each of the factors appearing in (6). Such an expression appears in [11], but slightly more detail is required here. Notice that for any finite place v and $x \in \mathfrak{R}_v^\times$,

$$|x^k - 1|_v \neq 1 \Leftrightarrow \omega(\overline{x})|k. \tag{13}$$

Lemma 4.3. Let $x \in \mathfrak{R}_v^\times$ and let $p = \mathrm{char}(\mathfrak{K}_v)$. Set $k(e,j) = \omega(\overline{x})p^e j$ and $y(x,e) = x^{k(j,e)/j} - 1$, where $e \geqslant 0$ and $j \geqslant 1$. If $p \nmid j$ then

$$|x^{k(e,j)} - 1|_v = |y(x,e)|_v$$

and if e_0 is any integer constant exceeding $v(p)$, for all $e > e_0$,

$$|y(x,e)|_v = p^{r_v(e_0 - e)}|y(x,e_0)|_v,$$

where $r_v = \deg(\mathfrak{K}_v|\mathbb{F}_p)v(p)$.

Proof. Let $j \in \mathbb{N}$. Since $|y(x,e)|_v < 1$,

$$|x^{k(j,e)} - 1|_v = |(1 + y(x,e))^j - 1|_v = |jy(x,e) + \mathrm{O}(y(x,e)^2)|_v. \tag{14}$$

If $p \nmid j$ then by the ultrametric property this is $|y(x,e)|_v$. If $j = p$, it follows from (14) that

$$|y(x,e+1)|_v = |x^{k(p,e)} - 1|_v = p^{-r_v}|y(x,e)|_v, \tag{15}$$

provided $|y(x,e)|_v < |p|_v$. But, for some $z(x,e) \in \mathfrak{R}_v$,

$$|y(x,e)|_v = |(1 + y(x,0))^{p^e} - 1|_v = |y(x,0)^{p^e} + p^e z(x,e)|_v$$

and the right hand side is at most $\max\{|y(x,0)|_v^{p^e}, |p|_v^e\} \leqslant |\mathfrak{K}_v|^{-e} < |p|_v$, whenever $e > v(p)$. Therefore, starting with $e = e_0$, (15) gives the required expression for $|y(x,e)|_v$ by induction. □

The next step is to rewrite $\delta_{\mathbf{n}}$ as in (5) and use (10) and Lemma 4.3 to evaluate $\mathsf{F}_k(\alpha^{\mathbf{n}})$.

Proof of Theorem 1.1. For any $\mathcal{S} \subset \mathcal{T}$, let

$$f(\mathcal{S}, \mathbf{n}, k) = \prod_{v \in \mathcal{S}} |\chi_v(\mathbf{n})^k - 1|_v^{-1}$$

and for any $\gamma = (\gamma_v) \in \Gamma$, set $\ell(\mathcal{S}, \gamma) = \mathrm{lcm}\{\omega(\gamma_v) : v \in \mathcal{S}\}$ and

$$N(\mathcal{S}, \gamma) = \{k \in \mathbb{N} : \ell(\mathcal{S}, \gamma)|k \text{ and } \omega(\gamma_v) \nmid k \text{ for all } v \in \mathcal{T} \setminus \mathcal{S}\}.$$

Set $\gamma = \vartheta(\mathbf{n})$. By (10), $\mathsf{F}_k(\alpha^{\mathbf{n}}) = f(\mathcal{T}, \mathbf{n}, k)$, so (5) and (13) give

$$\delta_{\mathbf{n}}(z) = \zeta(z+1)^{-1} \sum_{\mathcal{S} \subset \mathcal{T}} \sum_{k \in N(\mathcal{S}, \gamma)} \frac{f(\mathcal{S}, \mathbf{n}, k)}{k^{z+1}}. \tag{16}$$

Let $Q(\mathcal{S}) = \{\mathrm{char}(\mathfrak{K}_v) : v \in \mathcal{S}\}$ and $E(\mathcal{S}) = \mathbb{Z}_{\geqslant 0}^{Q(\mathcal{S})}$. For $\mathbf{e} \in E(\mathcal{S})$ and $P \subset Q(\mathcal{S})$, define

$$\varphi_P(\mathbf{e}) = \prod_{p \in P} p^{e_p}.$$

Let e_0 be a positive integer constant exceeding $\max_{v\in\mathcal{T}}\{v(\mathrm{char}(\mathfrak{K}_v)) + 1\}$ and let $\mathbf{e}_0 \in E(\mathcal{S})$ denote the vector with each entry equal to e_0. For any $P \subset Q(\mathcal{S})$, set

$$E(\mathcal{S}, P) = \{\mathbf{e} \in E(\mathcal{S}) : e_p \geqslant e_0 \text{ for all } p \in P, e_p < e_0 \text{ for all } p \in Q(\mathcal{S}) \setminus P\},$$

so the sets $\{E(\mathcal{S}, P) : P \subset Q(\mathcal{S})\}$ partition $E(\mathcal{S})$. For each q in the finite set $\varphi_{Q(\mathcal{S})\setminus P}(E(\mathcal{S}, P))$, let

$$E(\mathcal{S}, P, q) = \{\mathbf{e} \in E(\mathcal{S}, P) : \varphi_{Q(\mathcal{S})\setminus P}(\mathbf{e}) = q\},$$

and

$$J(\mathcal{S}, P, q, \gamma) = Q(\mathcal{S}) \cup \{\omega(\gamma_v)/\gcd(\omega(\gamma_v), \ell(\mathcal{S}, \gamma)\varphi_P(\mathbf{e_0})q) : v \in \mathcal{T} \setminus \mathcal{S}\}.$$

With P running through subsets of $Q(\mathcal{S})$ and q through $\varphi_{Q(\mathcal{S})\setminus P}(E(\mathcal{S}, P))$, if e_0 is also chosen to be at least as large as $\max_{p\in Q(\mathcal{T})}\{\mathrm{ord}_p(|\Gamma|)\}$ then the sets

$$\Big\{m\ell(\mathcal{S}, \gamma)\varphi_{Q(\mathcal{S})}(\mathbf{e}) : \mathbf{e} \in E(\mathcal{S}, P, q), m \in \mathbb{N}, j \nmid m \text{ for all } j \in J(\mathcal{S}, P, q, \gamma)\Big\}$$

partition $N(\mathcal{S}, \gamma)$. Also, if $\mathbf{e} \in E(\mathcal{S}, P, q)$ and $j \nmid m$ for all $j \in J(\mathcal{S}, P, q, \gamma)$,

$$f(\mathcal{S}, \mathbf{n}, m\ell(\mathcal{S}, \gamma)\varphi_{Q(\mathcal{S})}(\mathbf{e})) = f(\mathcal{S}, \mathbf{n}, \ell(\mathcal{S}, \gamma)\varphi_{Q(\mathcal{S})}(\mathbf{e})),$$

by (13) and Lemma 4.3. Let

$$\Lambda(\gamma) = \{(\mathcal{S}, P, q) : 1 \notin J(\mathcal{S}, P, q, \gamma)\}. \tag{17}$$

Then (16) may be rewritten as

$$\zeta(z+1)^{-1} \sum_{(\mathcal{S},P,q)\in\Lambda(\gamma)} \sum_{\mathbf{e}\in E(\mathcal{S},P,q)} \frac{f(\mathcal{S}, \mathbf{n}, \ell(\mathcal{S}, \gamma)\varphi_{Q(\mathcal{S})}(\mathbf{e}))}{(\ell(\mathcal{S}, \gamma)\varphi_{Q(\mathcal{S})}(\mathbf{e}))^{z+1}} \sum_m \frac{1}{m^{z+1}}, \tag{18}$$

where the inner sum runs through all $m \in \mathbb{N}$ such that $j \nmid m$ for all $j \in J(\mathcal{S}, P, q, \gamma)$. Using inclusion-exclusion, the inner sum is $\zeta(z + 1)$ times

$$g_1(\mathcal{S}, P, q, \gamma, z) = \sum_{I\subset J(\mathcal{S},P,q,\gamma)} \frac{(-1)^{|I|}}{\ell(I)^{z+1}},$$

where $\ell(I) = \mathrm{lcm}\{i : i \in I\}$. Setting $\mathcal{S}(P) = \{v \in \mathcal{S} : \mathrm{char}(\mathfrak{K}_v) \in P\}$, it follows that (18) is the sum over all $(\mathcal{S}, P, q) \in \Lambda(\gamma)$ of

$$a_1(\mathcal{S}, P, q, \mathbf{n}) \frac{g_1(\mathcal{S}, P, q, \gamma, z)}{(q\ell(\mathcal{S}, \gamma))^{z+1}} \sum_{\mathbf{e}\in E(\mathcal{S},P,q)} \frac{f(\mathcal{S}(P), \mathbf{n}, \ell(\mathcal{S}, \gamma)\varphi_P(\mathbf{e}))}{\varphi_P(\mathbf{e})^{z+1}}, \tag{19}$$

where

$$a_1(\mathcal{S}, P, q, \mathbf{n}) = f(\mathcal{S}(Q(\mathcal{S}) \setminus P), \mathbf{n}, q\ell(\mathcal{S}, \gamma)).$$

Using Lemma 4.3, the numerator in the summand in (19) is

$$\prod_{v\in\mathcal{S}(P)} p(v)^{r_v(\mathrm{ord}_{p(v)}(\ell(\mathcal{S},\gamma))+e_{p(v)}-e_0)}|y(\chi_v(\mathbf{n}),e_0)|_v^{-1},$$

where $p(v)=\mathrm{char}(\mathfrak{K}_v)$. Therefore, setting

$$a_2(\mathcal{S},P,q,\mathbf{n})=a_1(\mathcal{S},P,q,\mathbf{n})\prod_{v\in\mathcal{S}(P)}|y(\chi_v(\mathbf{n}),e_0)|_v^{-1},$$

and

$$g_2(\mathcal{S},P,q,\gamma,z)=\frac{g_1(\mathcal{S},P,q,\gamma,z)}{(q\ell(\mathcal{S},\gamma))^{z+1}}\prod_{v\in\mathcal{S}(P)} p(v)^{r_v\,\mathrm{ord}_{p(v)}(\ell(\mathcal{S},\gamma))},$$

(19) becomes

$$\underbrace{a_2(\mathcal{S},P,q,\mathbf{n})}_{a_g(\mathbf{n})}\underbrace{g_2(\mathcal{S},P,q,\gamma,z)\sum_{\mathbf{e}\in E(\mathcal{S},P,q)}\frac{\varphi_P(\Psi(\mathbf{e}))}{\varphi_P(\mathbf{e})^{z+1}}}_{g(z)},\tag{20}$$

where $\Psi(\mathbf{e})_p=(e_p-e_0)(\Delta(\mathcal{S},p)+1)$ with

$$\Delta(\mathcal{S},p)=-1+\sum_{v\in\mathcal{S}(p)} r_v.\tag{21}$$

Furthermore,

$$\sum_{\mathbf{e}\in E(\mathcal{S},P,q)}\frac{\varphi_P(\Psi(\mathbf{e}))}{\varphi_P(\mathbf{e})^{z+1}}=\frac{1}{\varphi_P(\mathbf{e}_0)^{z+1}\prod_{p\in P}1-p^{\Delta(\mathcal{S},p)-z}}.$$

Substituting this into (20) shows $\delta_{\mathbf{n}}(z)$ is of the form (3). □

To prove Theorems 1.2 and 1.3, it is useful to clarify the form of the functions $g(z)$ and the coefficients $a_g(\mathbf{n})$ appearing in (3). These are as shown in (20); the appropriate list $\mathcal{G}(\gamma)$ in (3) is obtained by letting $(\mathcal{S},P,q)$ run through all elements of $\Lambda(\gamma)$, the set defined in (17). Thus, for each $g\in\mathcal{G}(\gamma)$, there is a finite list of places $\mathcal{S}_g$, a finite set of primes P_g, positive integers A_g,B_g and finite sets J_g of integers greater than 1, such that

$$g(z)=\frac{A_g}{\prod_{p\in P_g}1-p^{\Delta(\mathcal{S}_g,p)-z}}\sum_{I\subset J_g}\frac{(-1)^{|I|}}{(B_g\ell(I))^{z+1}},\tag{22}$$

where $\Delta(\mathcal{S}_g,p)$ is given by (21) and $\ell(I)=\mathrm{lcm}\{i:i\in I\}$. Additionally, for each $v\in\mathcal{S}_g$, there exist positive integer constants $c_g(v)$ such that

$$a_g(\mathbf{n})=\prod_{v\in\mathcal{S}_g}|\chi_v(\mathbf{n})^{c_g(v)}-1|_v^{-1}.\tag{23}$$

Remark 4.4. It is helpful to note that $a_g(\mathbf{n})$ is always a positive integer. This follows from the fact that each χ_v is unitary and (9), which follows from the ergodicity assumption. We will also use the fact that (17) implies that, irrespective of γ, there is always a function $g\in\mathcal{G}(\gamma)$ with $\mathcal{S}_g=\mathcal{T}$ (the complete finite list of places appearing in (10)) and $P_g=\{\mathrm{char}(\mathfrak{K}_v):v\in\mathcal{T}\}$.

Proof of Theorem 1.2. Let

$$\sigma_g = \max\{\Delta(\mathcal{S}_g, p) : p \in P_g\} \text{ and } K_g = |\{p \in P_g : \Delta(\mathcal{S}_g, p) = \sigma_g\}|.$$

The Dirichlet series for $\prod_{p\in P_g}(1 - p^{\Delta(\mathcal{S}_g,p)-z})^{-1}$ has abscissa of convergence σ_g and this is a pole of order K_g. The coefficient of $(z-\sigma_g)^{K_g}$ in the corresponding Laurent series about σ_g is $D_g = \prod_{p\in P_g} D_g(p)$, where

$$D_g(p) = \begin{cases} (\log p)^{-1} & \text{if } \Delta(\mathcal{S}_g, p) = \sigma_g, \\ (1 - p^{\Delta(\mathcal{S}_g,p)-\sigma_g})^{-1} & \text{if } \Delta(\mathcal{S}_g, p) < \sigma_g. \end{cases}$$

Hence, the coefficient of $(z-\sigma_g)^{K_g}$ in the Laurent series for $g(z)$ is

$$\begin{aligned} A_g D_g \sum_{I\subset J_g} \frac{(-1)^{|I|}}{(B_g \ell(I))^{\sigma_g+1}} &= \frac{A_g D_g}{B_g^{\sigma_g+1}} \sum_{I\subset J_g} \frac{(-1)^{|I|}}{\ell(I)^{\sigma_g+1}} \\ &= \frac{A_g D_g}{B_g^{\sigma_g+1}} \sum_{I\subset J_g'} \frac{(-1)^{|I|}}{\ell(I)}, \end{aligned}$$

where $J_g' = \{j^{\sigma_g+1} : j \in J_g\}$. Since J_g' consists of integers strictly greater than 1, the result of the sum is positive and hence so too is the coefficient of $(z-\sigma_g)^{K_g}$ in the Laurent series for $g(z)$. Thus, by (3), the abscissa of convergence of $\delta_{\mathbf{n}}$ is

$$\sigma(\alpha) = \max\{\sigma_g : g \in \mathcal{G}(\vartheta(\mathbf{n}))\}$$

and $\delta_{\mathbf{n}}$ has a pole of order

$$K(\alpha) = \max\{K_g : \sigma_g = \sigma(\alpha)\}$$

at $z = \sigma(\alpha)$. To see that these constants depend only on α, as pointed out in Remark 4.4, irrespective of γ there is always a function $h \in \mathcal{G}(\gamma)$ with $\mathcal{S}_h = \mathcal{T}$ and $P_h = Q(\mathcal{T}) = \{\mathrm{char}(\mathfrak{K}_v) : v \in \mathcal{T}\}$. Morevover,

$$\sigma(\alpha) = \sigma_h$$

and

$$K(\alpha) = |\{p \in Q(\mathcal{T}) : \Delta(\mathcal{T}, p) = \sigma(\alpha)\}|.$$

□

Proof of Theorem 1.3. Let

$$\mathcal{G}'(\vartheta(\mathbf{n})) = \{g \in \mathcal{G}(\vartheta(\mathbf{n})) : g(z) \text{ has a pole of order } K(\alpha) \text{ at } \sigma(\alpha)\}.$$

Write $g(z) = \sum_{k\geqslant 1} a_k/k^z$. If $g \in \mathcal{G}'(\vartheta(\mathbf{n}))$ then [6, Sec. 7] shows that there exists $C_g > 0$ such that

$$\sum_{k\leqslant N} a_k \sim C_g N^{\sigma(\alpha)}(\log N)^{K(\alpha)-1},$$

and if $g \notin \mathcal{G}'(\vartheta(\mathbf{n}))$ then

$$\sum_{k\leqslant N} a_k \ll N^s(\log N)^k,$$

where either $s < \sigma(\alpha)$ or $s = \sigma(\alpha)$ with $k < K(\alpha) - 1$. It follows that

$$\pi_{\mathbf{n}}(N) \sim \left(\sum_{g \in \mathcal{G}(\vartheta(\mathbf{n}))} C_g a_g(\mathbf{n}) \right) N^{\sigma(\alpha)} (\log N)^{K(\alpha)-1},$$

where $C_g = 0$ for all $g \notin \mathcal{G}'(\vartheta(\mathbf{n}))$. Note that $\mathcal{G}'(\vartheta(\mathbf{n}))$ is non-empty by Remark 4.4 and the coefficient on the right hand side is at least

$$\sum_{g \in \mathcal{G}(\vartheta(\mathbf{n}))} C_g > 0.$$

Hence to obtain the lower bound given in the statement of the theorem, simply set $C = \min_{\gamma \in \Gamma} \{ \sum_{g \in \mathcal{G}'(\gamma)} C_g \}$.

Noting that $d \geqslant 2$ by assumption, given any homomorphism ξ from $\mathbb{Z}^d$ into the multiplicative group of an algebraic number field $\mathbb{K}^\times$, Yu's p-adic bounds for linear forms in logarithms [19] show that there exist positive constants $D(\xi)$, $E(\xi)$ such that

$$|\xi(\mathbf{n}) - 1|_v^{-1} \leqslant D(\xi) \max_{1 \leqslant i \leqslant d} \{n_i\}^{E(\xi)},$$

where v is any finite place of $\mathbb{K}$. Therefore, it follows from (23) that there exist positive constants D, E independent of $\mathbf{n}$ such that

$$\sum_{g \in \mathcal{G}(\vartheta(\mathbf{n}))} C_g a_g(\mathbf{n}) \leqslant D \max_{1 \leqslant i \leqslant d} \{n_i\}^{E}.$$

□

5. A more representative example

Example 3.2 has much simpler asymptotics for $\pi_{\mathbf{n}}(N)$, $\mathbf{n} \in \mathcal{E}(\alpha)$, than those suggested by Theorem 1.3. An example which is more representative of the general situation is the following member of the $\times 2 \times 3$ family.

Example 5.1. Consider the quadratic extension $\mathbb{Q}(\sqrt{5})|\mathbb{Q}$, choose finite places $\widetilde{7}, \widetilde{11}, \widetilde{13}$ lying above the rational primes $7, 11, 13$ and let $\mathfrak{R}_{\widetilde{7}}, \mathfrak{R}_{\widetilde{11}}, \mathfrak{R}_{\widetilde{13}}$ be the corresponding valuations rings in $\mathbb{Q}(\sqrt{5})$. The maps $x \mapsto 2x$ and $x \mapsto 3x$ are invertible on the local ring $\mathfrak{R}_{\widetilde{7}} \cap \mathfrak{R}_{\widetilde{11}} \cap \mathfrak{R}_{\widetilde{13}}$. Dualizing, we obtain an algebraic $\mathbb{Z}^2$-action α in the $\times 2 \times 3$ family.

Using (10), for all $\mathbf{n} = (n_1, n_2) \neq 0$,

$$\mathsf{F}_k(\alpha^{\mathbf{n}}) = \prod_{p \in \{7,11,13\}} |\chi_p(\mathbf{n})^k - 1|_{\widetilde{p}}^{-1},$$

where $\chi_p(\mathbf{n}) = \chi_{\widetilde{p}}(\mathbf{n}) = 2^{n_1} 3^{n_2}$. Since $r_{\widetilde{7}} = r_{\widetilde{13}} = 2$ and $r_{\widetilde{11}} = 1$ (see Lemma 4.3), the proof of Theorem 1.2 shows that $\delta_{\mathbf{n}}(z)$ has abscissa of convergence $\sigma = 1$ and that this is a double pole.

Theorem 1.3 shows that $\pi_{\mathbf{n}}(N)$ is asymptotic to a constant (depending on $\mathbf{n}$) times $N \log N$. To be precise, appealing to (11), we must consider the image of $\mathbf{n}$ in

$$\Gamma = \overline{\chi}_7(\mathbb{Z}^2) \times \overline{\chi}_{11}(\mathbb{Z}^2) \times \overline{\chi}_{13}(\mathbb{Z}^2) = \mathbb{F}_7^\times \times \mathbb{F}_{11}^\times \times \mathbb{F}_{13}^\times$$

under the map ϑ given by (12). In particular,

$$\vartheta(\mathbf{n})_p = \overline{\chi}_p(\mathbf{n}) = 2^{n_1}3^{n_2} \mod p.$$

According to the proof of Theorem 1.3, there is a contribution to the coefficient of $N \log N$ from at most two elements of $\mathcal{G}(\vartheta(\mathbf{n}))$. The first of these always appears and arises from (see (22)),

$$g(z) = \frac{1}{(1-7^{1-z})(1-11^{-z})(1-13^{1-z})} \sum_{I \subset \{7,11,13\}} \frac{(-1)^{|I|}}{(B_g(\vartheta(\mathbf{n}))\ell(I))^{z+1}},$$

where $\ell(I) = \operatorname{lcm}\{i : i \in I\}$ and

$$B_g(\vartheta(\mathbf{n})) = \operatorname{lcm}(\omega_7(\mathbf{n}), \omega_{11}(\mathbf{n}), \omega_{13}(\mathbf{n})).$$

Here, $\omega_p = \omega \cdot \overline{\chi}_p$. By (17), provided

$$j(\vartheta(\mathbf{n})) = \frac{\omega_{11}(\mathbf{n})}{\gcd(\omega_{11}(\mathbf{n}), \operatorname{lcm}(\omega_7(\mathbf{n}), \omega_{13}(\mathbf{n})))} \in \{1, 5\}$$

is not equal to 1, a second contribution arises from

$$h(z) = \frac{1}{(1-7^{1-z})(1-13^{1-z})} \sum_{I \subset \{5,7,13\}} \frac{(-1)^{|I|}}{(B_h(\vartheta(\mathbf{n}))\ell(I))^{z+1}},$$

where

$$B_h(\vartheta(\mathbf{n})) = \operatorname{lcm}(\omega_7(\mathbf{n}), \omega_{13}(\mathbf{n})).$$

The structure of the partition of $\mathcal{E}(\alpha) = \mathbb{Z}^2 \setminus \{0\}$ induced by $j \cdot \vartheta$ is shown below.

1	5	5	5	5	1	5	5	5	5	1
5	5	5	1	5	5	5	5	1	5	5
5	1	5	5	5	5	1	5	5	5	5
5	5	5	5	1	5	5	5	5	1	5
5	5	1	5	5	5	5	1	5	5	5
1	5	5	5	5	☐	5	5	5	5	1
5	5	5	1	5	5	5	5	1	5	5
5	1	5	5	5	5	1	5	5	5	5
5	5	5	5	1	5	5	5	5	1	5
5	5	1	5	5	5	5	1	5	5	5
1	5	5	5	5	1	5	5	5	5	1

Following the method of [6, Sec. 7.3] to evaluate the contributions from $g(z)$ and $h(z)$, we have

$$\pi_{\mathbf{n}}(N) \sim \frac{1}{\log 7 \log 13} \left(a_g(\mathbf{n}) \frac{6912}{5915\, B_g(\vartheta(\mathbf{n}))^2} + a_h(\mathbf{n}) \frac{C(\vartheta(\mathbf{n}))}{B_h(\vartheta(\mathbf{n}))^2} \right) N \log N,$$

where

$$C(\vartheta(\mathbf{n})) = \begin{cases} 0 & \text{if } j(\vartheta(\mathbf{n})) = 1, \\ \frac{27648}{29575} & \text{if } j(\vartheta(\mathbf{n})) = 5, \end{cases}$$

$$a_g(\mathbf{n}) = \prod_{p \in \{7,11,13\}} |\chi_p(\mathbf{n})^{\omega_p(\mathbf{n})} - 1|_{\widetilde{p}}^{-1},$$

and

$$a_h(\mathbf{n}) = |\chi_{11}(\mathbf{n})^{B_h(\vartheta(\mathbf{n}))} - 1|_{\widetilde{11}}^{-1} \prod_{p \in \{7,13\}} |\chi_p(\mathbf{n})^{\omega_p(\mathbf{n})} - 1|_{\widetilde{p}}^{-1}.$$

6. Concluding remarks

The main results reveal highly structured closed orbit subdynamics for the class of algebraic $\mathbb{Z}^d$-actions on solenoids satisfying the prime counting condition (2). For any given action in the class this structure is governed by a distinguished finite homomorphic image of $\mathbb{Z}^d$. The method used here shows that (2) is crucial for the finiteness of this image.

Without the assumption of connectedness, the usefulness of (2) breaks down. For example, Ledrappier's example, which corresponds to the R_2-module $R_2/(2, 1 + u_1 + u_2)$, arises from a $\mathbb{Z}^2$-shift invariant subgroup of $\{0,1\}^{\mathbb{Z}^2}$. It is not difficult to see that $\mathcal{P}(\alpha^{\mathbf{n}}) = \{2\}$, for all $\mathbf{n} \in \mathcal{E}(\alpha)$. However, this is an expansive action whose pattern of closed orbit subdynamics is more akin to Example 3.1 (see [11, Ex. 4.5] for details). For general $\mathbb{Z}^d$-actions on compact abelian groups, to capture the aspect of combinatorial finiteness that underlies (2), one needs an approach similar to that used in [6, Sec. 2] based on a partial ordering of periodic point sequences. Even with this, the erratic orbit growth demonstrated by examples such as [6, Ex. 4.4] suggest that the disconnected case would be somewhat more difficult to study.

If X is a solenoid of dimension one and (2) is satisfied for some algebraic $\mathbb{Z}^d$-action α on X, then in fact this condition is satisfied for all algebraic $\mathbb{Z}^d$-actions on X. To see this, one simply needs to note that for one-dimensional solenoids $M = \widehat{X}$ is a subgroup of $\mathbb{Q}$ and the method of Proposition 4.1 delivers a single set of p-adic places in (6) that is independent of the choice of generating automorphisms. In general, it would be interesting to know to what extent (2) depends on the underlying group rather than the choice of generating automorphisms for the action. Unfortunately, approaches to this problem appear to be hampered by the well-known classification problem for finite-rank torsion free abelian groups of rank greater than one. For details regarding the complexity of this problem, see [9].

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