ORBIT-COUNTING IN NON-HYPERBOLIC DYNAMICAL SYSTEMS

G. EVEREST, R. MILES, S. STEVENS, AND T. WARD

Abstract. There are well-known analogs of the prime number theorem and Mertens' theorem for dynamical systems with hyperbolic behaviour. Here we consider the same question for the simplest non-hyperbolic algebraic systems. The asymptotic behaviour of the orbit-counting function is governed by a rotation on an associated compact group, and in simple examples we exhibit uncountably many different asymptotic growth rates for the orbit-counting function. Mertens' Theorem also holds in this setting, with an explicit rational leading coefficient obtained from arithmetic properties of the non-hyperbolic eigendirections.

1. Introduction

A closed orbit τ of length $|\tau| = n$ for a continuous map $T : X \to X$ on a compact metric space X is a set of the form

$$\{x, T(x), T^2(x), \dots, T^n(x) = x\}$$

with cardinality n. A dynamical analog of the prime number theorem concerns the asymptotic behaviour of expressions like

$$\pi_T(N) = |\{\tau \colon |\tau| \leqslant N\}|, \tag{1}$$

and a dynamical analog of Mertens' Theorem concerns asymptotic estimates for expressions like

$$\mathcal{M}_T(N) = \sum_{|\tau| \leqslant N} \frac{1}{e^{h(T)|\tau|}} \tag{2}$$

where $h(T)$ denotes the topological entropy of the map. Results about the asymptotic behaviour of both expressions under the assumption that X has a metric structure with respect to which T is hyperbolic may be found in the works of Parry [12], Parry and Pollicott [13], Sharp [15] and others. An orbit-counting result on the asymptotic behavior of (1) for quasi-hyperbolic toral automorphisms has been found by Waddington [17], and an analog of Sharp's dynamical Mertens' Theorem for

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quasi-hyperbolic toral automorphisms has been found by Noorani [11]. Both the current state of these kinds of results and the seminal early work on geodesic flows is described in the book of Margulis [9] which also has a survey by Sharp on periodic orbits of hyperbolic flows.

One of the tools used in studying orbit-growth properties of hyperbolic maps is the dynamical zeta function. This may be viewed as a generalization of the Weil zeta function, which corresponds to the dynamical zeta function of the action of the Frobenius map on the extension of an algebraic variety over a finite field to the field's algebraic closure. Writing

$$\mathcal{F}_T(n) = |\{x \in X\colon T^n x = x\}|$$

for the number of points fixed by T^n, the dynamical zeta function is defined by

$$\zeta_T(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n}\, \mathcal{F}_T(n) \tag{3}$$

which has a formal expansion as an Euler product,

$$\zeta_T(z) = \prod_{\tau} \left(1 - z^{|\tau|}\right)^{-1}, \tag{4}$$

where the product is taken over all orbits of T. Just as the classical Euler product relates analytic properties of the Riemann zeta function to asymptotic counting properties of the prime numbers, the Euler expansion (4) relates analytic properties of the dynamical zeta function to orbit-counting asymptotics. In the hyperbolic case, the zeta function (3) has radius of convergence $e^{-h(T)}$ and, crucially, has a meromorphic extension to a strictly larger radius.

Our purpose here is on the one hand to study a very special class of maps of arithmetic origin, while on the other relaxing the hyperbolicity or quasi-hyperbolicity assumption. In this setting, the simplest non-trivial example is the map ϕ dual to the map $r \mapsto 2r$ on $\mathbb{Z}[\frac{1}{3}]$. This map is an isometric extension of the circle-doubling map $t \mapsto 2t$ (mod 1) on the additive circle $\mathbb{T}$ by a cocycle taking values in the 3-adic integers $\mathbb{Z}_3$; it is non-expansive and has topological entropy $\log 2$. The dynamical zeta-function associated to the map ϕ is shown to have a natural boundary by Everest, Stangoe and Ward [5], making it impossible to find a meromorphic extension beyond the radius of convergence. The radius of convergence is $e^{-h(\phi)} = \frac{1}{2}$ since easy estimates show that

$$\frac{1}{n} \log \mathcal{F}_\phi(n) \to \log 2 \text{ as } n \to \infty.$$

The bounds

$$\frac{1}{3} \leqslant \liminf_{N\to\infty} \frac{N\pi_\phi(N)}{2^{N+1}} \leqslant \limsup_{N\to\infty} \frac{N\pi_\phi(N)}{2^{N+1}} \leqslant 1 \tag{5}$$

were found in [5]. A problem left open there is to describe the asymptotics exactly, and in particular to show that $\frac{N\pi_\phi(N)}{2^{N+1}}$ does not converge as $N \to \infty$.

A similar result is found for the dynamical analog of Mertens' Theorem. Write

$$\mathcal{O}_T(n) = |\{\tau \colon \tau \text{ is a closed orbit of } T \text{ of length } |\tau| = n\}|$$

for the number of orbits of length n under T. Then

$$\frac{1}{2}\log N + \mathrm{O}(1) \leqslant \sum_{n\leqslant N} \frac{\mathcal{O}_\phi(n)}{2^n} \leqslant \log N + \mathrm{O}(1) \tag{6}$$

is shown in [5].

A consequence of the results in this paper is a better explanation of the sequences along which the expressions in (5) converge, and a proof that there is a single asymptotic in (6). The map considered in [5] is a special case of a more general construction of S-integer maps described in [3]. These are parameterized by an $\mathbb{A}$-field $\mathbb{K}$ (for example, $\mathbb{Q}$ or $\mathbb{F}_q(t)$), a subset S of the set of places of $\mathbb{K}$, and an element $\xi \in \mathbb{K}^*$ of infinite order (see the start of Section 3 for the construction; the assumption that ξ has infinite multiplicative order is equivalent to ergodicity for the resulting map). For the map ϕ above, these parameters are chosen with $\mathbb{K} = \mathbb{Q}$, $S = \{3\} \subset \{2, 3, 5, 7, 11, \dots\}$ and $\xi = 2$. If the $\mathbb{A}$-field $\mathbb{K}$ has characteristic zero, then the resulting map is an endomorphism of a solenoid.

The essential starting point is to note from [3] that if $T : X \to X$ is an S-integer map with S finite and X connected, then

$$\frac{1}{n}\log \mathcal{F}_T(n) \longrightarrow h(T) > 0,$$

so the dynamical zeta function has radius of convergence $e^{-h(T)}$. This suggests that the natural function to compare $\pi_T(N)$ with is $\frac{e^{h(T)(N+1)}}{N}$, so define

$$\Pi_T(N) = \frac{N\pi_T(N)}{e^{h(T)(N+1)}}.$$

Theorem 1.1. *Let $T : X \to X$ be an ergodic S-integer map with X connected and S finite. Then $(\Pi_T(N))$ is a bounded sequence, and*

$$\liminf_{N\to\infty} \Pi_T(N) > 0.$$

Moreover, there is an associated pair (X^*, a_T), where X^* is a compact group and $a_T \in X^*$, with the property that if $a_T^{N_j}$ converges in X^* as $j \to \infty$, then $\Pi_T(N_j)$ converges in $\mathbb{R}$ as $j \to \infty$.

Thus the pair (X^*, a_T) detects limit points in the orbit-counting problem. In the hyperbolic case, the group X^* is trivial, reflecting the fact that $(\Pi_T(N))_{N\geqslant 1}$ itself converges.

Example 1.2. The most familiar examples of non-hyperbolic automorphisms are the quasi-hyperbolic toral automorphisms (see Lind [8] for a detailed account of their dynamical properties.) Let $k = \mathbb{Q}(\xi)$ where $\xi = -(1+\sqrt{2}) - \sqrt{2\sqrt{2}+2}$, and $S = \emptyset$. Then the corresponding map T is the quasi-hyperbolic automorphism of the 4-torus defined by the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & 2 & -4 \end{bmatrix}.$$

There is a pair of eigenvalues $\lambda, \overline{\lambda}$ with $|\lambda| = 1$. The corresponding system (X^*, a_T) is the rotation $z \mapsto \lambda z$ on $\mathbb{S}^1$, and any sequence (N_j) for which $\left(\lambda^{N_j}\right)$ converges has the property that $(\Pi_T(N_j))$ converges as $j \to \infty$. This recovers in part a result of Waddington [17], who explicitly identifies $\Pi_T(N)$ as an almost-periodic function of N.

In some cases the correspondence between convergent subsequences seen in the detector group X^* and the orbit-counting problem is exact. For simplicity we state this for the case $\mathbb{K} = \mathbb{Q}$, $\xi = 2$, $S = \{3\}$; the same method gives a similar conclusion whenever $\mathbb{K} = \mathbb{Q}$ and $|S| = 1$. The full extent of the phenomena (and, in particular, of the appearance of uncountably many limit points) is not clear.

Theorem 1.3. *For the map ϕ dual to the map $x \mapsto 2x$ on $\mathbb{Z}[\frac{1}{3}]$, the sequence $(\Pi_\phi(N_j))$ converges as $j \to \infty$ if and only if the sequence $\left(2^{N_j}\right)$ converges in the group $\mathbb{Z}_3$. In particular, the sequence $(\Pi_\phi(N))$ has uncountably many limit points. Moreover, the upper and lower limits are both transcendental.*

The dynamical analog of Mertens' Theorem concerns the expression (2). In the simplest case (an endomorphism of a 1-dimensional solenoid) precise results are readily found, with a rational coefficient of the leading term.

Theorem 1.4. *For an ergodic S-integer map T with $\mathbb{K} = \mathbb{Q}$ and S finite, there are constants $k_T \in \mathbb{Q}$ and C_T such that*

$$\mathcal{M}_T(N) = k_T \log N + C_T + \mathrm{O}\left(1/N\right).$$

Example 1.5. Let $\xi = 2$ in Theorem 1.4, so the map T is the map dual to $x \mapsto 2x$ on the ring $R_S = \{\frac{p}{q} \in \mathbb{Q}\colon$ primes dividing q lie in $S\}$. The constant k_T for various simple sets S is given in Table 1.

Table 1. Leading coefficients in Mertens' Theorem

S	value of k_T
$\emptyset$	1
$\{3\}$	$\frac{5}{8}$
$\{3,5\}$	$\frac{55}{96}$
$\{3,7\}$	$\frac{269}{576}$
co-finite	0

In the general case there is less control of the error term (the error term in the dynamical Mertens' Theorem of Sharp [15] for the hyperbolic setting is improved to $\mathrm{o}(1/N)$ by Pollicott [14]).

Theorem 1.6. *Let $T : X \to X$ be an ergodic S-integer map with X connected and with S finite. Then there are constants $k_T \in \mathbb{Q}$, C_T and $\delta > 0$ with*

$$\mathcal{M}_T(N) = k_T \log N + C_T + \mathrm{O}(N^{-\delta}). \tag{7}$$

Recent work of Miles [10] shows that the number of periodic points for an automorphism of a solenoid is a finite product of the numbers of periodic points for connected S-integer systems, so the results above apply to those automorphisms of solenoids whose resulting product only involves finitely many valuations.

The class of S-integer systems with $|S|$ infinite provides a range of subtle behaviors that cannot readily be treated in this way. Possibilities include $\mathcal{F}(n)$ growing much slower than exponentially; the 'generic' behavior for S chosen randomly is discussed in [18] and [19]. Some results on systems with S co-finite may be found in the thesis of Stangoe [16].

Example 1.7. Let T be an S-integer map dual to $x \mapsto \xi x$ with $\mathbb{K} = \mathbb{Q}$ and S co-finite. For any finite place $w \in S$ there are constants $A, B > 0$ with $|\xi^n - 1|_w > A/n^B$, so by the product formula there is a constant $C > 0$ with $\mathcal{F}_T(n) \leqslant n^C$. It follows that $\mathcal{M}_T(N)$ is bounded for all N.

Little can be said about compact group automorphisms in general. For example, it is shown in [20] that for any $C \in [0, \infty]$ there is a compact group automorphism T with $\lim_{n\to\infty} \frac{1}{n}\mathcal{F}_n(T) \longrightarrow C$.

Allowing the compact group X to be infinite-dimensional is problematical for a different reason: the following example may be found in [16], Theorem 8.1.

Example 1.8. For any sequence $a_1, a_2, \dots$ there is an automorphism T of a compact connected group with

$$a_n \leqslant \mathcal{F}_T(n) < \infty \text{ for all } n \geqslant 1.$$

To see this, define a sequence of maps $T_1, T_2, \dots$ as follows. Let T_1 be the map dual to $x \mapsto 3x$ on $\mathbb{Z}$. Let T_2 be the map dual to $x \mapsto 2x$ on $\mathbb{Z}$. Let T_3 be the map dual to $x \mapsto 3x$ on $\mathbb{Z}[\frac{1}{2}]$. By Zsigmondy's Theorem (see [22] for the original result; a convenient modern source is [6]),

$$\{p \colon p \big| 3^n - 1 \text{ for some } n \le k\} \subsetneq \{p \colon p \big| 3^n - 1 \text{ for some } n \le k+1\}$$

unless $k = 1$. This allows the sequence of maps to be continued: Let T_4 be the map dual to $x \mapsto 3x$ on $\mathbb{Z}[\frac{1}{2}, \frac{1}{13}]$ and, similarly T_k will be the map dual to $x \mapsto 3x$ on $\mathbb{Z}[\frac{1}{s_1}, \dots, \frac{1}{s_t}]$, where

$$\{s_1, \dots, s_t\} = \{p \colon p \text{ is a prime with } p \big| 3^n - 1 \text{ for some } n < k\}.$$

Using the periodic point formula (17) from [3], the choice of primes ensures that $\mathcal{F}_{T_k}(j) = 1$ for $j < k$ and $\mathcal{F}_{T_k}(k) > 1$. Finally define the map T to be the infinite product

$$T = \underbrace{(T_1 \times T_1 \times \cdots \times T_1)}_{\text{so that } \infty > \mathcal{F}_T(1) > a_1} \times \underbrace{(T_2 \times T_2 \times \cdots \times T_2)}_{\text{so that } \infty > \mathcal{F}_T(2) > a_2} \times \cdots .$$

For any $k \geqslant 1$, all but finitely many terms in the product giving $\mathcal{F}_T(n)$ are 1, so the product is finite and exceeds a_n.

The paper is organized as follows. Theorem 1.3 and Theorem 1.4 for the same map ϕ dual to $x \mapsto 2x$ on $\mathbb{Z}[\frac{1}{3}]$ are proved in Section 2; this example illustrates some of the issues that arise in the more general setting while avoiding the Diophantine subtleties. Theorem 1.1 is proved in Section 3. Theorem 1.6 without an error term is proved in Section 5; this result may be found using soft methods. Theorem 1.4 is proved in Section 5, with the essential combinatorial step generalized to allow other fields. Finally, Section 6 assembles the additional Diophantine ingredients for Theorem 1.6. Since the relevant zeta functions do not have meromorphic extensions, we are unable to use Tauberian or complex-analytic methods. Instead the proofs use the theorems of Abel, Baker and Dirichlet.

2. Proof of Theorems 1.3 and 1.4 in a special case

The specific map ϕ dual to $x \mapsto 2x$ on $\mathbb{Z}[\frac{1}{3}]$ already reveals some of the essential features of these systems. In addition, the relatively simple nature of the map allows very precise results. This section contains a self-contained proof of Theorem 1.3 which may be read on its own or used to motivate some of the arguments in Section 3. It also contains a self-contained proof of Theorem 1.4 for the case $S = \{3\}$ and $\xi = 2$.

By [3], Lemma 5.2, the number of points fixed by ϕ^n is

$$\mathcal{F}_\phi(n) = (2^n - 1)|2^n - 1|_3,$$

so the number of orbits of length n is given by

$$\mathcal{O}_\phi(n) = \frac{1}{n}\sum_{d|n}\mu(\tfrac{n}{d})(2^d - 1)|2^d - 1|_3$$

by Möbius inversion, and hence

$$\pi_\phi(N) = \sum_{n \leqslant N}\frac{1}{n}\sum_{d|n}\mu(\tfrac{n}{d})(2^d - 1)|2^d - 1|_3. \tag{8}$$

We begin by replacing (8) with a more manageable expression. Let

$$G(N) = \sum_{n \leqslant N}\frac{1}{n}2^n|2^n - 1|_3. \tag{9}$$

Then

$$\begin{aligned} |\pi_\phi(N) - G(N)| &\leqslant \sum_{n \leqslant N}\frac{1}{n}\left(\sum_{d|n}\underbrace{|2^d - 1|_3}_{\leqslant 1} + \sum_{d|n, d<n} 2^d|2^d - 1|_3\right) \\ &\leqslant \sum_{n \leqslant N}\frac{1}{n}\left(n + \sum_{d \leqslant \lfloor n/2 \rfloor} 2^d\right) = \mathrm{O}\left(2^{N/2}\right), \end{aligned}$$

so for the purposes of the asymptotic sought we can use $G(N)$ in place of $\pi_\phi(N)$.

We next give a simple proof of the orbit-counting asymptotic for the circle-multiplication by $a \geqslant 2$, that is for the map $\psi_a(x) = ax \pmod 1$; for this map $\mathcal{F}_{\psi_a}(n) = a^n - 1$. Results like these are special cases of the more general picture in the work of Parry and Pollicott [13]. We give an elementary proof here because the argument used presages the estimates needed later.

Lemma 2.1. $\pi_{\psi_a}(N) \sim \frac{a^{N+1}}{N(a-1)}$.

Proof. By Möbius inversion

$$\pi_{\psi_a}(N) = \sum_{n \leqslant N} \frac{1}{n} \sum_{d|n} \mu(\tfrac{n}{d})(a^d - 1) = \sum_{n \leqslant N} \frac{1}{n} \sum_{d|n} \mu(\tfrac{n}{d}) a^d - 1.$$

Subtracting the dominant terms,

$$\begin{aligned}\left|\pi_{\psi_a}(N) - \sum_{n \leqslant N} \frac{1}{n} a^n\right| &= 1 + \left|\sum_{n \leqslant N} \frac{1}{n} \sum_{d|n, d<n} \mu(\frac{n}{d}) a^d\right| \\ &= \mathrm{O}\left(\sum_{n \leqslant N} \sum_{d \leqslant \lfloor n/2 \rfloor} a^d\right) \\ &= \mathrm{O}\left(\sum_{n \leqslant N} a^{n/2}\right) = \mathrm{O}\left(a^{N/2}\right). \qquad (10)\end{aligned}$$

To estimate the dominant terms, let $K(N) = \lfloor N^{1/4} \rfloor$. Then

$$\begin{aligned}\left|\sum_{n \leqslant N} \frac{1}{n} a^n - \sum_{N-K(N) \leqslant n \leqslant N} \frac{1}{n} a^n\right| &\leqslant \sum_{n \leqslant N-K(N)} a^n \\ &= \mathrm{O}\left(a^{N-K(N)}\right).\end{aligned}$$

Finally,

$$\begin{aligned}\sum_{N-K(N) \leqslant n \leqslant N} \frac{1}{n} a^n &= \frac{a^N}{N} \sum_{r=0}^{K(N)} a^{-r} \left(1 - \frac{r}{N}\right)^{-1} \\ &= \frac{a^N}{N} \left[\frac{a}{a-1} - \mathrm{O}(a^{-K(N)}) + \mathrm{O}(\sum_{r=0}^{K(N)} r/N)\right] \\ &= \frac{a^{N+1}}{N(a-1)} + \mathrm{O}\left(\frac{a^N}{N^2} \sum_{r=0}^{K(N)} r\right) \\ &= \frac{a^{N+1}}{N(a-1)} + \mathrm{O}\left(\frac{a^N}{N^{3/2}}\right).\end{aligned}$$

Together with (10), this proves the lemma. □

Returning to the main problem, write

$$I(N) = \sum_{n \leqslant N, 2|n} \frac{1}{n} 2^n |2^n - 1|_3$$

and

$$J(N)=\sum_{n\leqslant N, 2\nmid n}\frac{1}{n}2^n|2^n-1|_3,$$

so $G(N)=I(N)+J(N)$. Splitting into odd and even terms further simplifies the expressions since an easy calculation shows that

$$|2^n-1|_3=\begin{cases}\frac{1}{3}|n|_3 & \text{if } n \text{ is even;}\\ 1 & \text{if } n \text{ is odd,}\end{cases} \tag{11}$$

so

$$J(N)=\sum_{n\leqslant N, 2\nmid n}\frac{1}{n}2^n.$$

Lemma 2.2. $J(N)\sim\frac{1}{3}\cdot\frac{2^{N+1}}{N}$.

Proof. Lemma 2.1 applied to the maps ψ_2 and ψ_4 shows that

$$\sum_{n\leqslant N}\frac{1}{n}2^n\sim\frac{2^{N+1}}{N} \text{ and } \sum_{k\leqslant K}\frac{1}{k}4^k\sim\frac{4^{K+1}}{3K}.$$

Hence

$$\begin{aligned} J(N) &= \sum_{n\leqslant N}\frac{1}{n}2^n-\sum_{n\leqslant N, 2|n}\frac{1}{n}2^n\\ &= \sum_{n\leqslant N}\frac{1}{n}2^n-\sum_{k\leqslant N/2}\frac{1}{2k}4^k\\ &\sim \frac{2^{N+1}}{N}-\frac{2}{3}\cdot\frac{2^{N+1}}{N}\\ &= \frac{1}{3}\cdot\frac{2^{N+1}}{N}. \end{aligned}$$

□

We are therefore left with the expression

$$\begin{aligned} I(N)=\sum_{n\leqslant N, 2|n}\frac{1}{n}2^n|2^n-1|_3 &= \frac{1}{3}\sum_{n\leqslant N, 2|n}\frac{1}{n}2^n|n|_3\\ &= \frac{1}{6}\sum_{k\leqslant N/2}\frac{1}{k}2^{2k}|k|_3. \end{aligned}$$

Define

$$L(M)=\sum_{n\leqslant M}\frac{1}{n}4^n|n|_3$$

and

$$a_M=\frac{ML(M)}{4^M}.$$

Again it is enough to look only at the large terms, since

$$\left| \sum_{M-K(M)\leqslant n\leqslant M} \frac{4^n}{n}|n|_3 - \sum_{n\leqslant M} \frac{4^n}{n}|n|_3 \right| \leqslant \sum_{n\leqslant K(M)} 4^n = \mathrm{O}\left(4^{K(M)}\right).$$

Expanding from the last term gives

$$\begin{aligned} a_M &= \frac{|M|_3}{1} + \frac{4^{-1}|M-1|_3}{1-1/M} + \frac{4^{-2}|M-2|_3}{1-2/M} + \cdots \\ &\qquad + \frac{4^{-K(M)}|M-K(M)|_3}{1-K(M)/M} \\ &= \frac{|M|_3}{1} + \frac{|M-1|_3}{4} + \frac{|M-2|_3}{4^2} + \cdots + \frac{|M-K(M)|_3}{4^{K(M)}} \\ &\qquad + \mathrm{O}\left(\sum_{r=1}^{K(M)} r/M\right), \end{aligned}$$

and the error term is $\mathrm{O}\left(M^{-1/2}\right)$. Thus the limit points mentioned in Theorem 1.3 come from limit points of the sequence (b_M) defined by

$$b_M = \frac{|M|_3}{1} + \frac{|M-1|_3}{4} + \frac{|M-2|_3}{4^2} + \cdots + \frac{|M-K(M)|_3}{4^{K(M)}}. \tag{12}$$

Clearly

$$b_M \leqslant 1 + \frac{1}{4} + \frac{1}{4^2} + \cdots = \frac{4}{3}$$

and

$$b_M \geqslant \frac{1}{4}$$

because $3\big|M$ implies that $3\not\big|(M-1)$. These upper and lower bounds imply upper and lower bounds of $\frac{5}{9}$ and $\frac{9}{24}$ respectively in (5).

The shape of the expression (12) suggests that the lower limit will be seen along sequences highly divisible by 3, and the upper limit along sequences not divisible by 3, and this indeed turns out to be the case. To find limit points, it is easier to work with the infinite sum rather than (12), so notice first that if

$$c_M = \sum_{j=0}^{\infty} \frac{|M-j|_3}{4^j}$$

then $|b_M - c_M| = \mathrm{O}\left(2^{K(M)-M}\right)$. Now let $|M_k|_3 = 3^{-k}$ so that (by the ultrametric inequality)

$$\begin{aligned} c_{M_k} + t_k &= \frac{1}{3^k} + \frac{|1|_3}{4} + \frac{|2|_3}{4^2} + \frac{|3|_3}{4^3} + \cdots \\ &= \frac{1}{3^k} + \sum_{j=1}^{\infty} \frac{1}{4^j} - \frac{2}{3}\sum_{j=1}^{\infty} \frac{1}{4^{3j}} - \frac{2}{9}\sum_{j=1}^{\infty} \frac{1}{4^{9j}} - \frac{2}{27}\sum_{j=1}^{\infty} \frac{1}{4^{27j}} - \cdots \\ &= \frac{1}{3^k} + \frac{1}{3} - 2\sum_{r=1}^{\infty} \frac{1}{3^r(4^{3^r}-1)} \end{aligned}$$

where

$$t_k = \sum_{j=3^k}^{\infty} \frac{|j|_3 - |M-j|_3}{4^j} = \mathrm{O}(4^{-3^k}).$$

Thus c_{M_k} converges as $k \to \infty$. Moreover, the limiting value is transcendental.

Lemma 2.3. *The sum* $C = \sum_{r=1}^{\infty} \frac{1}{3^r(4^{3^r}-1)}$ *is transcendental, and*

$$\liminf_{M\to\infty} c_M = \frac{1}{3} - 2C.$$

Proof. Let $q_s = 3^s(4^{3^s} - 1)$. Then there is an integer p_s such that

$$C_s = C - \frac{p_s}{q_s} = \sum_{r=s+1}^{\infty} \frac{1}{3^r(4^{3^r}-1)}.$$

Thus $C_s = \mathrm{O}\left(3^{-s+1}4^{-3^{s+1}}\right)$, so

$$0 < |C - \frac{p_s}{q_s}| = \mathrm{O}\left(q_s^{-3}\right)$$

showing that C is too well-approximable to be algebraic.

To see that this does give the lower limit, notice that

$$c_{M_k} = \frac{1}{3^k} + \frac{1}{3} - 2C - t_k.$$

Any limit point along a sequence (M_k) with $\mathrm{ord}_3(M_k)$ bounded infinitely often is larger, and any limit point with $\mathrm{ord}_3(M_k) \to \infty$ must be this one. □

Essentially the same argument choosing M_k with $|M_k + 1|_3 = 3^{-k}$ shows that

$$\limsup_{M\to\infty} c_M = 4 \liminf_{M\to\infty} c_M,$$

completing the proof of the first part of Theorem 1.3.

We now turn our attention to the remaining part of Theorem 1.3.

Lemma 2.4. *Fix $M, N \in \mathbb{N}$ with $0 < \varepsilon = |M - N|_3$. Then*

$$\frac{\varepsilon}{3 \cdot 4^{3/\varepsilon}} < |c_M - c_N| \leqslant \frac{4}{3}\varepsilon$$

Proof. The second inequality is straightforward: By the reverse triangle inequality

$$\big|\, |M - j|_3 - |N - j|_3 \,\big| \leqslant |M - N|_3 = \varepsilon \tag{13}$$

for any j, so that

$$|c_M - c_N| \leqslant \sum_{j=0}^{\infty} \frac{\big|\, |M - j|_3 - |N - j|_3 \,\big|}{4^j} \leqslant \sum_{j=0}^{\infty} \frac{\varepsilon}{4^j} = \frac{4}{3}\varepsilon.$$

For the first inequality a more careful analysis of where the series in c_M and c_N differ is needed. Write $\varepsilon = 3^{-k}$, with $k \geqslant 0$. There exist unique integers $0 \leqslant j_M, j_N < 3^{k+1}$ such that

$$|M - j_M|_3 \leqslant 3^{-(k+1)} \ \text{ and } \ |N - j_N|_3 \leqslant 3^{-(k+1)}.$$

Since $|M - N|_3 = 3^{-k}$ we have $|j_M - j_N|_3 = 3^{-k}$ also and we may assume that $j_M < j_N$ without loss of generality. By the ultrametric inequality,

$$|M - j|_3 = |N - j|_3, \quad \text{for } j < j_M,$$

so the series in c_M and c_N differ first at the term $j = j_M$. Thus

$$|M - j_M|_3 \leqslant 3^{-(k+1)} < |N - j_M|_3 = 3^{-k}$$

and so

$$\begin{aligned}
|c_M - c_N| &\geqslant \frac{|N - j_M|_3 - |M - j_M|_3}{4^{j_M}} - \sum_{j=j_M+1}^{\infty} \frac{\big|\, |M - j|_3 - |N - j|_3 \,\big|}{4^j} \\
&\geqslant \frac{3^{-k} - 3^{-(k+1)}}{4^{j_M}} - \frac{3^{-k}}{4^{j_M}} \sum_{j=1}^{\infty} \frac{1}{4^j} \\
&\geqslant \frac{3^{-k}}{4^{j_M}} \left(1 - \frac{1}{3} - \frac{1}{3}\right) = \frac{3^{-k}}{3.4^{j_M}} > \frac{\varepsilon}{3.4^{3/\varepsilon}}
\end{aligned}$$

by (13). □

An immediate consequence of Lemma 2.4 is the following corollary, from which the remainder of Theorem 1.3 follows.

Corollary 2.5. *Given any $\alpha \in \mathbb{Z}_3$ and sequence of natural numbers (M_k) converging to α in $\mathbb{Z}_3$, define c_α to be $\lim_{M_k \to \infty} c_{M_k}$. Then c_α*

is well-defined (the limit exists and is independent of the choice of approximating sequence). Moreover, if $\beta \in \mathbb{Z}_3$ *and* $\varepsilon = |\alpha - \beta|_3$ *then*

$$\frac{\varepsilon}{3.4^{3/\varepsilon}} \leqslant |c_\alpha - c_\beta| \leqslant \frac{4}{3}\varepsilon.$$

This completes the proof of Theorem 1.3.

Theorem 1.4 for the map ϕ concerns the sum

$$\mathcal{M}_\phi(N) = \sum_{n \leqslant N} \frac{\mathcal{O}_\phi(n)}{2^n} \tag{14}$$

where $\mathcal{O}_\phi(n)$ is the number of orbits of length n under ϕ, so

$$\mathcal{O}_\phi(n) = \frac{1}{n} \sum_{d|n} \mu(n/d)(2^d - 1)|2^d - 1|_3.$$

Let

$$F(N) = \sum_{n \leqslant N} \frac{|2^n - 1|_3}{n}, \tag{15}$$

and notice that

$$\begin{aligned}
\mathcal{M}_\phi(N) - F(N) &= \sum_{n \leqslant N} \frac{1}{n} \left(\sum_{d|n} \mu(\tfrac{n}{d})|2^d - 1|_3 \frac{2^d - 1}{2^n} - |2^n - 1|_3 \right) \\
&= \sum_{n \leqslant N} \frac{1}{n} \left(\frac{|2^n - 1|_3}{2^n} + \sum_{d|n, d<n} \mu(\tfrac{n}{d})|2^d - 1|_3 \frac{2^d - 1}{2^n} \right) \\
&= \sum_{n \leqslant N} \frac{1}{n} \cdot \frac{|2^n - 1|_3}{2^n} + \mathrm{O}\left(2^{-N/2}\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{|2^n - 1|_3}{2^n} + \mathrm{O}\left(2^{-N/2}\right).
\end{aligned}$$

In particular, the difference between $F(N)$ and the sum in (14) is a constant plus $\mathrm{O}\left(2^{-N/2}\right)$.

Some well-known partial sums related to the classical Mertens' Theorem will be needed. For $x > 0$,

$$\sum_{n \leqslant x} \frac{1}{n} = \log x + c_2 + \mathrm{O}(1/x),$$

where the constant c_2 is the Euler–Mascheroni constant. It follows that

$$\sum_{k \leqslant x, \gcd(p,k)=1} \frac{1}{k} = \left(\frac{p-1}{p} \right) \log x + c_3(p) + \mathrm{O}(1/x) \tag{16}$$

for any prime p, where $c_3(p)$ is a constant depending on p (the implied constant in the $\mathrm{O}(1/x)$ term also depends on p).

The sum in (15) can be estimated using (11) as follows. The sum over the odd terms is

$$\sum_{n\leqslant N, 2\nmid n}\frac{1}{n}=\frac{1}{2}\log N+c_4+\mathrm{O}(1/N)$$

by (16), with $c_4=c_3(2)$. The sum over the even terms collapses just as before to give

$$\sum_{2k\leqslant N}\frac{|3k|_3}{2k}.$$

Now

$$\sum_{k\leqslant N}\frac{|k|_3}{k}=\sum_{r=0}^{\log N/\log 3}\frac{1}{3^{2r}}\sum_{k=1,\gcd(3,k)=1}^{N/3^r}\frac{1}{k}.$$

By (16), this is

$$\sum_{r=0}^{\log N/\log 3}\frac{2}{3^{2r+1}}[\log N-r\log 3+c_6+\mathrm{O}(3^r/N)],$$

where the constant in the $\mathrm{O}(3^r/N)$ term is independent of r. The computation of each term involves summing a geometric series. In each case the sum differs from the full series with an error that is $\mathrm{O}(1/N)$; we deduce that

$$\sum_{k\leqslant N}\frac{|k|_3}{k}=\frac{3}{4}\log N+c_5+\mathrm{O}(1/N).$$

The sum over the odd and even terms gives

$$\frac{1}{6}\cdot\frac{3}{4}\log N+\frac{1}{2}\log N+c_7+\mathrm{O}(1/N)=\frac{5}{8}\log N+c_7+\mathrm{O}(1/N),$$

completing the proof of Theorem 1.4 for the case $\xi=2$ and $S=\{3\}$.

3. Proof of Theorem 1.1

We are given an algebraic number field $\mathbb{K}$ with set of places $P(\mathbb{K})$ and set of infinite places $P_\infty(\mathbb{K})$, an element of infinite multiplicative order $\xi\in\mathbb{K}^*$, and a finite set $S\subset P(\mathbb{K})\setminus P_\infty(\mathbb{K})$ with the property that $|\xi|_w\leqslant 1$ for all $w\notin S\cup P_\infty(\mathbb{K})$. The associated ring of S-integers is

$$R_S=\{x\in\mathbb{K}\colon |x|_w\leqslant 1 \text{ for all } w\notin S\cup P_\infty(\mathbb{K})\}.$$

The compact group X is the character group of R_S, and the endomorphism T is the dual of the map $x\mapsto\xi x$ on R_S. Examples of this construction may be found in [3]. Following Weil [21], Chap. IV, write $\mathbb{K}_w$ for the completion at w, and for w finite, write r_w for the maximal compact subring of $\mathbb{K}_w$.

Define the compact group X_w^* by

$$X_w^* = \begin{cases} \mathbb{S}^1 & \text{if } w \in P_\infty(\mathbb{K}) \text{ and } |\xi|_w = 1; \\ r_w^* & \text{if } w \notin P_\infty(\mathbb{K}) \text{ and } |\xi|_w = 1; \\ \{1\} & \text{in all other cases.} \end{cases}$$

Finally, let $X^* = \prod_w X_w^*$. The element $a_T = (a_{T,w})_w$ of X^* is defined by $a_{T,w} = \imath_w(\xi)$ where $\imath_w$ is the corresponding embedding of $\mathbb{K}$ into $\mathbb{C}$ or $\mathbb{K}_w$ whenever X_w^* is non-trivial, and $a_{T,w} = 1$ in all other cases.

By [3], Lemma 5.2, the number of points in X fixed by T^n is

$$\mathcal{F}_T(n) = \prod_{w \in S \cup P_\infty(\mathbb{K})} |\xi^n - 1|_w, \tag{17}$$

so the number of orbits of length n is

$$\mathcal{O}_T(n) = \frac{1}{n} \sum_{d|n} \mu(\tfrac{n}{d}) \prod_{w \in S \cup P_\infty(\mathbb{K})} |\xi^n - 1|_w$$

by Möbius inversion, and hence

$$\pi_T(N) = \sum_{n \leqslant N} \frac{1}{n} \sum_{d|n} \mu(\tfrac{n}{d}) \prod_w |\xi^n - 1|_w \tag{18}$$

where w is restricted to run through the places in $S \cup P_\infty(\mathbb{K})$ only (both here and below).

We begin by replacing (18) with a more manageable expression just as in (9). Let

$$G(N) = \sum_{n \leqslant N} \frac{1}{n} \prod_{|\xi|_w > 1} |\xi|_w^n \prod_{|\xi|_w \leqslant 1} |\xi^n - 1|_w.$$

By [3], the topological entropy of T is

$$h(T) = \sum_{|\xi|_w > 1} \log |\xi|_w > 0. \tag{19}$$

It follows that $(\Pi_T(N))$ is a bounded sequence. Let $h'(T)$ denote the maximum value of $\frac{1}{2}h(T)$ and the expression (19) with one term omitted; notice in particular that $h' = h'(T) < h = h(T)$. Write

$$C_{\mathbb{K}} = 4^{|P_\infty(\mathbb{K})|}$$

Now

$$\begin{aligned}
|G(N)-\pi_T(N)| &= \sum_{n\leqslant N}\frac{1}{n}\left(\sum_{d|n}\mathrm{O}\left(e^{nh'}\right)\underbrace{\prod_{|\xi|_w\leqslant 1}|\xi^d-1|_w}_{\leqslant C_{\mathbb{K}}}\right.\\
&\qquad\left.+\sum_{d|n,d<n}\prod_w|\xi^n-1|_w\right)\\
&= \sum_{n\leqslant N}\frac{1}{n}\left(\,n\,\mathrm{O}\left(\,e^{nh'}\,\right)+\underbrace{\sum_{d\leqslant\lfloor n/2\rfloor}\prod_w|\xi^n-1|_w}_{\mathrm{O}(e^{nh/2})}\right)\\
&= \mathrm{O}\left(e^{Nh'}\right).
\end{aligned}$$

Since $h'<h$, this means that $(\Pi_T(N_j))$ converges if and only if

$$\frac{N_jG(N_j)}{e^{h(T)(N_j+1)}}$$

converges. Write

$$G(N)=\sum_{n\leqslant N}\frac{1}{n}A(n)B(n)$$

where

$$A(n)=\prod_{|\xi|_w>1}|\xi|_w^n,$$

and

$$B(n)=\prod_{|\xi|_w\leqslant 1}|\xi^n-1|_w.$$

Notice that $A(n)=e^{hn}$, $B(n)\leqslant C_{\mathbb{K}}$, and a subsequence $(B(N_j))$ of $(B(N))$ converges whenever $\left(a_T^{N_j}\right)$ converges in X^* (since the terms in $B(N)$ with $|\xi|_w<1$ simply converge to 1).

As before, let $K(N)=\lfloor N^{1/4}\rfloor$, and consider the expression

$$\begin{aligned}
a_N &= \sum_{n=N-K(N)}^{N}\frac{N}{e^{h(N+1)}}\cdot\frac{1}{n}\cdot A(n)B(n)\\
&= \sum_{t=0}^{K(N)}\frac{N}{e^{h(N+1)}}\cdot\frac{1}{N-t}A(N-t)B(N-t).
\end{aligned}$$

Now

$$\begin{aligned}\left|a_N - \frac{G(N)N}{e^{h(N+1)}}\right| &= \sum_{t=K(N)+1}^{N} \frac{NA(N-t)B(N-t)}{(N-t)e^{h(N+1)}} \\ &\leqslant \sum_{t=K(N)+1}^{N} \frac{N\cdot C_{\mathbb{K}}}{e^{h(t+1)}} \\ &= \mathrm{O}\left(Ne^{-K(N)}\right) \qquad (20)\end{aligned}$$

so in order to show that $(\Pi_T(N_j))$ converges it is enough to show that the subsequence (a_{N_j}) converges. The expression for a_N can be further simplified, since

$$\begin{aligned}a_N &= \sum_{t=0}^{K(N)} \frac{N}{e^{h(N+1)}}\cdot\frac{1}{N-t}A(N-t)B(N-t) \\ &= \sum_{t=0}^{K(N)} \frac{1}{e^{h(t+1)}}\cdot\frac{1}{1-t/N}B(N-t) \\ &= a_N^* + \mathrm{O}\left(\sum_{t=0}^{K(N)}\frac{t}{N}C_{\mathbb{K}}\right) = a_N^* + \mathrm{O}\left(N^{-1/2}\right), \qquad (21)\end{aligned}$$

where

$$a_N^* = \sum_{t=0}^{K(N)} \frac{1}{e^{h(t+1)}}\cdot B(N-t).$$

Choose δ with

$$0 < \delta = \tfrac{1}{2}\min\{|\xi^j - 1|_w\colon |\xi|_w = 1, 1 \leqslant j \leqslant |S|, w \in S\}.$$

If $|\xi^N - 1|_w < \delta$, then

$$|\xi^{N-j}-1|_w = |\xi^{-j}(\xi^N-1)+\xi^{-j}-1|_w \geqslant |\xi^{-j}-1|_w-\delta > \delta \text{ for } 1 \leqslant j \leqslant |S|.$$

Notice that a_N^* can only be small if $B(N), B(N-1), \dots, B(N-|S|)$ are small, but if

$$|B(N-j)| < \delta^{|S|} \text{ for } j = 0, \dots, |S|-1$$

then $|B(N)-|S|)| > \delta^{|S|}$. It follows that there is no sequence (N_j) with

$$\prod_{|\xi|_w\leqslant 1} |\xi^{N_j+k} - 1|_w \to 0 \text{ for } k = 0, 1, 2, \dots,$$

and, indeed $\liminf_{N\to\infty} a_N^* \geqslant \delta^{|S|} > 0$.

Assume now that (N_j) is a sequence with the property that $\left(a_T^{N_j}\right)$ converges in X^*, so in particular each sequence $\left(|\xi^{N_j}-1|_w\right)$ is Cauchy for $w \in S$, $|\xi|_w \leqslant 1$, hence $\left(|\xi^{N_j-t}-1|_w\right)$ and $(B(N_j-t))$ are Cauchy for each t. Moreover, these sequences are uniformly Cauchy in t, since $|\xi^{N_j-t}-\xi^{N_k-t}|_w = |\xi^{N_j}-\xi^{N_k}|_w$ for all t. We claim that $(a^*_{N_j})$ also converges, which (by the estimates (20) and (21)) will complete the proof of Theorem 1.1. Let $k<j$ be fixed. Then

$$\begin{aligned}
|a^*_{N_j}-a^*_{N_k}| &\leqslant \left|\sum_{t=0}^{K(N_j)}\frac{1}{e^{h(t+1)}}B(N_j-t)\right. \\
&\qquad \left. -\sum_{t=0}^{K(N_k)}\frac{1}{e^{h(t+1)}}B(N_k-t)\right| \\
&\leqslant \sum_{t=0}^{K(N_k)}\frac{1}{e^{h(t+1)}}|B(N_j-t)-B(N_k-t)| \\
&\qquad + \underbrace{\sum_{t=K(N_k)+1}^{K(N_j)}\frac{1}{e^{h(t+1)}}B(N_j-t)}_{\mathrm{O}\left(e^{-hK(N_k)}\right)} \\
&\longrightarrow 0 \text{ as } k\to\infty,
\end{aligned}$$

since

$$\begin{aligned}
&\sum_{t=0}^{K(N_k)}\frac{1}{e^{h(t+1)}}\,|B(N_j-t)-B(N_k-t)| \\
\leqslant &\left(\sum_{t=0}^{\infty}\frac{1}{e^{h(t+1)}}\right)\underbrace{\max_{0\leqslant t\leqslant K(N_k)}|B(N_j-t)-B(N_k-t)|}_{\to 0 \text{ as } k\to\infty \text{ by the uniform Cauchy property}}.
\end{aligned}$$

4. Mertens' Theorem without error term

The setting is an S-integer map $T: X\to X$ with X connected and S finite. We first give a simple argument to show a form of Theorem 1.6 without error term, and then consider how an error term is obtained. Recall that

$$\mathcal{M}_T(N)=\sum_{n\leqslant N}\frac{1}{n}\sum_{d|n}\mu(n/d)\left(\frac{\prod_w|\xi^d-1|_w}{e^{hn}}\right).$$

Let

$$C(n) = \prod_{|\xi|_w \neq 1} |\xi^n - 1|_w$$

and

$$D(n) = \prod_{|\xi|_w = 1} |\xi^n - 1|_w.$$

Define

$$F(N) = \sum_{n \leqslant N} \frac{1}{n} D(n),$$

and write

$$h^* = \prod_{\substack{|\xi|_w > 1, \\ w | \infty}} |\xi|_w$$

for the Archimedean contribution to the entropy. Then

$$\begin{aligned}
\mathcal{M}_T(N) - F(N) &= \sum_{n \leqslant N} \frac{1}{n} \left(\sum_{d|n} \mu\left(\tfrac{n}{d}\right) e^{-hn} \prod_{w} |\xi^d - 1|_w - D(n) \right) \\
&= \sum_{n \leqslant N} \frac{1}{n} \sum_{d|n} \mu\left(\tfrac{n}{d}\right) D(d) \prod_{\substack{|\xi|_w > 1, \\ w|\infty}} |\xi^d - 1|_w |\xi|_w^n \\
&\quad - \sum_{n \leqslant N} \frac{1}{n} D(n) \\
&= \sum_{n \leqslant N} \frac{1}{n} \Big(D(n) \left(1 - \mathrm{O}(e^{-h^* n})\right) - D(n) \Big) \\
&\quad + \sum_{n \leqslant N} \frac{1}{n} \mathrm{O} \Big(\underbrace{\sum_{d < n/2} D(d) e^{h^*(d-n)}}_{\mathrm{O}(e^{-h^* n/2})} \Big) \\
&= \sum_{n \leqslant N} \frac{1}{n} D(n) \, \mathrm{O}(e^{-h^* n}) + \sum_{n \leqslant N} \frac{1}{n} \mathrm{O}(e^{-h^* n/2})
\end{aligned}$$

in which the implied constants are uniformly bounded. It follows that $\mathcal{M}_T(N) - F(N)$ may be written as the difference between a sum of a convergent series and the sum from N to ∞ of that series, and this tail of the series is $\mathrm{O}(e^{-h^* N})$. Thus in order to prove Theorem 1.6 it is enough to consider $F(N)$.

Lemma 4.1. *Let g be an element of a compact abelian group G. Then the sequence (g^n) is uniformly distributed in the smallest closed subgroup of G containing g.*

Proof. This is essentially the Kronecker–Weyl lemma. Write X for the closure of the set $\{g^n : n \in \mathbb{Z}\}$ and μ_X for Haar measure on X. In order to show that

$$\frac{1}{N}\sum_{n=1}^{N} f(g^n) \to \int f \,\mathrm{d}\,\mu_X$$

for all continuous functions $f : X \to \mathbb{C}$, it is enough to show this for characters. If $\chi : X \to \mathbb{S}^1$ is a non-trivial character on X, then

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=0}^{N-1}\chi(g^n)\right| &= \left|\frac{1}{N}\sum_{n=0}^{N-1}\chi(g)^n\right| \\ &= \left|\frac{1}{N}\cdot\frac{1-\chi(g)^N}{1-\chi(g)}\right| \\ &\leqslant \frac{1}{N}\cdot\frac{2}{1-\chi(g)} \to 0 \text{ as } N \to \infty,\end{aligned}$$

so the sequence is uniformly distributed. □

Lemma 4.1 may be applied to the element $a_T \in X^*$: the function

$$x \mapsto \prod_{|\xi|_w=1} |x-1|_w$$

is continuous on X^*, so

$$\frac{1}{N}\sum_{n=1}^{N} D(n) \to k_T \text{ as } N \to \infty$$

where

$$k_T = \int_{X^*} \prod_{|\xi|_w=1} |x-1|_w \,\mathrm{d}\,\mu_{X^*}.$$

Thus

$$\begin{aligned}F(N) &= \sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)\sum_{m=1}^{n} D(m) + \frac{1}{N+1}\sum_{m=1}^{N} D(m) \\ &\sim k_T \log N,\end{aligned}$$

giving Theorem 1.6 without error term.

5. Mertens' Theorem with $\mathbb{K} = \mathbb{Q}$

Section 2 contains a proof of Theorem 1.4 for the case $S = \{3\}$ and $\xi = 2$. In this section we prove Theorem 1.4; the essential difference between this and Theorem 1.6 is that the assumption $\mathbb{K} = \mathbb{Q}$ does not permit ξ to induce an ergodic map (that is, ξ is not a unit root) while exhibiting non-hyperbolicity in an infinite place. The argument in this section, with simple modifications, would give Theorem 1.4 under the assumption that $\mathbb{K}$ does not contain any Salem numbers ($[\mathbb{K}:\mathbb{Q}] \leqslant 3$ would suffice, for example).

Fix a finite set S of primes, a rational $r \in \mathbb{Q}$ with $r \neq \pm 1$ and

$$|r|_p < 1 \implies p \in S.$$

Consider the map $T : X \to X$ dual to the map $x \mapsto rx$ on the additive group of the ring

$$R_S = \{r \in \mathbb{Q}\colon |r|_p \leqslant 1 \text{ for all } p \notin S\}.$$

By [3], Lemma 5.2, the number of points fixed by T^n is

$$\mathcal{F}_T(n) = |r^n - 1| \prod_{p \in S} |r^n - 1|_p = (r^n - 1)|r^n - 1|_S,$$

where we write $|x|_S$ for $\prod_{p\in S} |x|_p$, and so

$$\mathcal{O}_T(n) = \frac{1}{n} \sum_{d|n} \mu(\tfrac{n}{d}) |r^d - 1||r^d - 1|_S.$$

Just as in Section 4, it is sufficient to work with the sum $F(N)$.

The analog of Mertens' Theorem in this setting is most easily proved by isolating the following arithmetic argument. A function f is called totally multiplicative if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

Lemma 5.1. *Let $f : \mathbb{N} \to \mathbb{C}$ be a totally multiplicative function with*

$$\sum_{n \leqslant N} f(n) = k_f \log N + c_f + \mathrm{O}(1/N),$$

for constants c_f and k_f. Let E be a finite set of natural numbers and, for $D \subseteq E$, let $n_D = \mathrm{lcm}\{n : n \in D\}$. Then there is a constant $c_{f,E}$ for which

$$\sum_{n \leqslant N, k \nmid n \text{ for } k \in E} f(n) = k_{f,E} \log N + c_{f,E} + \mathrm{O}(1/N),$$

where

$$k_{f,E} = k_f \sum_{D \subseteq E} (-1)^{|D|} f(n_E).$$

Proof. Notice that

$$\begin{aligned} \sum_{n \leqslant N, n_D | n} f(n) &= f(n_D) \sum_{n \leqslant N/n_D} f(n) \\ &= f(n_D) \left(k_f \log(N/n_D) + c_f + \mathrm{O}(1/N)\right) \\ &= k_f f(n_D) \log N + c_{f,n_D} + \mathrm{O}(1/N), \end{aligned}$$

for some constant c_{f,n_D}. The result follows by an inclusion-exclusion argument. □

Notice that, if E is a set of pairwise coprime natural numbers, then

$$k_{f,E} = k_f \prod_{n \in E} (1 - f(n)).$$

Now let $\mathcal{P}$ be a finite set of (rational) primes. For $\boldsymbol{r} = (r_p)_{p \in \mathcal{P}} \in \mathbb{Z}^{|\mathcal{P}|}$, write

$$\boldsymbol{p}^{\boldsymbol{r}} = \prod_{p \in \mathcal{P}} p^{r_p}$$

and abbreviate $\boldsymbol{p} = \boldsymbol{p}^{(1,\dots,1)} = \prod_{p \in \mathcal{P}} p$. Define a partial order on $|\mathcal{P}|$-tuples by

$$\boldsymbol{r} = (r_p)_{p \in \mathcal{P}} \leqslant \boldsymbol{s} = (s_p)_{p \in \mathcal{P}} \iff r_p \leqslant s_p \ \forall p \in \mathcal{P}$$

and write $\boldsymbol{0} = (0)_{p \in \mathcal{P}}$.

For $\boldsymbol{t} = (t_p)_{p \in \mathcal{P}} \in \mathbb{N}^{|\mathcal{P}|}$, write

$$f_{\mathcal{P},\boldsymbol{t}}(n) = \frac{1}{n} \prod_{p \in \mathcal{P}} |n|_p^{t_p};$$

notice that this is a totally multiplicative function.

Proposition 5.2. *There is a constant $c_{\mathcal{P},\boldsymbol{t}}$ for which*

$$\sum_{n<N} f_{\mathcal{P},\boldsymbol{t}}(n) = k_{\mathcal{P},\boldsymbol{t}} \log N + c_{\mathcal{P},\boldsymbol{t}} + \mathrm{O}(1/N).$$

where $k_{\mathcal{P},\boldsymbol{t}}$ is the product $\prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p^{t_p+1}}\right)^{-1}$.

Note that, since $f_{\mathcal{P},\boldsymbol{t}}$ is totally multiplicative, Lemma 5.1 may be applied to this result to get asymptotics for sums over subsets of $\mathbb{N}$.

Proof. The proof is by induction on $m = |\mathcal{P}|$, the case $m = 0$ being the familiar statement

$$\sum_{n<N} \frac{1}{n} = \log N + c + \mathrm{O}(N^{-1}).$$

Write $p^r \| n$ if $r = \mathrm{ord}_p(n)$ is the exact order with which p divides n. Put $\mathcal{P} = \{p_1, \dots, p_m\}$, $\mathcal{P}_1 = \{p_2, \dots, p_m\}$, $t_1 = t_{p_1}$ and $\boldsymbol{t}_1 = (t_{p_2}, \dots, t_{p_m})$.

Then

$$\begin{aligned}\sum_{n\leqslant N} f_{\mathcal{P},\boldsymbol{t}}(n) &= \sum_{r_1=0}^{\log N/\log p_1} \sum_{n\leqslant N, p_1^{r_1}\| n} f_{\mathcal{P},\boldsymbol{t}}(n) \\ &= \sum_{r_1=0}^{\log N/\log p_1} \frac{1}{p_1^{(t_1+1)r_1}} \sum_{n\leqslant N/p_1^{r_1}, p_1\nmid n} f_{\mathcal{P}_1,\boldsymbol{t}_1}(n) \\ &= \sum_{r_1=0}^{\log N/\log p_1} \frac{1}{p_1^{(t_1+1)r_1}} \left(1-\frac{1}{p_1}\right) k_{\mathcal{P}_1,\boldsymbol{t}_1} \cdot \\ &\qquad [\log N - r_1\log p_1 + c' + \mathrm{O}(p_1^{r_1}/N)]\end{aligned}$$

using the inductive hypothesis and Lemma 5.1 (applied to $f = f_{\mathcal{P}_1,\boldsymbol{t}_1}$ and $E = \{p_1\}$). Note that the implied constants in the $\mathrm{O}(p_1^{r_1}/N)$ terms are independent of r_1. The computation of each term involves summing some geometric series, and in each case the sum differs from the full series with an error term that is $\mathrm{O}(1/N)$. □

The next argument will be needed again in Section 6 in a more general setting, so we now allow $\mathbb{K}$ to be a number field. Theorem 1.4 will follow at once, since the sum considered here is the $F(N)$ from Section 4.

Proposition 5.3. *Let $\mathbb{K}$ be a number field, $\xi \in \mathbb{K}$ and S a finite set of non-Archimedean places of $\mathbb{K}$ such that $|\xi|_v = 1$ for all $v \in S$. Write $|x|_S = \prod_{v\in S} |x|_v$ for $x \in \mathbb{K}$. Then there are constants $k_S \in \mathbb{Q}$ and $c_S \in \mathbb{R}$ such that*

$$\sum_{n<N} \frac{|\xi^n - 1|_S}{n} = k_S \log N + c_S + \mathrm{O}(1/N).$$

Proof. For $v \in S$, let o_v denote the order of ξ in the residue field at v, that is, the least positive integer o such that $|\xi^o - 1|_v < 1$. Then

$$|\xi^n - 1|_v = 1 \iff o_v \nmid n.$$

Let p be the rational prime such that $v\big|p$. It is sometimes more convenient to use the extension of the p-adic absolute value $|\cdot|_p$, which is related to $|\cdot|_v$ by

$$|\cdot|_v = |\cdot|_p^{[K_v:\mathbb{Q}_p]},$$

where K_v is the completion of K at v.

Let m_v be the least positive integer m such that

$$|\xi^m - 1|_p < \frac{1}{p^{1/p-1}}.$$

Then $m_v = p^{r_v} o_v$, for some $r_v \geqslant 0$. Moreover, if $m_v | n$ then

$$|\xi^n - 1|_v = |n|_v |\log \xi|_v,$$

where log is here the p-adic logarithm.

Finally, if $n = kp^r o_v$, with $(k, p) = 1$, then

$$|\xi^n - 1|_v = |\xi^{p^r o_v} - 1|_v.$$

For T a subset of S, put $o_T = \operatorname{lcm}\{o_v : v \in T\}$. Split up the sum according to the subsets of S, giving

$$\sum_{n<N} \frac{|\xi^n - 1|_S}{n} = \sum_{T \subset S} \sum_{n<N,\, o_T | n,\, o_v \nmid n \forall v \notin T} \frac{|\xi^n - 1|_T}{n}.$$

We show that each internal sum has the required form and, since there are only a finite number of subsets of S, we will be done.

So let T be a subset of S and let $\mathcal{P}$ be the set of rational primes divisible by some $v \in T$. Putting $m_T = \operatorname{lcm}\{m_v : v \in T\}$, there exists $\boldsymbol{r} = (r_p) \geqslant \boldsymbol{0}$ such that $m_T = \boldsymbol{p}^{\boldsymbol{r}} o_T$. Then we have

$$\begin{aligned} &\sum_{n<N,\, o_T | n,\, o_v \nmid n \forall v \notin T} \frac{|\xi^n - 1|_T}{n} \\ &\qquad = \sum_{\boldsymbol{0} \leqslant \boldsymbol{s} \leqslant \boldsymbol{r}} \frac{|\xi^{\boldsymbol{p}^{\boldsymbol{s}} o_T} - 1|_T}{\boldsymbol{p}^{\boldsymbol{s}} o_T} \sum_{n<N/\boldsymbol{p}^{\boldsymbol{s}} o_T,\, (n,\boldsymbol{p})=1,\, o_v \nmid n\boldsymbol{p}^{\boldsymbol{s}} o_T \forall v \notin T} \frac{1}{n} \\ &\qquad\qquad + \frac{|\xi^{m_T} - 1|_T}{m_T} \sum_{n<N/m_T,\, o_v \nmid n m_T \forall v \notin T} \frac{|n|_T}{n}. \end{aligned}$$

Now $|n|_T = \prod_{p \in \mathcal{P}} |n|_p^{t_p}$, where $t_p = \sum_{v \in T,\, v|p} [K_v : \mathbb{Q}_p]$, so $\frac{|n|_T}{n} = f_{\mathcal{P},\boldsymbol{t}}(n)$. So this again gives a finite number of sums, each of which has the required form, by applying Lemma 5.1 to Proposition 5.2. □

This completes the proof of Theorem 1.4. The constants appearing in Theorem 1.4 may be found explicitly for any given set S, by following the recipe in the proof of Proposition 5.3 and using Proposition 5.2, leading to Example 1.5.

6. Allowing infinite places

The estimate in (7) requires several improvements to the argument above. From now on S denotes a finite set of non-Archimedean valuations on the number field $\mathbb{K}$ and $\xi \in \mathbb{K}^*$ is an element of infinite multiplicative order with $|\xi|_v = 1$ for all $v \in S$.

Lemma 6.1. *Let $M \in \mathbb{N}$ denote any integral S-unit. The solutions of the equation*

$$|\xi^n - 1|_S = \frac{1}{M}$$

consist of $\mathrm{O}(M^{1-1/d})$ cosets mod M' where $M' = \rho M$ for some fixed integer ρ and some $d > 0$, both independent of M.

Proof. For each $v \in S$, the set $U_k = \{n \in \mathbb{Z} \colon \mathrm{ord}_v(\xi^n - 1) \geqslant k\}$ is a subgroup of $\mathbb{Z}$. For sufficiently large k, the cosets of U_{k+1} in U_k are defined by either 1 or p congruence classes modulo sp^{k+1} for a uniform constant s. Now for $n \in U_k \setminus U_{k+1}$, $|n|_v = sp^{-kd}$ for $d = [\mathbb{K} : \mathbb{Q}]$, so n lies in $\mathrm{O}(p^{kd-k}) = \mathrm{O}(M^{1-1/d})$ classes. Choose $\rho = m_v$ in the notation of the proof of Proposition 5.3. The Chinese Remainder Theorem then gives the same bound for the product of the finitely many valuations in S. □

Write $\sum'$ for a sum taken only over integral S-units.

Lemma 6.2. *For any $c > 0$, the series*

$$\sum_{M}{}' \frac{\log M}{M^c} \tag{22}$$

converges. The tail of the series satisfies

$$\sum_{M>X}{}' \frac{\log M}{M^c} = \mathrm{O}(1/X^e),$$

for any $e < c$.

Proof. Let $p_1, \dots, p_r$ be the distinct rational primes dividing the elements of S. Write each integral S-unit M in the form $p_1^{e_1} \dots p_r^{e_r}$ with $0 \leqslant e_i$ for $i = 1, \dots, r$. The sum in (22) is then a finite sum of terms, each of which may be written as a finite product of convergent geometric progressions and their squares, showing the convergence. To estimate the error notice that if $M > X$ then at least one term $e_i > \kappa \log X$ for some uniform constant κ, depending on S only. Hence the error is bounded above by

$$\sum_{i=1}^{r} K_i \sum_{t > \kappa \log X} \frac{t}{p_i^{ct}},$$

for some constants K_i, and this sum is $\mathrm{O}(\log X / X^c)$ by Euler Summation. □

Theorem 6.3. *Let a denote a complex algebraic number with $|a| = 1$ and a not a root of unity. Then for some $\delta > 0$ and constant ℓ,*

$$\sum_{n<N} \frac{a^n|\xi^n - 1|_S}{n} = \ell + \mathrm{O}(N^{-\delta}).$$

Proof. Decompose the sum according to the integral S-units M with

$$|\xi^n - 1|_S = \frac{1}{M}.$$

Consider the sum

$$F_N(X) = {\sum_{M<X}}' \frac{1}{M} \sum_{n<N:|\xi^n-1|_S=\frac{1}{M}} \frac{a^n}{n}.$$

We claim that there is a constant ℓ for which

$$F_N(X) = \ell + \mathrm{O}(\max\{X^B/N, 1/X^e\}), \tag{23}$$

where $e > 0$ is a constant depending on S and ξ only and B is a constant depending on ξ only. To see this, we use Lemma 6.1: Let $\{\alpha_i\}$ be representatives for the $\mathrm{O}\left(M^{1-1/d}\right)$ cosets modulo $M' = \rho M$ which are solutions to $|\xi^n - 1|_S = \frac{1}{M}$. Then each of the sums

$$\sum_{n<N:n\equiv\alpha_i \pmod{M'}} \frac{a^n}{n}$$

can be written using Dirichlet characters in the form

$$\sum_{n<N} \sum_{j=1}^{M'} c_{ij} \frac{\zeta_j^n a^n}{n}$$

where $|c_{ij}| = 1/M'$ and each ζ_j is an M'th root of unity (see Apostol [1], Chap. 6 for example). We can rearrange this double sum to get

$$\sum_{j=1}^{M'} c_{ij} \sum_{n<N} \frac{\zeta_j^n a^n}{n}.$$

The inner sum is a partial sum of a convergent power series for the logarithm since $\zeta_j a \neq 1$ (convergence to the logarithm is an instance of Abel's Theorem; see [7], Th. 2.6.4). Thus

$$\sum_{n<N:n\equiv\alpha_i \pmod{M'}} \frac{a^n}{n} = -\sum_{j=1}^{M'} c_{ij} \log(1 - \zeta_j a) + \sum_{j=1}^{M'} c_{ij} \sum_{n>N} \frac{\zeta_j^n a^n}{n}.$$

Applying Abel Summation to the last sum gives

$$\sum_{n<N:n\equiv\alpha_i \pmod{M'}} \frac{a^n}{n} = -\sum_{j=1}^{M'} c_{ij}\log(1-\zeta_j a) + \mathrm{O}\left(\frac{1}{N\min_j|1-\zeta_j a|}\right),$$

using the bound $|c_{ij}| \leqslant 1/M'$. Thus the sum sought is

$$\begin{aligned} F_N(X) &= -\sum_{M<X}\frac{1}{M}\sum_{\alpha_i}\sum_{j=1}^{M'} c_{ij}\log(1-\zeta_j a) \\ &\quad + \sum_{M<X}\sum_{\alpha_i}\frac{1}{M}\,\mathrm{O}\left(\frac{1}{N\min_j|1-\zeta_j a|}\right) \end{aligned} \tag{24}$$

in which there are $\mathrm{O}(M^{1-1/d})$ terms α_i.

Both sums in (24) require a lower bound for $|1-\zeta a|$ for ζ an M'th root of unity. A bound of the form $|1-\zeta a| > A/M'^B$ for constants $A, B > 0$ when ζ is an M'th root of unity follows from Baker's Theorem [2]: writing $a = e^{2\pi i\theta}$ and $\zeta = e^{2\pi ij/M'}$, the quantity $|1 - e^{2\pi ij/M'}e^{2\pi i\theta}|$ is small if and only if $\frac{j}{M'} + \theta$ is close to some integer K, in which case $e^{2\pi i(j/M'+\theta)} - 1$ is close to $2\pi i\left(\frac{j}{M'} + \theta - K\right)$; by Baker's Theorem there are constants $A, C > 0$ with

$$|M'\log(e^{2\pi ij/M'}) - M'\log e^{2\pi i\theta}| = |2\pi iR - M'\log a| \geqslant \frac{A}{M'^C}$$

for any choice of branches of the logarithm (here $R - j \in M'\mathbb{Z}$). It follows that there are constants $A, B > 0$ with $|1-\zeta a| > A/M'^B$.

The first sum in (24) is bounded in absolute value by

$$\sum_{M<X}\frac{1}{M}\sum_{\alpha_i}\sum_{j=1}^{M'}|c_{ij}||\log(1-\zeta_j a)|$$

$$= \mathrm{O}\left(\sum_{M<X}\frac{1}{M^{1/d}}\max_{j=1\dots M'}|\log(1-\zeta_j a)|\right),$$

using the existence of an absolute bound on the number of the α_i from Lemma 6.1 as well as the bound $|c_{ij}| \leqslant 1/M'$. Thus this term is $\mathrm{O}(\sum_{M<X}\log M'/M^{1/d})$ and we obtain convergence by comparison with the series

$$\sum_M{}' \frac{\log M}{M^{1/d}}$$

since M' and M are commensurate. Thus at this point, in relation to (23), any $e < 1/d$ will do.

To estimate the second sum in (24) use Baker's Theorem in the same way to get an estimate

$$\mathrm{O}\left(\sum_{\alpha_i}\sum_{M<X}\frac{1}{M}.\frac{M'^B}{N}\right)=\mathrm{O}(X^B/N).$$

This concludes the proof of claim (23). To complete the proof of Theorem 6.3, note that the sum over those n with

$$|\xi^n-1|_S\leqslant\frac{1}{N^\epsilon}$$

is $\mathrm{O}(N^{-\delta})$ since

$$\sum_{|\xi^n-1|_S\leqslant N^{-\epsilon}}\left|\frac{a^n|\xi^n-1|_S}{n}\right|\leqslant N^{-\epsilon}\sum_{n<N}\frac{1}{n}=\mathrm{O}(N^{-\delta})\text{ for any }\delta<\epsilon.$$

Thus in estimating the error term, we are allowed to assume that

$$\frac{1}{M}=|\xi^n-1|_S>\frac{1}{N^\epsilon}.$$

In other words, we may write $X=N^\epsilon$ in claim (23), where $\epsilon=\frac{1}{B+1/d}$. This finally gives an error term $\mathrm{O}(1/N^{\epsilon/d})=\mathrm{O}(1/N^{1/dB+1})$. □

As we saw in Proposition 5.3, a similar result holds for the case $a=1$. We have assembled the material needed to prove Theorem 1.6. By the arguments of Section 4 above, it it enough to show that

$$F(N)=k_T\log N+C_T+\mathrm{O}\left(N^{-\delta}\right)$$

for some $\delta>0$, where $F(N)=\sum_{n<N}\frac{1}{n}D(n)$ and

$$\begin{aligned}D(n)&=\prod_{|\xi|_w=1}|\xi^n-1|_w\\&=\prod_{|\xi|_w=1,w|\infty}|\xi^n-1|_w\times\prod_{|\xi|_w=1,w<\infty}|\xi^n-1|_w\\&=f(a_1^n,\dots,a_r^n)\times\prod_{|\xi|_w=1,w<\infty}|\xi^n-1|_w\end{aligned}$$

where f is an integral polynomial in r variables, and $a_i\in\mathbb{S}^1$ for $i=1,\dots,r$ are multiplicatively independent.

This reduces the problem to expressions of the form

$$\sum_{n<N}\frac{1}{n}a^n|\xi^n-1|_S$$

with a an algebraic number of modulus one that is not a root of unity, to which Theorem 6.3 can be applied, or of the same form with $a=1$, to which Proposition 5.3 may be applied. Notice in particular that the coefficient of the leading term comes entirely from the case $a = 1$ covered by Proposition 5.3, and is therefore rational.

Remark 6.4. The leading coefficient in Theorem 1.6 can also be described as $\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} |\xi^n - 1|_S$, which is redolent of an integral. There is a sophisticated theory showing that many p-adic integrals must be rational (see Denef [4] for example); is it possible to identify the limit with an S-adic integral, and is it possible to extend that theory to handle finitely many valuations?

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School of Mathematics, University of East Anglia, Norwich NR4 7TJ, United Kingdom

E-mail address: t.ward@uea.ac.uk