

## ENTROPY RANGE PROBLEMS AND ACTIONS OF LOCALLY NORMAL GROUPS

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**ABSTRACT.** This paper deals with the problem of finding the range of entropy values resulting from actions of discrete amenable groups by automorphisms of compact abelian groups. When the acting group  $G$  is locally normal, we obtain an entropy formula and show that the full range of entropy values  $[0, \infty]$  occurs for actions of  $G$ . We consider related entropy range problems, give sufficient conditions for zero entropy and, as a consequence, verify the known relationship between completely positive entropy and mixing for these actions.

**1. Introduction.** Let  $G$  be a countably infinite discrete amenable group and  $X$  a compact metrizable abelian group with Borel  $\sigma$ -algebra  $\mathcal{B}(X)$  and normalized Haar measure  $\mu_X$ . Suppose  $\alpha$  is a  $G$ -action by  $\mu_X$ -preserving transformations  $\alpha^g$  of  $X$  and consider the metric entropy  $h(\alpha)$  with respect to  $\mu_X$ . The definition of  $h(\alpha)$  is given in Section 2; detailed background material may be found in the seminal article [8]. In this paper, the focus is on *algebraic  $G$ -actions*, that is each  $\alpha^g$  is a continuous automorphism of  $X$ . Thus, there is also a notion of topological entropy whose correspondence with metric entropy is summarized in [1].

For a given class of discrete amenable groups  $\mathcal{G}$ , a natural problem is to determine the range of possible entropy values,

$$\mathcal{H}(\mathcal{G}) = \{h(\alpha) : \alpha \text{ is an algebraic } G\text{-action for some } G \in \mathcal{G}\}.$$

Clearly,  $\mathcal{H}(\mathcal{G})$  is dense in  $[0, \infty]$  if and only if zero is an accumulation point of  $\mathcal{H}(\mathcal{G})$ . Standard results for algebraic actions also show that if  $\mathcal{G}$  is a singleton, then density can be replaced by equivalence in this statement. Thus, we are interested in whether or not arbitrarily small non-trivial entropies exist for actions of a given class of groups. This question has received special attention for the cases  $\mathcal{G} = \{\mathbb{Z}\}$  and  $\mathcal{G} = \{\mathbb{Z}^d : d \geq 1\}$ . In particular, the results of Lind, Schmidt and Ward [7] show that for both these classes, the problem is equally difficult, being equivalent to the well-known Lehmer problem, which has been open for over seventy years. For further background, see [3].

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2000 *Mathematics Subject Classification.* 37A35, 37B40.

*Key words and phrases.* Entropy, entropy range, algebraic action, locally normal group, Lehmer's problem.

The second author was supported by grant KAW 2005.0098 from the Knut and Alice Wallenberg Foundation. Both authors wish to thank the referee for improving the clarity of this article.

From a slightly different perspective, it is often useful to develop a formula for calculating entropy. The origins of such formulae lie in the work of Sinai [12] and Rokhlin [9]. The search for entropy formulae for algebraic actions was continued by Yuzvinskiĭ [14], Lind and Ward [6], Lind, Schmidt and Ward [7] and more recently by Deninger [1] and Deninger and Schmidt [2]. In Section 3, a formula is developed that applies to all algebraic actions of locally normal groups on finite-dimensional compact abelian groups. An outline of the method is as follows. First, using Ward and Zhang's version of the Abramov-Rokhlin formula [13] and a uniform bound on the size of finite subgroups of  $GL(m, \mathbb{Q})$ , it is shown that one may pass to a zero-dimensional factor which yields the entire contribution to entropy (Theorem 3.1(1)). Then, since a version of Yuzvinskiĭ's addition formula holds for  $G$ -actions on zero-dimensional groups, this reduces the problem to that of  $G$ -actions corresponding to cyclic modules over the group ring  $\mathbb{Z}G$ . For these systems, a simple entropy formula is established that arises from a sequence of finite normal subgroups in  $G$  (Theorem 3.2).

In Section 4, some applications of the techniques used in Section 3 are considered. Suppose  $G$  is a countably infinite amenable group. We show that if  $G$  contains arbitrarily large finite normal subgroups, then  $\mathcal{H}(\{G\}) = [0, \infty]$ . For example, this is the case if  $G$  is locally normal. We also show that if  $G$  contains a non-trivial finite normal subgroup and  $\mathcal{G} = \{G^d : d \geq 1\}$  then  $\mathcal{H}(\mathcal{G})$  is dense in  $[0, \infty]$ . When  $G$  is locally normal, we deduce sufficient conditions for zero entropy (Theorem 4.3) and, as a corollary, show that completely positive entropy actions are also mixing. This is an instance of a much deeper result due to Rudolph and Weiss [10] that shows that completely positive entropy implies mixing of all orders for measure-preserving actions of amenable groups. We conclude by posing a new entropy range problem that does not yield to our methods.

**2. Preliminaries.** Throughout,  $G$  is a countably infinite discrete group. Recall that  $G$  is locally normal if every finite subset of  $G$  has a finite normal closure. For example, a torsion abelian group is always locally normal and a locally normal group is always locally finite.

A *left Følner sequence*  $(F_i)$  in  $G$  is a sequence of finite sets  $F_i \subset G$  for which, given any  $g \in G$ ,

$$\lim_{i \rightarrow \infty} |F_i \triangle gF_i|/|F_i| = 0.$$

If  $G$  has a left Følner sequence, then  $G$  is *amenable*. If  $G$  is locally normal, there is a sequence  $(F_i)$  of finite normal subgroups of  $G$  with  $\bigcup_{i \geq 1} F_i = G$  and such a sequence is also a left Følner sequence in  $G$ .

Let  $X$  be a compact abelian group with Borel  $\sigma$ -algebra  $\mathcal{B}(X)$  and normalized Haar measure  $\mu_X$  (we always assume  $X$  is metrizable). If  $\alpha$  is a  $G$ -action by  $\mu_X$ -preserving transformations  $\alpha^g$  of  $X$ , the *metric entropy* of  $\alpha$  with respect to  $\mu_X$  is the quantity

$$h(\alpha) = \sup_{\xi} h(\alpha, \xi),$$

where  $\xi$  runs over all finite measurable partitions of  $X$  and

$$h(\alpha, \xi) = \lim_{i \rightarrow \infty} \frac{1}{|F_i|} H(\xi^{F_i}), \quad (1)$$

$H(\cdot)$  being the entropy of a partition with respect to  $\mu_X$  and  $\xi^{F_i} = \bigvee_{g \in F_i} \alpha^g \xi$ . For detailed background concerning these definitions, for example the existence of the limit in (1) and the independence of the definition of entropy from choice of

Følner sequence, see [8] and [13]. Note that for any finite measurable partitions  $\xi, \eta$  of  $X$ ,  $h(\alpha, \xi) \leq h(\alpha, \eta) + H(\xi|\eta)$ , and so the Rokhlin inequality holds.

Suppose  $\alpha$  is an algebraic action, that is each  $\alpha^g$  is a continuous automorphism of  $X$ . If  $Y$  is a closed  $\alpha$ -invariant subgroup of  $X$ , then there are induced algebraic  $G$ -actions  $\alpha_Y$  and  $\alpha_{X/Y}$  on  $Y$  and  $X/Y$  respectively. Furthermore,

$$h(\alpha_{X/Y}) \leq h(\alpha_X) \text{ and } h(\alpha_Y) \leq h(\alpha_X).$$

If  $Y_1 \supset Y_2 \supset \dots$  is a sequence of closed  $\alpha$ -invariant subgroups of  $X$  with  $\bigcap_{n \geq 1} Y_n = \{0\}$ , then

$$h(\alpha) = \lim_{n \rightarrow \infty} h(\alpha_{X/Y_n}). \quad (2)$$

These results may be obtained as for  $\mathbb{Z}^d$ -actions [11, Sec. 13].

Finally, observe that there is a correspondence between algebraic  $G$ -actions and countable left  $\mathbb{Z}G$ -modules, facilitated by Pontryagin duality as follows (standard results from duality theory may be found in [5]). Let  $r = \sum_{g \in G} c_g(r)g \in \mathbb{Z}G$ , where  $c_g(r) = 0$  for all but finitely many  $g \in G$ , and for any  $a \in M = \widehat{X}$  set

$$ra = \sum_{g \in G} c_g(r) \widehat{\alpha}^{g^{-1}}(a).$$

Then  $M$  becomes a countable left  $\mathbb{Z}G$ -module. Conversely, if  $M$  is a countable left  $\mathbb{Z}G$ -module, then the *natural action*  $\alpha_M$  of  $G$  on  $X_M = \widehat{M}$  is obtained by setting  $\alpha^g$  to be the automorphism dual to multiplication by  $g^{-1}$ .

**3. Entropy formulae.** First we reduce our entropy calculation from the setting of a finite-dimensional group to that of a zero-dimensional one; we then observe that on zero-dimensional groups, Yuzvinskiĭ's addition formula holds.

**Theorem 3.1.** *Let  $\alpha$  be an algebraic action of a locally normal group  $G$  on a compact finite-dimensional abelian group  $X$ .*

1. *If  $Y$  is the connected component of the identity then*

$$h(\alpha_X) = h(\alpha_{X/Y}).$$

2. *(Yuzvinskiĭ's addition formula) If  $X$  is zero-dimensional and  $Y$  is a closed  $\alpha$ -invariant subgroup of  $X$  then*

$$h(\alpha_X) = h(\alpha_{X/Y}) + h(\alpha_Y).$$

*Proof.* To prove (1), first note that  $Y$  is a closed  $\alpha$ -invariant subgroup of  $X$ . Set  $Z = X/Y$  and take a Borel cross section  $s : Z \rightarrow X$ , so that  $\pi \cdot s$  is the identity on  $Z$ , where  $\pi : X \rightarrow Z$  is the quotient map. This technique is explained in [11, Sec. 14] for actions of  $\mathbb{Z}^d$  and works equally well for actions of other groups. For each  $g \in G$ , define  $\tau^g : Z \rightarrow Y$  by

$$\tau^g(z) = \alpha_X^g(s(z)) - s(\alpha_Z^g(z)).$$

Then  $\alpha = \alpha_X$  is measurably isomorphic to the measure preserving  $G$ -action  $T$  on  $(Z \times Y, \mu_Z \times \mu_Y)$  given by

$$T^g(z, y) = (\alpha_Z^g(z), \tau^g(z) + \alpha_Y^g(y)),$$

so  $h_{\mu_X}(\alpha_X) = h_{\mu_Z \times \mu_Y}(T)$ .

For  $y \in Y$  and any finite measurable partition  $\eta$  of  $Y$ , let  $y + \eta = \{y + B : B \in \eta\}$ . We can apply Ward and Zhang's version of the Abramov-Rokhlin formula [13] to the skew product action  $T$ , and follow the method in [11, Sec. 14] to obtain

$$h_{\mu_Z \times \mu_Y}(T) = h_{\mu_Z}(\alpha_Z) + \lim_{n \rightarrow \infty} \lim_{i \rightarrow \infty} \frac{1}{|F_i|} \int_Z H_{\lambda_Y}(\xi_n^{F_i}(z)) d\mu_Z(z), \quad (3)$$

where  $(\xi_n)$  is any increasing sequence of finite measurable partitions of  $Y$  satisfying  $\sigma\left(\bigvee_{n \geq 1} \bigvee_{g \in G} \alpha_Y^g \xi_n\right) = \mathcal{B}(Y)$  and

$$\xi_n^{F_i}(z) = \bigvee_{g \in F_i} \tau^g(z) + \alpha_Y^g(\xi_n).$$

Since  $Y$  is a finite-dimensional compact connected abelian group,  $\text{Aut}(Y) \subset GL(m, \mathbb{Q})$  for some  $m \in \mathbb{N}$ . By a theorem of Minkowski, any finite subgroup of  $GL(m, \mathbb{Q})$  has cardinality bounded by a constant depending only on  $m$  (for a precise bound, see [4]). Therefore, since  $\alpha(G) \subset \text{Aut}(Y)$  is locally finite, it follows that  $\alpha(G)$  must be finite. Hence, without loss of generality,  $(\xi_n)$  may be replaced by a refined sequence of finite measurable partitions that are  $\alpha_Y$ -invariant. Then

$$\xi_n^{F_i}(z) = \bigvee_{g \in F_i} \tau^g(z) + \xi_n.$$

Furthermore, by considering  $Y$  as a closed subgroup of  $\mathbb{T}^{\mathbb{N}}$ , it follows that  $\xi_n$  may be chosen so that the number of non-empty sets in  $\xi_n^{F_i}(z)$  does not exceed  $(C|F_i|)^C$ , where  $C = C(n) \in \mathbb{N}$  is independent of  $i$ . Therefore,

$$H_{\lambda_Y}(\xi_n^{F_i}(z)) \ll \log |F_i|,$$

and so by (3),  $h_{\mu_Z \times \mu_Y}(T) = h_{\mu_Z}(\alpha_Z)$ .

To establish Yuzvinskiĭ's addition formula, note that with the aid of [13], since  $Y$  is zero-dimensional, the method in [11, Sec. 14] is easily adapted when the dual  $\mathbb{Z}G$ -module for  $\alpha_Y$  is cyclic. Therefore, Yuzvinskiĭ's formula holds in this case. A standard induction argument using (2) then allows the result to be extended to more general  $Y$ .  $\square$

By duality, Theorem 3.1(1) shows that to calculate the entropy of an algebraic action of a locally normal group  $G$  on a compact finite-dimensional abelian group  $X$ , it is enough to consider the submodule  $M'$  of  $M = \widehat{X}$  consisting of all elements of finite order. Furthermore,  $M'$  contains a chain of submodules

$$\{0\} = M_0 \subset M_1 \subset M_2 \subset \dots$$

such that  $L_i = M_i/M_{i-1}$  is a cyclic module,  $i \geq 1$ , and  $\bigcup_{i \geq 1} M_i = M'$ . Therefore, using Theorem 3.1(2), induction and (2),

$$h(\alpha_M) = h(\alpha_{M'}) = \lim_{n \rightarrow \infty} \sum_{i=1}^n h(\alpha_{L_i}),$$

if this limit exists and  $h(\alpha_M) = \infty$  otherwise. Hence, it remains to consider the case of a cyclic module  $L \cong \mathbb{Z}G/I$ , where  $I \subset \mathbb{Z}G$  is a left ideal with  $I \cap \mathbb{Z} \neq \{0\}$ .

**Theorem 3.2.** *Let  $G$  be a locally normal group and let  $I \subset \mathbb{Z}G$  be a left ideal such that  $I \cap \mathbb{Z} \neq \{0\}$ . Then the entropy of the natural action  $\alpha = \alpha_{\mathbb{Z}G/I}$  of  $G$  on*

$X = X_{\mathbb{Z}G/I}$  is finite and is given by

$$h(\alpha) = \lim_{i \rightarrow \infty} \frac{\log |R_i|}{|G_i|}, \quad (4)$$

where  $G_i$  is any sequence of finite normal subgroups of  $G$  with  $\bigcup_{i \geq 1} G_i = G$  and  $R_i = \mathbb{Z}G_i/I \cap \mathbb{Z}G_i$ .

To prove the theorem and for use in Section 4, it will be useful to express a discrete amenable group  $G$  in terms of a finite normal subgroup  $F$ . Suppose  $F$  is such a subgroup, let  $K = G/F$  and let  $\pi : G \rightarrow K$  denote the quotient map. By choosing a set of left coset representatives, we may obtain an injection  $\tilde{\cdot} : K \rightarrow G$  with  $\tilde{1} = 1$  and  $\pi(\tilde{\cdot})$  equal to the identity map. Hence, there is a bijection  $G \rightarrow F \times K$  given by

$$g \mapsto (\pi(\tilde{g})^{-1}g, \pi(g)). \quad (5)$$

With respect to this representation of elements of  $G$ , the group operation is given by

$$(f, k)(g, l) = (\tilde{k}\tilde{l}^{-1}\tilde{k}f\tilde{l}g, kl), \quad (6)$$

for all  $(f, k), (g, l) \in F \times K$ .

Note that  $K$  is also amenable and if  $(K_i)$  is a left Følner sequence in  $K$ , for any  $(f, k) \in F \times K$ ,

$$(F \times K_i) \triangle (f, k)(F \times K_i) \subset F \times (K_i \triangle kK_i).$$

This implies the following.

**Lemma 3.3.** *In terms of (5), the sequence  $(F_i) = (F \times K_i)$  corresponds to a left Følner sequence in  $G$ .*

We now calculate the entropy of the natural  $G$ -action arising from a cyclic module of a fundamental form. If  $I \subset \mathbb{Z}G$  is a left ideal such that  $I \cap \mathbb{Z}$  is generated by  $n \neq 0$ , then  $\mathbb{Z}G/I$  may be written in the form  $RG/J$  and considered as a left  $RG$ -module, where  $R = \mathbb{Z}/(n)$  and  $J$  is the image of  $I$  in  $RG$ , obtained by reducing coefficients modulo  $n$ .

**Proposition 1.** *Let  $G$  be a discrete amenable group, and suppose  $R$  and  $J$  are as above. If there is a finite normal subgroup  $F \subset G$  such that  $RF$  contains a set of generators for  $J$  then*

$$h(\alpha_{RG/J}) = \frac{\log |RF/J \cap RF|}{|F|}$$

*Proof.* Using (5), write  $RG$  as  $\bigoplus_{F \times K} R$  and the dual group  $X_{RG}$  as  $R^G \cong R^{F \times K}$ , with the pairing

$$\langle (x_{(g,l)}), (c_{(g,l)}) \rangle = \phi \left( \sum_{(g,l) \in F \times K} c_{(g,l)} x_{(g,l)} \right),$$

where  $\phi$  sends  $R$  isomorphically to the group of complex  $n$ -th roots of unity. The action  $\alpha_{RG}$  then has the following description. For each  $x = (x_{(g,l)}) \in R^{F \times K}$  and  $(f, k) \in F \times K$ ,

$$\alpha^{(f,k)}(x)_{(g,l)} = x_{(f,k)^{-1}(g,l)}. \quad (7)$$

Furthermore,  $X_{RG/J} \cong J^\perp \subset X_{RG}$  and under this identification,  $\alpha_{RG/J}$  is given by the appropriate restriction of  $\alpha_{RG}$ .

Let  $\mathcal{J}$  be a set of generators for  $J$  and write each  $s \in \mathcal{J}$  in the form  $(c_{(g,l)}(s)) \in \bigoplus_{F \times K} R$ . Then,  $J^\perp$  can be described as the set of  $x = (x_{(g,l)}) \in R^{F \times K}$  satisfying

$$\sum_{(g,l) \in F \times K} c_{(g,l)}(s) x_{(f,k)(g,l)} = 0 \quad (8)$$

for all  $s \in \mathcal{J}$  and all  $(f,k) \in F \times K$ . Since  $\mathcal{J} \subset RF$ ,  $c_{(g,l)}(s) = 0$  for all  $l \neq 1$  and  $s \in \mathcal{J}$ . Furthermore, (6) implies

$$(f,k)(g,1) = (\tilde{k}^{-1} \tilde{k} f g, k) = (fg, k).$$

So, setting  $c_g(s) = c_{(g,1)}(s)$ , the relation (8) may be rewritten as

$$\sum_{g \in F} c_g(s) x_{(fg,k)} = 0,$$

and  $X_{RG/J}$  can thus be viewed as  $Q^K$ , where

$$Q = \left\{ (y_g) \in R^F : \sum_{g \in F} c_g(s) y_{fg} = 0 \text{ for all } s \in \mathcal{J} \text{ and } f \in F \right\}.$$

The action  $\alpha_{RG/J}$  is described on  $Q^K$  as follows. Let  $r = (r_g) \in R^F$ , define  $\beta \in \text{Aut}(R^F)$  by  $\beta^f(r)_g = r_{f^{-1}g}$  and note that  $Q$  is  $\beta$ -invariant. Then, for each  $q = (q_l) \in Q^K$  and  $(f,k) \in F \times K$ , it follows from (6) that

$$\alpha^{(f,k)}(q)_l = \beta^{e(f,k,l)} q_{k^{-1}l}, \quad (9)$$

where  $e(f,k,l) = \tilde{l}^{-1} \tilde{k} f \tilde{k}^{-1} l$ .

For any finite subset  $E$  of  $K$ , let  $\pi_E : Q^K \rightarrow Q^E$  be the coordinate projection and let  $\xi_E$  be the pre-image under  $\pi_E$  of the point partition of  $Q^E$ . By using (9), we obtain

$$\alpha^{(f,1)} \xi_{\{1\}} = \xi_{\{1\}} \text{ and } \alpha^{(1,k)} \xi_{\{1\}} = \xi_{\{k^{-1}\}},$$

for all  $f \in F$  and  $k \in K$ . Hence, if  $(F_i) = (F \times K_i)$  is the Følner sequence defined in Lemma 3.3, then

$$\xi_{\{1\}}^{F_i} = \bigvee_{k \in K_i} \alpha^{(1,k)} \bigvee_{f \in F} \alpha^{(f,1)} \xi_{\{1\}} = \bigvee_{k \in K_i} \alpha^{(1,k)} \xi_{\{1\}} = \bigvee_{k \in K_i} \xi_{\{k^{-1}\}} = \xi_{K_i^{-1}},$$

where  $K_i^{-1} = \{k^{-1} : k \in K_i\}$ . So,

$$H(\xi_{\{1\}}^{F_i}) = H(\xi_{K_i^{-1}}) = |K_i| \log |Q|.$$

Thus, since  $\xi_{\{1\}}$  is a generator,

$$h(\alpha_{RG/J}) = h(\alpha_{RG/J}, \xi_{\{1\}}) = \lim_{i \rightarrow \infty} \frac{|K_i| \log |Q|}{|F| |K_i|} = \frac{\log |Q|}{|F|}.$$

The required result follows, since  $Q$  is isomorphic to the self-dual finite abelian group  $RF/J \cap RF$ .  $\square$

*Proof of Theorem 3.2.* First note that since  $I \cap \mathbb{Z} \neq \{0\}$ , we can write  $\mathbb{Z}G/I$  as  $RG/J$ , where  $R = \mathbb{Z}/I \cap \mathbb{Z}$  and  $J$  is the image of  $I$  in  $RG$ . Set  $J'_i = J \cap RG_i$ ,  $J_i = RGJ'_i$  and  $M_i = RG/J_i$ . The natural chain of projections

$$M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow \dots$$

dualizes to a chain of inclusions

$$X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \dots \quad (10)$$

where each  $X_i = X_{M_i}$  may be regarded as a closed  $\alpha$ -invariant subgroup of  $X_{RG}$  and  $\bigcap_{i \geq 1} X_i = X$ . Note that Proposition 1 shows  $h(\alpha_{X_1}) < \infty$  and (10) implies  $(h(\alpha_{X_i}))$  is decreasing. By Theorem 3.1(2),

$$h(\alpha_X) + h(\alpha_{X_1/X}) = h(\alpha_{X_1}) = h(\alpha_{X_i}) + h(\alpha_{X_1/X_i}),$$

and (2) shows  $h(\alpha_{X_1/X_i}) \rightarrow h(\alpha_{X_1/X})$  as  $i \rightarrow \infty$ . Therefore,

$$h(\alpha_X) = \lim_{i \rightarrow \infty} h(\alpha_{X_i}).$$

By Proposition 1,

$$h(\alpha_{X_i}) = h(\alpha_{M_i}) = \frac{\log |RG_i/J'_i|}{|G_i|},$$

and since  $RG_i/J'_i \cong \mathbb{Z}G_i/I \cap \mathbb{Z}G_i$ , (4) follows.  $\square$

**4. Applications.** In this section, we apply results from Section 3 to solve some entropy range problems, answer some related questions and describe a new problem for actions of locally normal groups that does not yield to our methods.

**Theorem 4.1.** *If  $G$  is a countably infinite amenable group containing arbitrarily large finite normal subgroups, then  $\mathcal{H}(\{G\}) = [0, \infty]$ .*

*Proof.* Suppose  $F$  is a finite non-trivial normal subgroup of  $G$ . The left ideal  $I \subset \mathbb{F}_2G$  generated by the augmentation ideal in  $\mathbb{F}_2F$  satisfies

$$|\mathbb{F}_2F/I \cap \mathbb{F}_2F| = |\mathbb{F}_2| = 2.$$

Hence, applying Proposition 1, the natural  $G$ -action arising from the  $\mathbb{Z}G$  module  $\mathbb{F}_2G/I$  has entropy  $|F|^{-1} \log 2$ . Since  $|F|$  can be arbitrarily large, by forming products of such systems and using (2), it follows that  $(0, \infty] \subset \mathcal{H}(\mathcal{G})$ . A zero entropy example is given by the  $G$ -action on  $\mathbb{T}$  corresponding to the trivial map  $\alpha : G \rightarrow \{1_{\text{Aut}(\mathbb{T})}\}$ , so  $\mathcal{H}(\{G\}) = [0, \infty]$ .  $\square$

**Theorem 4.2.** *Suppose  $G$  is a countably infinite amenable group containing a finite non-trivial normal subgroup and let  $\mathcal{G} = \{G^d : d \geq 1\}$ . Then  $\mathcal{H}(\mathcal{G})$  is dense in  $[0, \infty]$ .*

*Proof.* Let  $d \in \mathbb{N}$ , let  $H$  be a normal subgroup of  $G$ , regard  $H^d$  as a normal subgroup of  $G^d$  and replace  $G$  by  $G^d$  and  $F$  by  $H^d$  in the proof of Theorem 4.1. Then, we obtain a  $G^d$ -action with entropy  $|H|^{-d} \log 2$ . By varying  $d$ , we see arbitrarily small entropies are possible and it follows that  $\mathcal{H}(\mathcal{G})$  is dense in  $[0, \infty]$ .  $\square$

**Remark 1.** Theorem 4.2 remains true if  $\mathcal{G}$  is replaced by any class of groups for which, given any  $n \in \mathbb{N}$ , there exists some  $G \in \mathcal{G}$  possessing a finite normal subgroup  $F$  with  $|F| > n$ .

For actions of locally normal groups, beyond a simple statement concerning the convergence of (4), it seems difficult to characterize the property of zero entropy algebraically. However, using the results of the previous section, we are able to deduce the following.

**Theorem 4.3.** *Let  $\alpha_M$  be an algebraic action of a locally normal group  $G$  on a compact finite-dimensional abelian group  $X_M$ . If any element  $a \in M$  of finite order is annihilated by the augmentation ideal in  $\mathbb{Z}G^{(a)}$ , for some infinite subgroup  $G^{(a)}$  of  $G$ , then  $h(\alpha_M) = 0$ .*

*Proof.* It is enough to prove the theorem in the case that  $M$  is itself cyclic and each of its elements has finite order; the more general result follows from the discussion after the proof of Theorem 3.1. So, we may assume  $M = RGa$  for some non-trivial quotient ring  $R$  of  $\mathbb{Z}$  and some non-zero  $a \in M$ . Furthermore, if  $I$  is the ideal of  $RG$  generated by the augmentation ideal in  $RG^{(a)}$ , by hypothesis there is a surjective  $RG$ -module homomorphism from  $RG/I$  onto  $M$ . Therefore, it remains to show that  $h(\alpha_{RG/I}) = 0$ .

Using the same notation as in the statement of Theorem 3.2, let  $\phi_i$  be the homomorphism from  $G_i$  to the unit group of  $R_i$  obtained by restricting the natural map  $RG_i \rightarrow R_i$  to  $G_i$ . Then,

$$|R_i| \leq |R\phi_i(G)| = |R|^{|G_i|}. \quad (11)$$

Hence, by Theorem 3.2

$$h(\alpha_{RG/I}) = \lim_{i \rightarrow \infty} \frac{\log |R_i|}{|G_i|} \leq \lim_{i \rightarrow \infty} \frac{|\phi_i(G_i)| \log |R|}{|G_i|} = \lim_{i \rightarrow \infty} \frac{\log |R|}{|\ker \phi_i|}. \quad (12)$$

The result follows, since by hypothesis  $\bigcup_{i \geq 1} \ker \phi_i$  contains  $G^{(a)}$ .  $\square$

**Example 1.** Given a finite alphabet  $A$  and a surjective homomorphism of locally normal groups  $\phi : G \rightarrow H$ , there is a very natural action of  $G$  on the shift space  $A^H$ , given by setting  $\alpha^g = \beta^{\phi(g)}$ , where  $g \in G$  and  $\beta$  is the full  $H$ -shift on  $A^H$ . We can realize this as an algebraic action via the left  $RG$ -module  $R\phi(G) = RH$ , where  $R = \mathbb{Z}/(|A|)$ . To calculate  $h(\alpha)$ , apply Theorem 3.2 and proceed as in the proof above, noting that  $\phi_i$  is simply the restriction of  $\phi$  to  $G_i$ , so equality holds in (11) and (12). Therefore,

$$h(\alpha) = |\ker \phi|^{-1} \log |R|.$$

Recall that a  $G$ -action  $\alpha$  on a compact abelian group  $X$  has *completely positive entropy* if for any non-trivial finite measurable partition  $\xi$  of  $X$ ,  $h(\alpha, \xi) \neq 0$ , and that  $\alpha$  is (strongly) *mixing* if, for all  $A, B \in \mathcal{B}(X)$ ,

$$\mu_X(A \cap \alpha^{g_n}(B)) \rightarrow \mu_X(A)\mu_X(B) \text{ as } n \rightarrow \infty,$$

for any sequence  $(g_n) \in G^{\mathbb{N}}$  with the property that, given any finite set  $F \subset G$ , there exists  $N \in \mathbb{N}$  such that  $n > N$  implies  $g_n \notin F$ . There is a fundamental result in classical ergodic theory which states that transformations with completely positive entropy are also mixing; this extends to actions of amenable groups [10]. The result also surfaces as a corollary to Theorem 4.3.

**Corollary 1.** *Let  $\alpha$  be an algebraic action of a locally normal group  $G$  on a compact finite-dimensional abelian group  $X$ . If  $\alpha$  has completely positive entropy, then  $\alpha$  is also mixing.*

*Proof.* Since  $\alpha$  has completely positive entropy, Theorem 3.1(1) shows that every element of the dual module  $M = \widehat{X}$  has finite order. By [11, Th. 1.6],  $\alpha$  is mixing if and only if the group

$$G^{(a)} = \{g \in G : (g-1)a = 0\}$$

is finite all non-zero  $a \in M$ . Suppose  $\alpha$  is not mixing. Then  $G^{(a)}$  is infinite for some non-zero  $a \in M$  and the augmentation ideal in  $\mathbb{Z}G^{(a)}$  annihilates the left  $\mathbb{Z}G$ -module  $L = \mathbb{Z}Ga$ . By Theorem 4.3, it follows that  $h(\alpha_L) = 0$ . Since  $\alpha_L$  is a factor of  $\alpha$ , we can find a non-trivial partition of  $X$  with zero entropy, contradicting the assumption of completely positive entropy.  $\square$

Let  $G$  be a locally normal group. An obvious way to construct examples of  $G$ -actions with infinite entropy is by gluing infinitely many systems with finite non-zero entropy together. More interesting examples arise from systems corresponding to cyclic modules. Suppose  $I \subset \mathbb{Z}G$  is a left ideal with  $I \cap \mathbb{Z} = \{0\}$ , so  $X_{\mathbb{Z}G/I}$  is connected. If the torsion-free rank of  $\mathbb{Z}G/I$  over  $\mathbb{Z}$  is finite, then  $X_{\mathbb{Z}G/I}$  is finite-dimensional and Theorem 3.1(1) shows  $h(\alpha_{\mathbb{Z}G/I}) = 0$ . However, if  $I$  is finitely generated then  $X_{\mathbb{Z}G/I}$  has infinite dimension and a calculation similar to that used in the proof of Proposition 1 shows that  $h(\alpha_{\mathbb{Z}G/I}) = \infty$ . Thus, it is possible to find examples of algebraic actions of locally normal groups on compact connected abelian groups with both zero and infinite entropy. An obvious question is, can a connected system with finite non-zero entropy be found? It is clear that such an example must arise from an infinitely generated ideal in  $\mathbb{Z}G$ . It may be useful to note that even if Yuzvinskiĭ's addition formula holds in this setting, the limiting technique used in the proof of Theorem 3.2 fails, because the leading terms in (10) correspond to systems with infinite entropy.

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