IMA Journal of Applied Mathematics (2005) 1–??
doi: 10.1093/imamat/hxh000

Gap solitary waves in two-layer fluids

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We present computations of solitary waves in a two-layered fluid flow. Both the effects of gravity and interfacial tension are considered. The top surface of the upper fluid is free. The solitary waves are found in a narrow 'gap' which exists between the two branches of the dispersion relation. Both in-phase and out-of-phase solitary waves are presented.

Keywords: solitary waves

1. Introduction

We consider a two-dimensional steady potential flow with a fluid of constant density lying on top of a heavier fluid. The lower fluid is of finite depth and the upper fluid is bounded above by a free surface. Fully-nonlinear boundary conditions are imposed on both interfaces. In this configuration there are two independent modes of oscillations about the state of equilibrium: a 'fast' and a 'slow' mode. There are only a small number of studies dealing with this configuration, as the interaction between the two modes makes the problem difficult to treat.

The effect of gravity on the fluid flow has been studied, computationally and analytically, by several authors in the last decades (see e.g. Peters & Stoker (1960), Kakutani & Yamasaki (1978), Moni & King (1995), Michallet & Dias (1999), Părău & Dias (2001), Iooss *et al.* (2002), Craig *et al.* (2005), Craig *et al.* (2011)). When the surface tension and/or the interfacial tension are included, the dispersion relation changes significantly (see Iooss (1999), Barrandon & Iooss (2005)). In a recent paper Woolfenden & Părău (2011) computed solitary waves in the cases when either only the surface tension was nonzero, or both the interfacial and surface tension were nonzero.

In this paper we calculate solitary waves in two-layer fluids with a free surface when only the effect of interfacial tension is considered. The study of the dispersion relation in this case reveals a gap between the two branches for some values of the parameters. The upper branch of the dispersion relation often has a minimum at a finite wavenumber which may be close to a maximum of the lower branch of the dispersion relation. The gap between these two extrema can be quite small in some cases. This was not the case when we assumed that the surface tension was present while the interfacial tension was zero (see Woolfenden & Părău (2011)).

In a three-layer stratified flow Grimshaw & Christodoulides (2001) have also found a spectral gap in the wavenumber space for a certain range of parameters. They used a weakly nonlinear analysis to derive a system of nonlinear evolution equations to describe the waves associated with this gap. The interfacial tension played an important role in obtaining solitary-wave solutions for this system. The properties of these solitary waves for this weakly nonlinear system were studied in Grimshaw & Christodoulides (2008), where they are called gap-soliton solutions. In their case the basic horizontal velocity of the flow in the middle layer was assumed to be different from that in the neighbouring layers, such that there is always a shear flow or a jet flow. The gap-solitons are known to be found in other physical problems, such as nonlinear optics (see Chen & Mills (1987)).

The problem is formulated in section 2. The dispersion is studied in section 3. The numerical scheme is briefly described in section 4 and the numerical results are presented in section 5, followed by a discussion in section 6.

2. Formulation

The two superposed fluids are assumed to be perfect, the flow to be irrotational in each layer and there is no shear between the layers. The lower fluid of density ρ_1 has a rest equilibrium thickness h_1 and the upper fluid of density ρ_2 ($< \rho_1$) has a rest equilibrium thickness h_2. Generally, the subscript 1 refers to quantities related to the lower fluid, and the subscript 2 to the upper fluid. As we are interested only in steady flows, we adopt a frame of reference moving with the solitary waves at a constant speed U. We use h_2 as unit length and U as velocity scale to make all the quantities dimensionless. The following dimensionless parameters are introduced:

$$H = \frac{h_2}{h_1}, \qquad F = \frac{U}{\sqrt{gh_2}}, \qquad \tau = \frac{\sigma}{\rho_2 g h_2^2}, \qquad R = \frac{\rho_2}{\rho_1},$$

where g is the acceleration due to gravity and σ is the interfacial tension.

We introduce Cartesian dimensionless coordinates x, y with the y-axis directed vertically upwards. We choose the level $y = 0$ to be the undisturbed level of the interface, so the botom is at $y = -\frac{1}{H}$ and the unperturbed free surface is at $y = 1$. We denote by $y = \eta(x)$ the position of the interface and by $y = 1 + \zeta(x)$ the equation of the free surface. In dimensionless form, the governing equation in each layer is

$$\nabla^2 \Phi_i = 0, \qquad i = 1, 2, \tag{1}$$

where Φ_i is the velocity potential in layer i. On the interface $y = \eta(x)$ there are two kinematic conditions

$$\frac{\partial \Phi_i}{\partial x}\eta' - \frac{\partial \Phi_i}{\partial y} = 0, \qquad i = 1, 2, \tag{2}$$

and a dynamic condition, obtained after the use of Bernoulli's equation in each fluid

$$\frac{1}{2}|\nabla \Phi_1|^2 - \frac{1}{2}R\,|\nabla \Phi_2|^2 - \frac{1-R}{2} + \frac{1-R}{F^2}\,\eta - \frac{\tau}{F^2}\frac{\eta''}{(1+(\eta')^2)^{\frac{3}{2}}} = 0, \tag{3}$$

The prime denotes differentiation with respect to x. Along the free surface $y = 1 + \zeta(x)$ there is the kinematic condition

$$\frac{\partial \Phi_2}{\partial x}\zeta' - \frac{\partial \Phi_2}{\partial y} = 0, \qquad i = 1, 2, \tag{4}$$

and the dynamic condition

$$\frac{1}{2}|\nabla \Phi_2|^2 - \frac{1}{2} + \frac{1}{F^2}\,\zeta = 0. \tag{5}$$

The system is closed with the kinematic condition at the bottom $y = -1/H$

$$\frac{\partial \Phi_1}{\partial y} = 0. \tag{6}$$

As we are looking for solitary waves, we have imposed $\eta \to 0$, $\zeta \to 0$ and $\frac{\partial \Phi_j}{\partial x} \to 1$ as $|x| \to \infty$.

3. Dispersion relation

Considering linear travelling waves with a dimensionless wavenumber k, we obtain the dispersion relation

$$F^4k^2\left(1+R\tanh\left(\frac{k}{H}\right)\tanh(k)\right)-F^2k\left(\tanh\left(\frac{k}{H}\right)+\tanh(k)+k^2\tau\tanh\left(\frac{k}{H}\right)\right)$$
$$+(1-R+\tau k^2)\tanh\left(\frac{k}{H}\right)\tanh(k)=0.$$

Solving this equation we obtain two positive roots for the Froude number, F,(see also Woolfenden & Părău (2011))

$$F_\pm^2(k)=\frac{b(k)\pm\sqrt{b^2(k)-4\,a(k)\,c(k)}}{2\,k\,a(k)} \tag{7}$$

where

$$\left.\begin{aligned} a(k)&=1+R\tanh\left(\tfrac{k}{H}\right)\tanh(k),\\ b(k)&=\tanh\left(\tfrac{k}{H}\right)+\tanh(k)+k^2\tau\tanh\left(\tfrac{k}{H}\right),\\ c(k)&=(1-R+\tau k^2)\tanh\left(\tfrac{k}{H}\right)\tanh(k).\end{aligned}\right\} \tag{8}$$

The solution F_+ corresponds to the 'fast' mode (or surface-wave mode) and F_- corresponds to the 'slow' mode (or internal-wave mode). It is worth noting that when the free-surface is replaced by a rigid lid the Froude number satisfying the dispersion relation is (see e.g. Laget & Dias (1997))

$$F_{lid}^2(k)=\frac{(1-R+\tau k^2)\tanh\left(\frac{k}{H}\right)\tanh(k)}{k\left(R\tanh\left(\frac{k}{H}\right)+\tanh(k)\right)}.$$

The 'fast' critical Froude number F_+ and F_{lid} have the same asymptotic behaviour for large k, and it can be shown that

$$F_{lid}^2\approx F_+^2=\frac{\tau}{1+R}k+\frac{1-R}{1+R}\frac{1}{k}+O\left(\frac{1}{k^2}\right),\qquad \text{as } k\to\infty.$$

In the long wave-limit ($k\to 0$) the 'slow' mode F_- and F_{lid} approach the same value when H is large:

$$F_{lid}^2\approx F_-^2=\frac{1-R}{H}-\frac{R(1-R)}{H^2}+O\left(\frac{1}{H^3}\right).$$

They also have the same asymptotic behaviour in the long wave-limit for $0<1-R\ll 1$:

$$F_{lid}^2\approx F_-^2=\frac{1-R}{H+1}+O\left((1-R)^2\right).$$

For different values of the parameters we observe that the upper curve F_+ has a minimum and the lower branch F_- has a maximum for finite values of k close to each other. Moreover, by decreasing τ, the gap between these two extrema may become very small (see Fig. 1). For values of F in the gap between these two extrema we search for solitary waves. Woolfenden& Părău (2011) showed that when the surface tension on the free-surface is nonzero and the interfacial tension is zero there is a gap between the two modes of the dispersion relation, for some parameters values. It was found however that the gap cannot be made small as in the case presented here. If both the surface tension and interfacial tension were assumed to be nonzero, there is no gap between the two branches as in this case F_+ and F_- approach infinity as k tends to infinity.

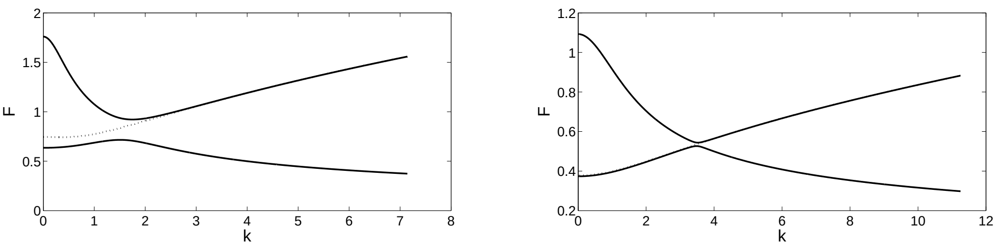

FIG. 1. Dispersion relation curves (solid lines) for $\tau = 0.5$, $H = 0.4$, $R = 0.5$ (left) and for $\tau = 0.1$, $H = 3$, $R = 0.5$ (right). The Froude number corresponding to the rigid-lid approximation is also shown (dotted line).

4. Numerical method

The numerical scheme used is described in detail in Woolfenden & Părău (2011), so it is reviewed only briefly here. The problem is solved using a boundary integral equation method based on Cauchy's integral formula. The complex potential is introduced in each layer $w_j = \Phi_j + i\Psi_j$, where Ψ_j is the stream function in layer $j = 1, 2$. The problem is reformulated in the inverse plane, as the complex potential is chosen as the independent variable. We choose $\Psi_1 = \Psi_2 = 0$ on the interface, so that $\Psi_1 = -1/H$ on the bed and $\Psi_2 = 1$ on the free-surface. We also choose the potentials $\Phi_1 = \Phi_2 = 0$ at $x = 0$. By applying the Cauchy's integral formula twice on the upper layer with the evaluation point on the interface and on the free-surface respectively, and once on the lower layer, three integro-differential equations are obtained (see equations (3.2)-(3.4) from Woolfenden & Părău (2011) for more details) . The unknowns are $x'(\Phi_0) = x'(\Phi + i0)$, $y'(\Phi_0) = y'(\Phi_0 + i0)$, $X'(\Phi_0) = x'(g(\Phi_0) + i)$, $Y'(\Phi_0) = y'(g(\Phi_0) + i)$ and $g'(\Phi_0)$, where $\Phi_2 = g(\Phi_1)$ is the (unknown) link between the two potentials and Φ_0 is a point on the interface. The Bernoulli equation at the interface (3) and on the free-surface (5) become

$$\frac{1}{2}F^2\left(\frac{1 - R(g')^2}{(x')^2 + (y')^2} - (1 - R)\right) + (1 - R)\,y - \tau\frac{x'y'' - x''y'}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}} = 0, \tag{9}$$

$$\frac{1}{2}F^2\left(\frac{(g')^2}{(X')^2 + (Y')^2} - 1\right) + (Y - 1) = 0. \tag{10}$$

We assume the symmetry of solitary waves and reduce the integration range to $[0, \infty)$. The interfacial potential Φ_1 is discretized using N equally spaced points. Finite-difference and interpolation formulae are used for the various quantities appearing on the problem. Following Woolfenden & Părău (2011) we reduce the number of unknowns and obtain a nonlinear system of $3N$ equations for $3N$ unknowns y', X' and Y', which is solved by Newton's method. To calculate solitary waves, a localised pressure is applied either on the interface, or on the free-surface. Forced waves are first computed for a given value of the Froude number in the gap between the two extrema discussed in the previous section. Using a continuation method (see Vanden-Broeck 2010), the pressure is gradually eliminated and solitary waves are obtained. Once a solitary wave is calculated, we use the continuation method by varying the parameters of the problem τ, R, H to obtain new solutions. The numerical accuracy of the method was tested by varying the number of mesh points and the domain size. Most of the results presented here were obtained with $N = 600$ and $0 \leqslant \Phi_1 \leqslant 40$, or in some cases with $N = 1200$.

5. Numerical results

Using the numerical scheme described above we have calculated various solitary waves. We were able to find out-of-phase solitary waves, when the free-surface is either of elevation, or of depression. We also obtained depression and elevation in-phase solitary waves.

In Figure 2 we show the amplitude of the free-surface and interface out-of-phase solitary waves for $H = 0.1$, $R = 0.5$, $\tau = 0.5$. The gap between the two extrema of the branches of the dispersion relation is given by $0.7158 < F < 0.9222$. The out-of-phase solitary waves bifurcate from the lower critical Froude number at $F_{-} = 0.7158$. It can be observed that the amplitudes of depression free-surface/elevation interface solitary waves at $x = 0$ are smaller then those of elevation free-surface/depression interface solitary waves.

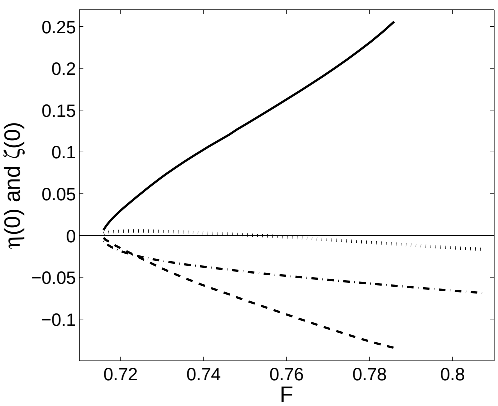

FIG. 2. Amplitudes of out-of-phase solitary waves for $H = 0.1$, $R = 0.5$, $\tau = 0.5$. The elevation free-surface/depression interface solitary waves are marked by solid (free-surface) and dashed (interface) lines. The depression free-surface/elevation interface solitary waves are marked by dashed-dotted (free-surface) and dotted (interface) lines.

The amplitude of the elevation of the interfacial wave in the case of depression free-surface/elevation interface solitary waves increases first with Froude number, then it starts to decreases. At some point the elevation interfacial wave becomes of depression at $x = 0$ (see Figure 2).

Some profiles of these out-of-phase waves are presented in Figure 3 and Figure 4 for values of F close to the bifurcation point (left graphs) and far from it (right graphs). It can be observed that for values of F not so close to the critical value $F_{-} = 0.7158$ the depression free-surface/elevation interface solitary waves have deep troughs and small crests on the interface,

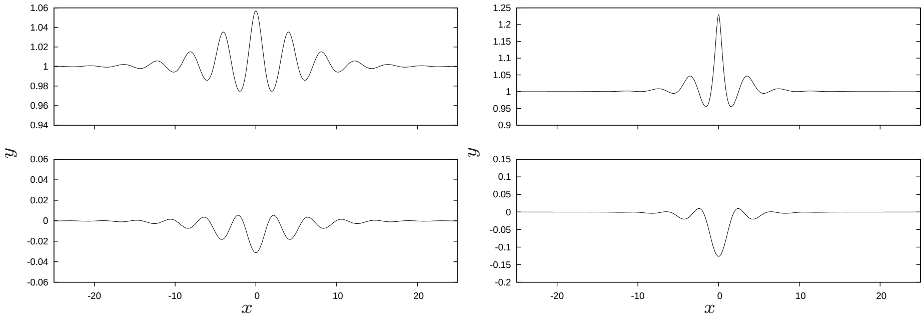

FIG. 3. Profiles of elevation free-surface/depression interface solitary waves for $F = 0.7265$ (left) and $F = 0.7798$ (right). The other parameters are $H = 0.1$, $R = 0.5$, $\tau = 0.5$.The free-surface is presented in the upper plots and the interface in the lower plots.

and shallow troughs and tall crests for the free-surface. The same is true for elevation free-surface/depression interface solitary waves.

The amplitudes of in-phase solitary waves are shown in Figure 5 for $H = 3$, $R = 0.5$, $\tau = 0.5$. The gap between the two extrema of the branches of the dispersion relation is given by $0.6717 < F < 0.7737$. Both elevation and depression waves are calculated. Some solutions are presented in Figure 6. This time the solutions bifurcate from the upper critical Froude number which in this case is $F_+ = 0.7737$.

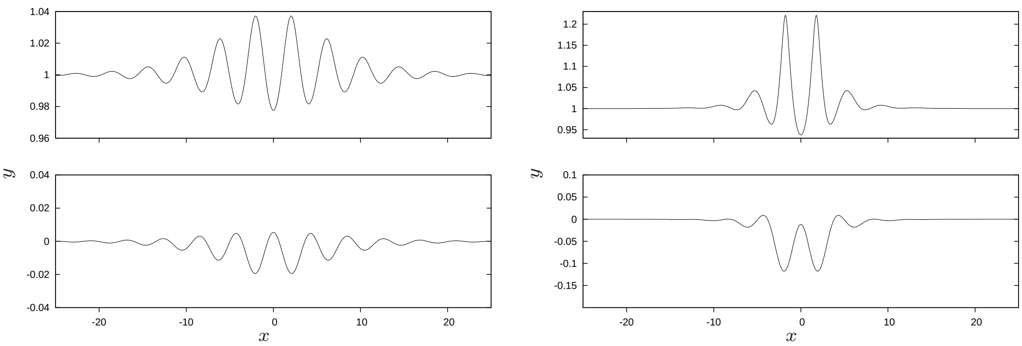

FIG. 4. Profiles of depression free-surface/elevation interface solitary waves for $F = 0.7224$(left) and $F = 0.7913$ (right). The other parameters are $H = 0.1$, $R = 0.5$, $\tau = 0.5$. The free-surface is presented in the upper plots and the interface in the lower plots.

We have presented out-of-phase and in-phase solitary waves for different parameters, but they also coexist for the same set of parameters H, R, τ. For example, we plot the amplitudes of both types of solutions for $H = 0.4$, $R = 0.5$, $\tau = 0.5$ (see Figure 7) together with some profiles (see Figure 8). In this case the gap in the spectrum is given by $0.71 < F < 0.925$. The branch of out-of-phase solitary waves bifurcating from the 'slow' Froude number cannot be followed

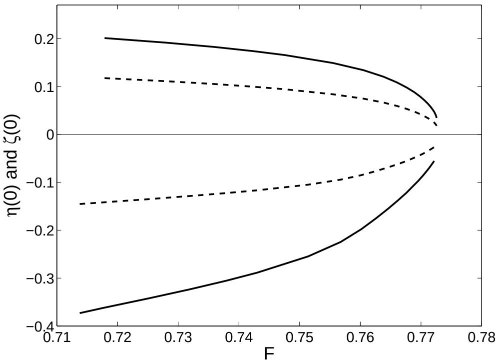

FIG. 5. Amplitudes of in-phase solitary waves for $H = 3$, $R = 0.5$, $\tau = 0.5$. Both the depression waves and elevation waves are shown. The free-surface is plotted in solid lines, the interface in dashed lines.

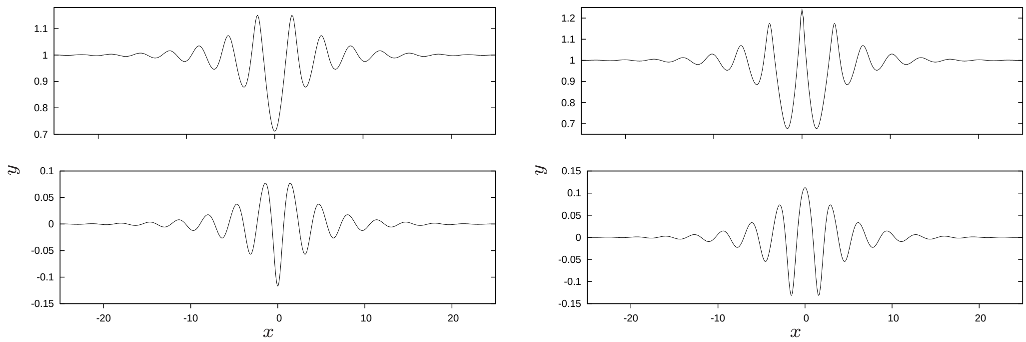

FIG. 6. Profiles of depression solitary waves for $F = 0.7430$ (left) and elevation solitary waves for $F = 0.7279$ (right). The other parameters are $H = 3$, $R = 0.5$, $\tau = 0.5$. The free-surface is presented in the upper plots and the interface in the lower plots.

up to values of F where the other branch of in-phase solitary waves bifurcated from the 'fast' Froude number is found, as the free surface approaches a limiting configuration at some point. The crests of the free surface waves become narrower and taller and they approach, presumably, the 120° limit configuration of gravity solitary waves in finite depth (see Stokes 1880, Amick & Toland 1981), as there is no surface tension. However, with our numerical method we cannot follow the waves up to this limiting configuration.

The solitary waves can be found for smaller value of τ and higher value of R, when the gap between the 'fast' and 'slow' mode becomes narrower. We were able to compute solitary waves down to $\tau = 0.29$ and up to $R = 0.9$. Here the branches of the solitary waves are shorter, and some of the calculations suggest that the waves bifurcate from a finite-amplitude, not from zero. A similar observation was made by Laget & Dias (1997) in the case of interfacial gravity-capillary waves between two infinite fluids. The waves have more oscillations in this case near the bifurcation point, as the bifurcation wavenumber corresponding to the gap between the modes increases and the numerical computations become more difficult to perform.

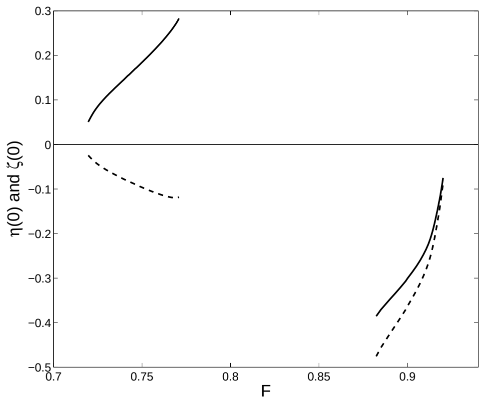

FIG. 7. Amplitudes of in-phase and out-of-phase solitary waves for $H = 0.4$, $R = 0.5$, $\tau = 0.5$. The free-surface is plotted in solid lines, the interface in dashed lines.

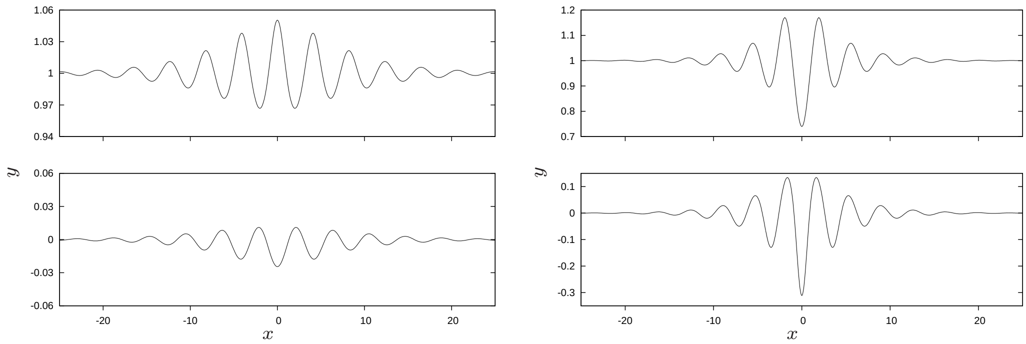

FIG. 8. Profiles of out-of-phase solitary waves for $F = 0.7196$ (left) and in-phase solitary waves for $F = 0.9072$ (right). The free-surface is presented in the upper plots and the interface in the lower plots.

6. Discussion

In-phase and out-of-phase solitary waves were computed for values of Froude number in the spectral gap between the maximum of the 'slow' branch and the minimum of the 'fast' branch of the dispersion relation. Gravity and interfacial tension have been included in the formulation. All the possible combinations of elevation/depression of the free-surface/interfacial waves were found to exist, depending on the values of the parameters.

The in-phase solitary waves bifurcate at a finite wavenumber from the 'fast' critical Froude number, while the out-of-phase solitary waves bifurcate from the 'slow' critical Froude number. The solitary waves resemble wave packets, having damped oscillations in their tails for small amplitudes. The branches of solutions can be followed up to large amplitudes, until some limiting configurations are obtained.

Similar results have been obtained by Grimshaw & Christodoulides (2008), where all type of in-phase/out-of-phase small-amplitudes gap-solitons are shown to exist for the nonlinear coupled equations which were derived using a weakly-nonlinear analysis of thee-layer stratified flows. However, they are interested in the case where the linear dispersion relation of the system has two distinct wave modes at the critical point when the gap is closed. This condition can be satisfied only if there is background shear (see Grimshaw & Christodoulides 2001, 2008 for more details), which is not the case in our problem. Grimshaw & Christodoulides (2008) have also observed that there is phase variation between the envelopes of the solitary waves in the two interfaces, which does not occur in our solutions.

Acknowledgement

This research was supported by the EPSRC under Grant No. EP/H008489/1. The research presented in this paper was carried out on the High Performance Computing Cluster supported by the Research Computing Service at the University of East Anglia. The authors also want to thank the anonymous referee for his valuable and constructive comments.

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