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Nonlinear three-dimensional gravity–capillary solitary waves

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Steady three-dimensional fully nonlinear gravity–capillary solitary waves are calculated numerically in infinite depth. These waves have decaying oscillations in the direction of propagation and monotone decay perpendicular to the direction of propagation. They travel at a velocity U smaller than the minimum velocity c_{min} of linear gravity–capillary waves. It is shown that the structure of the solutions in three dimensions is similar to that found by Vanden-Broeck & Dias (*J. Fluid Mech.* vol. 240, 1992, pp. 549–557) for the corresponding two-dimensional problem.

1. Introduction

Three-dimensional gravity–capillary solitary waves travelling at the surface of a fluid of infinite depth with a velocity $U < c_{min}$ are considered. Here c_{min} is the minimum phase velocity of linear gravity–capillary waves defined as

$$c_{min} = \left(\frac{4gT}{\rho}\right)^{1/4}, \tag{1.1}$$

where T is the constant coefficient of surface tension, g is the acceleration due to gravity and ρ is the fluid density. Vanden-Broeck & Dias (1992) and Dias Menasce & Vanden-Broeck (1996) considered the corresponding two-dimensional problem. They calculated numerically branches of solitary waves which have either a central depression or a central elevation. Iooss & Kirrmann (1996) have proved the existence of these two branches of solitary waves for $c_{min} - U > 0$ small. Analytical approximations for solitary waves with either a central elevation or depression were given by Dias & Iooss (1993). Solitary gravity–capillary waves for $U \ll c_{min}$ were also computed by Longuet-Higgins (1989).

In this paper we calculate fully nonlinear three-dimensional solitary waves by a boundary integral equation method. The numerical procedure is similar to that used by Forbes (1989) and by Părău & Vanden-Broeck (2002) for pure gravity waves. We show that the three-dimensional problem is qualitatively similar to the two-dimensional problem. In particular there are branches of fully localized three-dimensional gravity–capillary solitary waves. These waves have decaying oscillations in the direction of propagation and are monotonically decaying perpendicular to the direction of propagation. The curves obtained by cutting the free surface with planes parallel to the direction of propagation are qualitatively similar to the profiles for the two-dimensional problem obtained by Vanden-Broeck & Dias (1992) and by Dias *et al.* (1996).

Three-dimensional gravity–capillary waves do not appear to have been calculated before in the fully nonlinear regime. However Kim & Akylas (2005) derived asymptotic formulae for gravity–capillary solitary waves for values of U close to c_{min}. Three-dimensional solitary waves were also obtained for a weakly nonlinear model by Milewski (2005). These works complement our numerical investigations. Furthermore the theoretical existence of three-dimensional fully localized solitary waves, but on finite depth, was predicted in some very recent papers by Groves (2004) and Groves & Sun (2004). Fully localized solitary-wave solutions are also known to exist as solutions of the KP-I equation, which is a model equation used in the case of finite depth and strong surface tension. Zhang (1995) performed experiments on three-dimensional gravity–capillary waves in a wind-wave tank and observed isolated steep surface dips which resemble the gravity–capillary solitary waves computed in this paper.

The formulation of the problem is considered in § 2 and the numerical results are presented in § 3.

2. Formulation

We consider a three dimensional solitary wave travelling at a constant velocity U at the upper surface of a fluid of infinite depth. The fluid is incompressible and the flow is irrotational. We choose a frame of reference moving with the wave and assume that the flow is steady. We introduce Cartesian coordinates x, y, z with the z-axis directed vertically upwards (opposite to the direction of gravity) and the x-axis in the direction of the wave propagation. We denote by $z=\zeta(x,y)$ the equation of the free surface. We introduce dimensionless variables by using U as the unit of velocity and $T/\rho U^2$ as the unit of length. The velocity potential function $\Phi(x,y,z)$ satisfies Laplace's equation

$$\nabla^2\Phi = 0, \qquad x, y \in \boldsymbol{R}, z < \zeta(x,y), \tag{2.1}$$

in the flow domain.

The kinematic and dynamic boundary conditions can be written as

$$\Phi_x\zeta_x + \Phi_y\zeta_y = \Phi_z, \quad z = \zeta(x,y), \tag{2.2}$$

$$\frac{1}{2}\left(\Phi_x^2+\Phi_y^2+\Phi_z^2\right)+\alpha\zeta-\left[\frac{\zeta_x}{\sqrt{1+\zeta_x^2+\zeta_y^2}}\right]_x-\left[\frac{\zeta_y}{\sqrt{1+\zeta_x^2+\zeta_y^2}}\right]_y=\frac{1}{2}, \quad z=\zeta(x,y). \tag{2.3}$$

We have used the conditions

$$(\Phi_x, \Phi_y, \Phi_z) \to (1,0,0), \;\; \zeta \to 0 \text{ as } \quad (x^2+y^2)^{1/2} \to \infty \tag{2.4}$$

to fix the value of Bernoulli's constant in (2.3). The parameter α in (2.3) is defined by

$$\alpha = \frac{gT}{\rho U^4}.$$

We solve the problem numerically by a boundary integral equation method. The approach follows the work of Părău & Vanden-Broeck (2002) and Forbes (1989) in which pure gravity flows are calculated. One different feature for the numerical scheme is that the surface tension term introduces higher derivatives in (2.3). They are approximated by centred-difference formulae. Another new feature is that no radiation condition is needed. Instead the solutions are assumed to be symmetric about the x- and y-axes. The discretization involves a regular grid with N points in

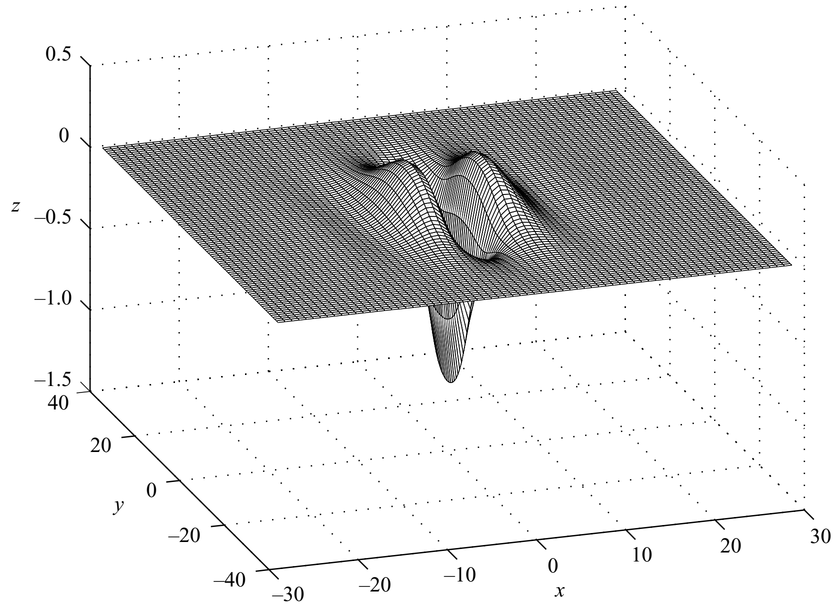

FIGURE 1. Solitary gravity–capillary wave with central depression for $\alpha = 0.35$. The surface elevation is vertically exaggerated by a factor 20.

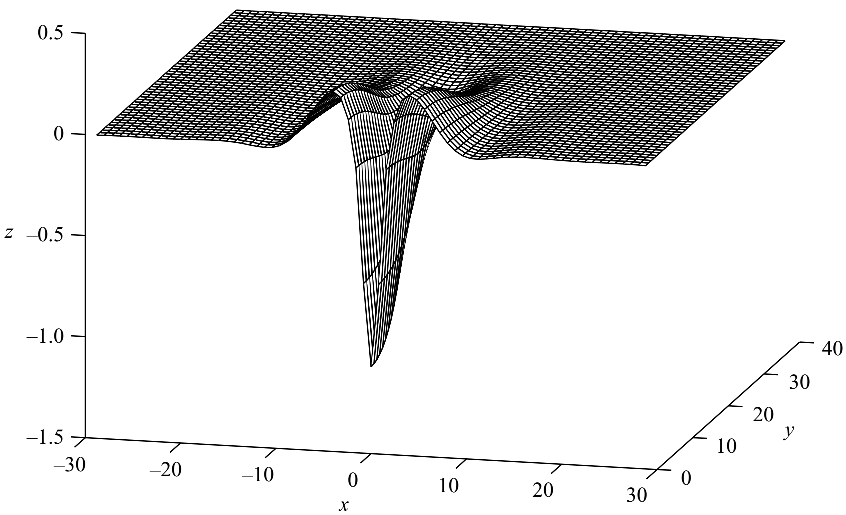

FIGURE 2. Central depression solitary gravity–capillary wave for $\alpha = 0.35$. Only half of the solution is shown. The surface elevation is vertically exaggerated by a factor 20.

the x-direction and M in the y-direction. The uniform mesh size on the x- and y-axes are denoted by Δx and Δy. The algebraic equations obtained after discretization are solved by Newton's method. A suitable initial guess to compute solitary waves is

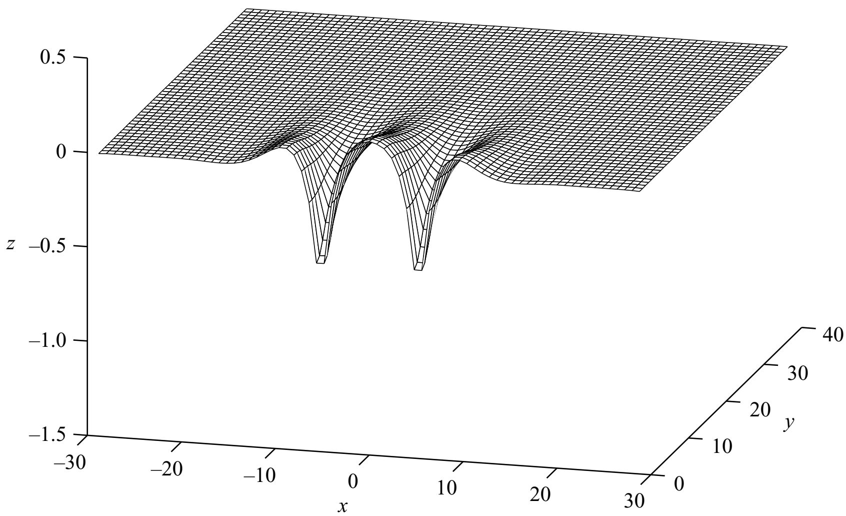

FIGURE 3. Central elevation solitary gravity–capillary wave for $\alpha = 0.35$. Only half of the solution is shown. The vertical exaggeration is by a factor 20.

obtained by adapting Vanden-Broeck & Dias's (1992) method from two dimensions to three dimensions. This consists of first obtaining solutions for the problem with an extra pressure term in equation (2.3) and then taking the limit as the magnitude of the pressure tends to zero.

3. Results

We computed solutions for various values of α by using the numerical scheme described in the previous section. In all cases α is assumed to be greater than 1/4, which corresponds to waves moving steadily with a constant velocity U smaller than the minimum phase speed c_{min} defined by (1.1). In this case only a highly localized disturbance of the water surface is predicted. In two dimensions, Vanden-Broeck & Dias (1992) computed capillary-gravity waves in the same regime of parameters ($\alpha > 1/4$).

Most of the computations were performed with $\Delta x = \Delta y = 0.8$ and $N = 40$, $M = 50$. The accuracy of the solutions has been tested by varying the number of grid points and the intervals Δx and Δy between grid points. To indicate the numerical consistency we have found that for $\alpha = 0.35$ if we change Δx and Δy from 0.8 to 0.6 the corresponding change in the wave surface elevation is less than 5% everywhere. The smallness of Δx and Δy is limited by memory capacity in order to obtain a truncated domain large enough to encompass nearly all of the wave disturbance.

We present typical free surface profiles in figures 1–3 and 5. As expected, there is a localized disturbance of the water surface. For each α there is a central depression wave ($\zeta(0, 0) < 0$, figure 1 for the full solution and figure 2 for half of the solution) and a central elevation wave ($\zeta(0, 0) > 0$, figure 3). The waves have decaying oscillations in the direction of propagation and monotonic decay in the direction perpendicular to the direction of propagation. Figure 4 shows the x and y cross-sections and a close-up in natural scaling for $\alpha = 0.35$. As α decreases and approaches 1/4 more and more oscillations appear in front of and behind the main disturbance.

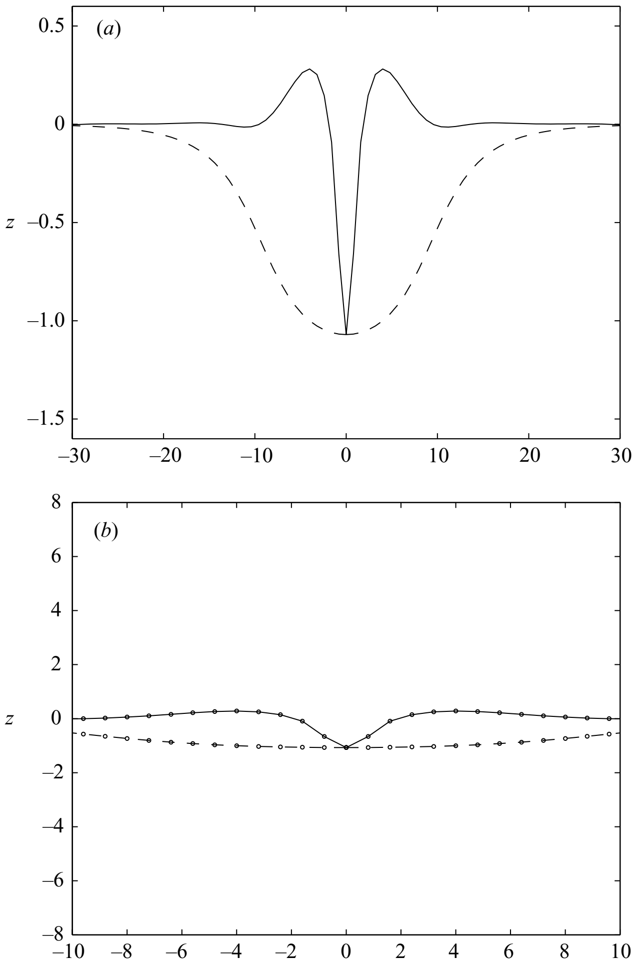

FIGURE 4. (*a*) The centreline in the Ox-direction (solid line) and Oy-direction (dashed line) for a central depression wave and $\alpha = 0.35$. The vertical exaggeration is by a factor 20. (*b*) A close-up with natural scaling.

The amplitude of the free capillary-gravity waves decreases to zero as α decreases to $1/4$. Their form (see figure 5) suggests that they approach a train of two-dimensional (constant in the y-direction) periodic waves in the limit as α decreases to $1/4$. When α is increased, the solitary capillary-gravity wave elevation decreases quickly in every direction and the surface has the form of a central depression or central elevation three-dimensional fully localized solitary wave (see figures 1–3).

The branches of solitary waves are shown in figure 6. These branches bifurcate from the uniform stream at $\alpha = 1/4$. For the corresponding two-dimensional problem Vanden-Broeck & Dias (1992) and Dias *et al.* (1996) found, as α increases, that the central depression waves reach a limiting configuration with a trapped bubble at the trough. Also the branch of the central elevation waves has a complicated structure with multiple turning points (see figure 3.2 in Dias *et al.* 1996). Further work is needed to reveal the related behaviour in the three-dimensional case.

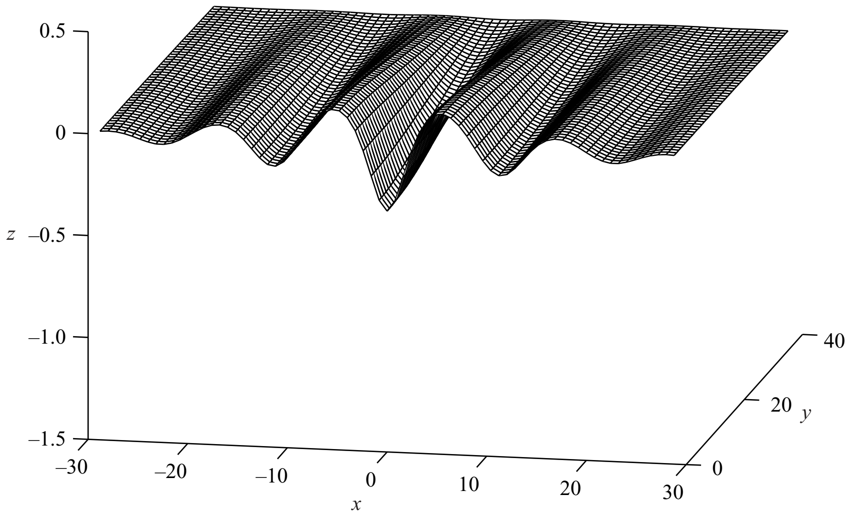

FIGURE 5. Solitary capillary-gravity waves for $\alpha = 0.266$. Only half of the solution ($y \geqslant 0$) is shown. The vertical exaggeration is by a factor 20.

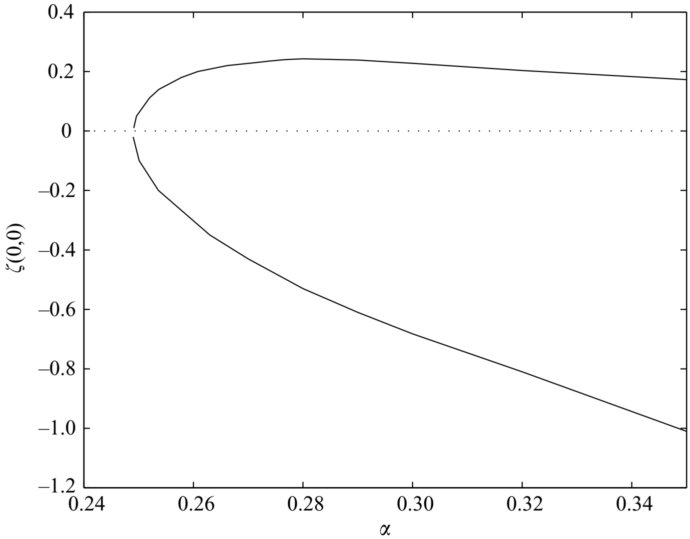

FIGURE 6. Values of the amplitude $\zeta(0, 0)$ versus α.

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