BOOK REVIEWS

SPECTRAL THEORY OF NON-SELF-ADJOINT TWO-POINT DIFFERENTIAL OPERATORS
(Mathematical Surveys and Monographs 73)

By John Locker 252 pp., US$65.00, isbn 0-8218-2049-4
(American Mathematical Society, Providence, RI, 2000).

There are, of course, many books on the subject of linear differential equations, including those by Titchmarsh, Coddington and Levinson, Hellwig, Dumford and Schwartz, Naimark, Edmunds and Evans, to mention only a few. Some of these authors restrict themselves to classical analytic methods, while others use methods of functional analysis which in many cases they combine with a classical approach to achieve their end. The book under review is in the latter class, and is an up-to-date account of the spectral theory of non-self-adjoint ordinary differential equations on a compact interval of the real line.

The early work in functional analysis, by Fredholm, Hilbert and Von Neumann, for example, was driven by problems in integral and differential equations, and a rich theory of self-adjoint operators was developed. This led to many elegant results in the spectral theory of self-adjoint differential equations, which included a study of the essential spectrum and eigenfunction expansion. However, when self-adjointness is removed, substantial difficulties appear: the eigenvalues may no longer be real, the essential spectrum need no longer be confined to the real line, and the question of an eigenfunction expansion is much more complex, depending on the notion of root subspace, and may even fail to exist. Thus the well-known self-adjoint operator theoretic approach becomes much more complex. This book contains a clear exposition of this theory and its application to two-point boundary value problems; it is divided into two sections.

First, the author collects relevant results from the spectral theory of operators in a Hilbert space setting. As a lot of this material is well known, proofs are often omitted, but references are provided. Basic notions of operator calculus are introduced, and the spectral theory of bounded linear operators is reviewed. The ascent and descent of an operator is defined, and the eigenvalues, generalized eigenfunctions and resolvents of such operators are introduced. The connection between the ascent and the descent of an operator and the order of a pole of the resolvent is established. Operators defined from ordinary differential equations are used to illustrate many of these topics. These illustrations provide examples of operators whose spectrum consists of a countable set, the empty set, or all of the complex plane, which, as the author remarks, are common occurrences in the spectral theory both of nth-order ordinary differential equations and of Fredholm operators in general. Chapter 2 deals with Fredholm theory, with special emphasis on the Hilbert–Schmidt operators and discrete Hilbert–Schmidt operators. The basic notions of Fredholm operators are introduced, and the spectral theory for these operators is developed. The expansion problem for a vector in terms of the generalized eigenfunctions is introduced.

In the second part of the book, the spectral theory of two-point boundary value problems is discussed in relation to the concepts introduced in Chapters 1 and 2. Chapter 3 introduces the spectral theory of linear two-point nth-order differential operators in a Sobolev space. Appropriate boundary conditions are defined, and the adjoint operator is calculated. This leads to the notion of maximal and minimal operators. The Fredholm theory introduced in Chapter 2 then leads to a definition of the resolvent set and spectrum of the differential equation. The characteristic determinant of an nth-order ordinary differential equation is constructed by Taylor expansion of the resolvent, and it is used to determine the eigenvalues of the operator. Of critical importance to the approach taken in this book is the idea of writing the nth-order differential operator L as

$$L = T + S,$$

where T is the principal part of L, that is, the nth-order differential operator $(1/i^n)(d/dt)^n$. Chapters 4–5 are devoted to the study of the spectral theory of T. The topics discussed include asymptotic formulae for the characteristic determinant of T and for its Green's function. The boundary values of T as an operator in some Sobolev space are characterized as being regular, irregular or degenerate, this being determined by differing behaviour of the characteristic determinant. The eigenvalues are calculated as the zeros of the characteristic determinant, and asymptotic formulae for them are derived. The family of spectral projections is calculated, and the decay rate of the resolvent is estimated. Finally, it is shown that when the boundary values are regular, each function in the Hilbert space can be expanded in a series of generalized eigenfunctions of T. By viewing S as a perturbation of T, Chapter 7 establishes similar results for L.

This book is well written and is accessible to all who have a rudimentary knowledge of functional analysis. It is well suited both to graduate students working in two-point boundary value problems and to other scientists seeking further information concerning them.

University of Wales, Cardiff MALCOLM BROWN

CHARACTERS OF CONNECTED LIE GROUPS
(Mathematical Surveys and Monographs 71)

By LAJOS PUKÁNSZKY 128 pp., US$59.00, ISBN 0-8218-1088-X
(American Mathematical Society, Providence, RI, 1999).

Lajos Pukánszky [**3**] devoted most of his life to the study of a single subject: the unitary representation theory of solvable, or even completely arbitrary, connected Lie groups. (Any connected Lie group is, at least locally, a semidirect product of a semisimple group and a solvable group.)

A locally compact group G is said to be 'type I' if each of its unitary representations generates a von Neumann algebra of type I. This condition is implied by the stronger condition of being 'CCR' ('liminaire' in French), or having the property that for each irreducible representation π, $\pi(C_c(G))$ is contained in the compact operators. Back in the 1950s, Harish-Chandra proved that semisimple Lie groups are CCR, and Dixmier proved that nilpotent groups are CCR. But even the simplest solvable Lie group, the '$ax + b$ group', or the two-dimensional affine group of the line, is not CCR, and starting in dimension 5, there are solvable Lie groups that are not type I. Pukánszky set himself the task of understanding why and how this is the case, and of trying to make order out of the seemingly 'wild' aspects of Lie group representation theory. This (posthumous) book is a summary of his main accomplishments, originally published in a long series of papers, of which the most notable are [**8**], [**9**] and [**10**].

Chapter I of this book, based largely on the technical results in [**9**], is devoted first to proving that *locally algebraic* connected Lie groups are type I. This class includes the nilpotent Lie groups, the semisimple Lie groups, and a few of the most familiar solvable groups, such as the '$ax+b$ group' and the 'diamond group'. Then Pukánszky goes on to prove a theorem of Dixmier, that the regular representation of a connected Lie group always generates a semifinite von Neumann algebra. (In other words, it has a central direct integral decomposition into irreducible representations and type II factor representations. Pukánszky eventually showed that the representations occurring in the central decomposition of the regular representation are among the *normal representations* studied in Chapter III.)

In contrast to semisimple Lie groups, which have a rigid structure theory, solvable Lie groups are quite 'flexible', and there is no good classification of them. Thus it is not possible to study their representation theory 'case by case', as is sometimes done with semisimple Lie groups. Nevertheless, Pukánszky discovered (though he never stated things in these terms) that the phenomenon of 'non-type-I-ness' in solvable Lie groups can arise for exactly two different reasons, which are typified by two basic examples:

(1) the *Mautner group* of dimension 5, the semidirect product $\mathbb{R} \ltimes \mathbb{C}^2$, where $\mathbb{R}$ acts on $\mathbb{C}^2$ by $t \cdot (z, w) = (e^{it}z, e^{i\lambda t}w)$, where λ is irrational;
(2) the *Dixmier group* of dimension 7, the simply connected Lie group with Lie algebra spanned by $e_1, \ldots, e_7$, satisfying the bracket relations

$$[e_1, e_2] = e_7, \quad [e_1, e_3] = e_4, \quad [e_1, e_4] = -e_3,$$
$$[e_2, e_5] = e_6, \quad [e_2, e_6] = -e_5, \quad [e_i, e_j] = 0 \text{ for } i, j \geqslant 3.$$

In both of these cases, it is natural to try to analyse the representation theory of the solvable Lie group G by applying the 'Mackey method' to the action of G on the Pontryagin dual $\widehat{N}$ of the (abelian) commutator subgroup N. (Since N acts trivially on $\widehat{N}$, the action factors through G/N.) In the case of the Mautner group, $N = \mathbb{C}^2$ and the action of $G/N \cong \mathbb{R}$ on $\widehat{N} \cong \mathbb{C}^2$ is given by $t \cdot (z, w) = (e^{-it}z, e^{-i\lambda t}w)$. The key feature here is that each torus $|z| = c_1$, $|w| = c_2$ $(c_1, c_2 > 0)$ is invariant, and on it $\mathbb{R}$ acts ergodically. Or, in Mackey's terminology, there are 'non-transitive quasi-orbits' giving rise to non-type-I behaviour. In Dixmier's example, something rather different happens. The action of $G/N \cong \mathbb{R}^2$ on $\widehat{N} \cong \mathbb{R}^5$ has nice orbits, but for generic points in $\widehat{N}$, the stabilizer is disconnected, and the character χ of N does not extend to a character of its stabilizer G_χ in G. The non-type-I-ness in this case arises from the fact that G_χ is a non-type-I central extension of a *discrete* abelian group.

Pukánszky showed that the Dixmier example typifies a feature of the general case: that the failure of Lie groups to be type I can always be traced to the representation theory of central extensions of abelian groups, the subject of Chapter II. From this analysis, Pukánszky arrives at the main results of the book, which are in Chapter III.

The basic result can be summarized by saying that connected Lie groups have a good representation theory, provided that one is willing to view the basic objects of this theory as being quasi-equivalence classes of *normal* representations rather than unitary equivalence classes of irreducible representations. A normal representation is a factor representation of type I or II, for which the image of the group C^*-algebra has non-trivial intersection with the trace-class operators in the sense of von Neumann algebras. The trace on such a representation makes possible a character formula of Kirillov type, which Pukánszky proved but does not discuss in detail in this book. However, he does show in Chapter IV how, in the solvable case, to parametrize the normal representations via generalized coadjoint orbits.

Unfortunately, Pukánszky did not live to finish and fully polish this book, so certain important parts of his theory are missing. Also, the book does everything from Pukánszky's own, rather idiosyncratic, point of view. Thus the author does not mention the general theory of multiplier representations of abelian groups, due to Baggett and Kleppner [**1**], from which the results of Chapter II follow easily, nor does he mention the alternative proof by Green [**4**] of the main results of Chapter III. The important results of Charbonnel [**2**] and Poguntke [**6**], which amplify many of Pukánszky's results, are mentioned only in passing.

Nevertheless, the book is a useful reference for Pukánszky's theory, and is somewhat more convenient than reading [**8**], [**9**] and [**10**], since some of the duplication between papers has been eliminated. The reader should be warned about two things, however. First, the style of this book is totally different from that of Pukánszky's earlier book [**7**] on nilpotent groups. Whereas that book was intended for beginners, this book is intended only for those who already know quite a bit about Lie group representations. Unlike Kirillov's text [**5**], which is rather informal and tries to avoid technicalities, this book almost relishes them. Secondly, Pukánszky's notation takes some getting used to. The notation $\overset{\barwedge}{\mathscr{G}}_g$ of [**8**] has here been typeset as $\overset{A}{\mathscr{G}}_g$. And the notation $G = (I)$ is supposed to mean not that G is equal to anything, but that it is a type I group.

References

1. L. BAGGETT and A. KLEPPNER, 'Multiplier representations of abelian groups', *J. Funct. Anal.* 14 (1973) 299–324.

2. J.-Y. CHARBONNEL, 'Sur les caractères des groupes de Lie', *J. Funct. Anal.* 72 (1987) 94–150.

3. J. DIXMIER, M. DUFLO, A. HAJNAL, R. KADISON, A. KORÁNYI, J. ROSENBERG and M. VERGNE, 'Lajos Pukánszky (1928–1996)', *Notices Amer. Math. Soc.* 45 (1998) 492–499.

4. P. GREEN, 'The local structure of twisted covariance algebras', *Acta Math.* 140 (1978) 191–250.

5. A. A. KIRILLOV, *Elements of the theory of representations* (translated by Edwin Hewitt), Grundlehren Math. Wiss. 220 (Springer, Berlin, 1976).

6. D. POGUNTKE, 'Simple quotients of group C^*-algebras for two-step nilpotent groups and connected Lie groups', *Ann. Sci. École Norm. Sup.* (4) 16 (1983) 151–172.

7. L. PUKÁNSZKY, *Leçons sur les représentations des groupes*, Monographs Soc. Math. France 2 (Dunod, Paris, 1967).

8. L. PUKÁNSZKY, 'Unitary representations of solvable Lie groups', *Ann. Sci. École Norm. Sup.* (4) 4 (1971) 457–608.

9. L. PUKÁNSZKY, 'Action of algebraic groups of automorphisms on the dual of a class of type I groups', *Ann. Sci. École Norm. Sup.* (4) 5 (1972) 379–395.

10. L. PUKÁNSZKY, 'Characters of connected Lie groups', *Acta Math.* 133 (1974) 81–137.

University of Maryland JONATHAN ROSENBERG

PARTIALLY ORDERED GROUPS
(Series in Algebra 7)

By A. M. W. Glass 307 pp., £26.00, ISBN 981-02-3493-7
(World Scientific, Singapore, 1999).

This is the latest addition to the series of books on partially ordered algebraic structures, an area which has been enriched by a number of prominent mathematicians such as Holder, Hahn, G. Birkoff, P. Hall, H. Wielandt, G. Higman, B. H. Neumann, P. F. Conrad, W. C. Holland and many others, since the beginning of 20th century. The author's style of writing is very lucid, and the material presented is self-contained. It is an excellent reference text for a graduate course in this area, as well as a source of material for individual reading. References are given but the list is not complete, and there are no exercises; there is a long discussion of open problems in the last chapter of the book.

The first chapter is devoted to the mandatory definitions followed by a number of generic examples forming a basis for the type of results which can be expected to hold. The second chapter deals with basic group theoretical properties of groups that satisfy various orderability conditions such as partial orders, lattice orders and total orders. This is followed by the reverse, where orderability properties are investigated for groups possessing various group theoretic properties. In the third chapter we are introduced to the convex subgroups of partially ordered groups, and basic results are discussed. Chapter 4 deals with Abelian ordered and lattice ordered groups: Holder's characterization of Archimedean ordered groups as well as Hahn's Theorem are established, together with their generalization to lattice ordered groups. These topics are discussed in greater depth in the following chapter. Chapter 6 is devoted to the structure of orderable groups which are soluble or satisfy the n-Engel law. Various recent results have been collected together, with proofs, for the first time. Chapter 7 is about order-preserving permutation groups. Notations and basic results are presented here, and Chapter 8 continues with applications to lattice ordered groups satisfying various conditions. Complete partially ordered groups are discussed in the following chapter. Varieties of lattice ordered groups and their structure, together with applications to the variety of residually ordered groups, form the content of Chapter 10. The last chapter is devoted to the discussion of fourteen open problems.

Compared to the the book by V. M. Kopytov and N. Ya. Medvedev [**2**], which contains a rather large number of errors including several places where the reader has to refer to the original papers, I found this book to be remarkably well written. Errors can be expected in research-level textbooks, which are often written at a raw stage of the subject when polished proofs are not yet available. The only error I could find in this book was in the proof of Theorem 6.J: the result is true, and a proof may be found in [**1**].

References

1. S. D. Brodskii, 'Equations over groups with one defining relation', *Sibirsk. Mat. Zh.* 25 (1984) 84–103.
2. V. M. Kopytov and N. Ya. Medvedev, *Right-ordered groups*, Siberian School Algebra Logic (Consultants Bureau, New York, 1996).

University of Alberta A. H. Rhemtulla

CLASSIFICATION AND ORBIT EQUIVALENCE RELATIONS
(Mathematical Surveys and Monographs 75)

By GREG HJORTH 195 pp., US$55.00, ISBN 0-8218-2002-8
(American Mathematical Society, Providence, RI, 2000).

The book under review presents a beautiful example of a theory whose creation and consequences, in one way or another, involve most fields of mathematics. To mention the ones involved most directly, we may single out set theory, model theory, ergodic theory, topology and group theory. The point of the study is to classify the equivalence relations on a Polish (that is, separable complete metric) space, with a particular emphasis on the equivalence relations arising as orbit equivalence relations given by an action of a Polish group on a Polish space. As a standard of classification, the author takes what is termed *classification by countable models*. An equivalence relation E defined on a Polish space X is said to admit such a classification if for some countable language $\mathscr{L}$ there is a Borel function which to each E-equivalence class assigns a countable $\mathscr{L}$-structure considered up to isomorphism. There is much evidence presented to confirm that this notion, which is less stringent than the previously considered notion of smoothness, isolates 'nice' equivalence relations appearing in various contexts.

One of the main achievements of the theory, due to Hjorth, is that in the realm of equivalence relations induced by a continuous action of a Polish group on a Polish space, the notion of classification by countable models is shown to have an internally recognizable nemesis, that is, a dynamical property of the action itself which guarantees that the classification by countable models is not possible. The property in question is called *turbulence*. (In fact, a deeper connection between turbulence and non-classification is established using the notion of ergodic genericity; see Theorem 3.21.)

In the world of classification theories, one often looks for *dichotomy* theorems. These are theorems which isolate a specific 'minimal' reason for non-classifiability, so that every object under consideration is either classifiable or has the minimal reason neatly embedded into itself. A specific example is the Harrington–Kechris–Louveau dichotomy theorem which isolates such a minimal example among the non-smooth Borel equivalence relations. It was at first hoped that the notion of turbulence would give rise to such dichotomy theorems, but this is not the case. The notion of turbulence combined with Borel reducibility gives rise to a rather complicated partial order among equivalence relations, as can be seen from the results of Farah quoted in the book; see, for example, Theorem 3.36. Although this shows that the most optimistic scenario for classification using turbulence is not correct, this does not lessen the importance of the classification scheme that turbulence gives rise to, much as the fact that there are more than two Turing degrees does not mean that one should not consider Turing reducibility as a viable classification. There are many other ways of looking at this, of course, such as relaxing the idea of Borel reducibility or restricting the kinds of spaces and actions under consideration, and for many of these it is shown here that the notion of turbulence indeed gives rise to a dichotomy theorem. In fact, the author says on page 177 that 'the main results of (the) manuscript could be considered as a reaction to the failure to find an analogue of Harrington–Kechris–Louveau for the notion of turbulence'.

While the first part of the book has the pleasing feature that it could be read by an enthusiastic reader with a minimal background, this is nicely complemented by the fine analysis of the failure of the Harrington–Kechris–Louveau dichotomy, which comprises the second part of the book. In particular, Chapter IX provides an application of modern set theory to the problem of classification when a more general reduction than the Borel reduction is allowed. This can be used to put the whole programme of the book into a context of much concern to set theorists, that of calculating effective cardinalities. The idea behind this is that when verifying that certain objects have the same size, one exhibits a bijection between the two sets, but often such bijections are not *effective* or explicit, as they rely on an application of the axiom of choice. One may ask when two objects can be effectively shown to have the same cardinality, where the notion of effectiveness can be precisely defined.

Space does not allow us to give a detailed description of the way that the material is presented in the book, but the reader can find such a presentation in Chapter II, which also gives a path diagram through the book. With the exception of Chapter IX, which requires some background in modern set theory, a mathematician with a postgraduate education will find the book to be self-contained.

To return to Chapter IX, although the book makes an interesting read even when one forgets about this chapter, it is exactly this part of the book that gives the real sense of completeness to the programme. In this chapter it is shown that if one gives up the requirement that reductions are Borel, and replaces it with a notion which is more complicated, but very *effective*, then the notions of classifiability and turbulence become the exact antipodes of each other. Even those who do not have a background required for reading this chapter can appreciate the interaction that the results here show between the seemingly abstract notions of infinitary mathematics and the notions present in everyday mathematical life, namely equivalence relations and group actions.

The book provides a good bibliography, and includes a chapter which gives a list of open problems, conjectures and directions. Although the problems presented in this chapter, Chapter X, might be the ones most central to the book, there are open problems scattered throughout the book; for example, one asks if the GE groups are the only ones with the strong Glimm–Effros property (page 125). It is worth mentioning that other connections between descriptive set theory and ergodic theory have been studied in addition to the ones offered by turbulence, and that a good reference is the recently published volume 277 of the LMS Lecture Note Series, *Descriptive set theory and dynamical systems*, edited by M. Foreman, A. S. Kechris, A. Louveau and B. Weiss.

University of East Anglia MIRNA DŽAMONJA

RANDOM WALKS ON INFINITE GRAPHS AND GROUPS
(Cambridge Tracts in Mathematics 138)

By WOLFGANG WOESS 334 pp., £40.00 (US$64.95), ISBN 0-521-55292-3
(Cambridge University Press, 2000).

The simple random walk on the integers is one of the simplest random processes that one can imagine. It generalizes to any finitely generated group Γ equipped with a finite set S of generators. If $X_n \in \Gamma$ denotes the position at time n, then $X_{n+1} = X_n \xi_{n+1}$, where ξ_n is chosen uniformly at random in S. More generally, ξ_n can be picked according to a given probability measure μ. Then the probability that the walk started at the neutral element $X_0 = e$ reaches x at time n is $\mathbf{P}_e(X_n = x) = \mu^{(n)}(x)$, where $\mu^{(n)}$ denotes the nth convolution power of μ. We always assume below that the generating set S is symmetric, that is, $S = S^{-1}$.

Will a random walk return infinitely many times to its starting point? This is the question discussed in Pólya's seminal article [**6**], in the case of integer lattices. If the answer is yes, then the walk is called recurrent; otherwise, it is called transient. Pólya's well-known finding is that the simple random walk on the integer lattice in Euclidean space is recurrent in one or two dimensions, and transient in dimension three or higher. Indeed, transience/recurrence is equivalent to the convergence/divergence of the series $\sum \mathbf{P}_e(X_n = e)$ and, in dimension d,

$$\mathbf{P}_e(X_{2n} = e) \sim c_d n^{-d/2} \quad \text{as } n \text{ tends to infinity.} \tag{1}$$

(For parity reasons, one cannot return to the starting point at odd time.) Spitzer's famous book [**7**] gives a thorough and beautiful treatment of random walks on integer lattices.

For general groups, one of the most basic and natural questions about random walks concerns the asymptotic behaviour of the probability of return to the starting point. How does (1) generalize to non-Abelian groups? For instance, can one characterize those groups which carry a recurrent simple random walk? The first work on random walks on general finitely generated groups is Kesten's thesis [**4**]. In the sequel [**5**], he proves the fundamental result that $\mathbf{P}_e(X_n = e)$ decays exponentially fast if and only if the group is non-amenable. The next crucial development concerning the behaviour of $\mathbf{P}_e(X_n = e)$ came more than twenty years later. To describe this, let $V(n)$ be the number of elements in the group that can be written as words of length at most n in the generators $s \in S$. Write $f(n) \approx g(n)$ if there are constants such that $c_1 f(c_2 n) \leqslant g(n) \leqslant c_3 f(c_4 n)$ for all n. During the 1980s, Varopoulos [**8**] proved that if $V(n) \geqslant cn^d$, then $\mathbf{P}_e(X_n = e) \leqslant Cn^{-d/2}$. This is remarkable because no further assumption on the group Γ is made. One celebrated consequence is that the only recurrent groups are the finite extensions of $\{0\}$, $\mathbb{Z}$ and $\mathbb{Z}^2$. Together with deep theorems concerning the algebraic structure of groups (theorems due to Malcev, Gromov, Tits, Wolf, and others), this leads to the following result. For a simple random walk on a discrete subgroup Γ of a connected Lie group, three and only three behaviours may occur: (i) $\mathbf{P}_e(X_{2n} = e) \approx n^{-d/2}$ for some integer d; (ii) $\mathbf{P}_e(X_{2n} = e) \approx \exp -n^{1/3}$; (iii) $\mathbf{P}_e(X_{2n} = e) \approx \exp(-n)$. Moreover: case (i) happens if and only if Γ is virtually nilpotent and $V(n) \approx n^d$; case (ii) happens if and only if Γ is virtually polycyclic and $V(n) \approx \exp(n)$; case (iii) happens if and only if Γ is non-amenable. Still, today, there are many finitely generated groups for which the behaviour of $\mathbf{P}_e(X_n = e)$ is not well understood, for instance, metabelian (that is, two-steps solvable) non-polycyclic groups.

Another fundamental aspect of the theory of random walks concerns the existence and behaviour of harmonic functions and the related boundary theories. (A function u is (μ)-harmonic if it satisfies the convolution equation $u * \mu = u$.) Indeed, this aspect played an important role at an early stage of the development of the theory, and is still an active area of study. A celebrated problem in this direction concerns the existence of bounded or positive harmonic functions: a measure μ has the Liouville property (respectively, the strong Liouville property) if any bounded (respectively, positive) harmonic function is constant. It is still an open problem today whether or not these Liouville properties for simple random walks on a group are, in general, independent of the generating set.

Woess' book gives a well-documented, informative and personal treatment of the theory of random walks as it has evolved since Spitzer's book [**7**]. Although it is not meant to be self-contained, it gives careful proofs of most of the results that are discussed. The book actually treats the more general theory of random walks on graphs, but manages always to stay close to the heart of the matter. It includes some beautiful results on random walks on planar graphs. Random walks on Cayley graphs (that is, simple random walks on groups) are treated as a special case of random walks on graphs having a vertex-transitive group of automorphisms. The first chapter studies the 'type problem', that is, whether or not a given walk is recurrent. It gives a thorough and detailed treatment, including many interesting specific examples of recurrent graphs. The second chapter concerns the amenable/non-amenable dichotomy, and the problem of computing the so-called spectral radius $\rho = \limsup_{n\to\infty} \sqrt[n]{\mathbf{P}_e(X_n = e)}$ of some walks. The third chapter treats the asymptotic behaviour of $\mathbf{P}_e(X_n = e)$. Although many satisfactory results concerning the rough asymptotic behaviour of $\mathbf{P}_e(X_n = e)$ (in the sense of the relation $\approx$) are known, obtaining precise asymptotic results such as (1) is, in many cases, an open problem. In this direction, Woess describes a rich collection of results concerning specific groups. A recent development in this direction that is not included in the book is [**1**]. The fourth and last chapter gives a nice treatment of certain aspects of boundary theory. (I was surprised not to find [**2**] in the bibliography, and I wish that the results of [**3**] had been included, at least in the further results section.) The book focuses chiefly on positive harmonic functions, leaving the task of giving a complete treatment of Poisson boundary theory to another author.

Random walk theory is connected with many other areas of mathematics. Without distracting the author from its main theme, these links appear all through the text. Some readers will find that certain connections (for example, to volume growth, isoperimetry, geometric group theory, algebraic structure, covering of compact manifolds) could have been developed more, but this would have led to a voluminous and very different book.

This is an excellent book, where beginners and specialists alike will find useful information. It will become one of the major references for all those interested directly or indirectly in random walks. I highly recommend it.

References

1. G. Alexopoulos, 'Convolution powers on discrete groups of polynomial volume growth', *Harmonic analysis and number theory* CMS Conf. Proc. 21 (ed. S. Drury and M. Ram Murty, Amer. Math. Soc., Providence, RI, 1997) 31–57.

2. R. Azencott, *Espaces de Poisson des groupes localement compacts*, Lecture Notes in Math. 148 (Springer, New York, 1970).

3. P. Bougerol and L. Elie, 'Existence of positive harmonic functions on groups and covering manifolds', *Ann. Inst. H. Poincaré Probab. Statist.* 31 (1995) 59–80.

4. H. Kesten, 'Symmetric random walks on groups', *Trans. Amer. Math. Soc.* 92 (1959) 336–354.

5. H. Kesten, 'Full Banach mean values on countable groups', *Math. Scand.* 7 (1959) 146–156.

6. G. Pólya, 'Über eine Aufgabe der Wahrscheinlichkeitstheory betreffend die Irrfahrt im Straßennetz', *Math. Ann.* 84 (1921) 149–160.

7. F. Spitzer, *Principles of random walks* (2nd edn, Springer, New York, 1976).

8. N. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Math. 100 (Cambridge University Press, 1992).

Cornell University L. Saloff-Coste

COHOMOLOGY OF NUMBER FIELDS
(Grundlehren der Mathematischen Wissenschaften 323)

By JÜRGEN NEUKIRCH, ALEXANDER SCHMIDT and KAY WINGBERG 699 pp., £61.50, ISBN 3-540-66671-0 (Springer, Berlin, 2000).

The book under review is a sequel to Jürgen Neukirch's *Algebraische Zahlentheorie* (1992), recently translated into English [**2**]. This text takes the reader through many of the summits of classical algebraic number theory, including class field theory and functional equations of Artin L-functions, with a minimal use of homological techniques. However, to get deeper into the modern theory, one requires such techniques, and Neukirch therefore began to write a sequel. He prepared the first 150 pages for publication before his untimely death in 1997. His beautiful introduction to the cohomology of profinite groups forms the basis for the first three chapters of the book under review. After Neukirch's death, Alexander Schmidt and Kay Wingberg set themselves the task of completing the volume.

The quest for higher-dimensional reciprocity laws, following Gauss's proof of the quadratic reciprocity theorem, led to the study of class field theory at the end of the nineteenth century, and the work of Kronecker, Weber, Hilbert, Furtwängler and others. These reciprocity laws were completed with the work of Artin (1927), who realised that they arose from an explicit isomorphism of the class group with a Galois group. The notes [**1**] of the Artin–Tate seminar in 1951–52 interpreted class field theory from a cohomological viewpoint, and this has proved to be the most fruitful way of considering the theory. Many classical results in number theory may be recast as statements involving Galois theory, cohomology or both.

For example, Dirichlet's theorem on primes in arithmetic progression may be viewed as a special case of the Čebotarev density theorem for the Galois groups of cyclotomic extensions, and one sometimes forgets the original arithmetic formulation of Hilbert's Theorem 90, now that it is almost always stated in its cohomological form.

Nowadays, algebraic number theory has developed into arithmetic algebraic geometry. The scheme theoretic point of view has allowed us to re-interpret algebraic number theory as the 1-dimensional case of a more general geometric setting, and consequently much current research involves the study of arithmetic properties of schemes over number fields, local fields, or their rings of integers, and related subjects. While the authors restrict themselves to this 1-dimensional case, largely for convenience and brevity—a full treatment of the higher-dimensional case would occupy several volumes—it seems to me that this book is written with the higher-dimensional geometric situation in mind. The abstract theory of Galois cohomology has already been developed by Serre [**3**]. However, Serre seems interested more in the algebraic properties of the theory, and gives few explicit number theoretic applications. The current book is thus the first to really treat the theory from the point of view of arithmetic algebraic geometry. As such, it contains many proofs which have not previously appeared in book form, as well as several original results.

But this is not to say that the book is just for specialists in arithmetic algebraic geometry. Indeed, the book is divided into two sections: the first presents the abstract algebraic theory of the cohomology of profinite groups, and the second provides the applications to arithmetic. I would certainly recommend the first part of the book to any graduate student in algebra wishing to learn about cohomology. The first chapter gives a lovely introduction to group cohomology, with particular reference to profinite groups, and the subsequent two chapters, on homological algebra (including spectral sequences and derived functors) and duality (also discussing cohomological dimension) are also likely to be of interest more generally. Perhaps the rest of the first part of the book is more likely to be useful only to number theorists: it is here that the algebraic foundations of Iwasawa theory (completed group rings, and so on) are laid.

The second half of the book is devoted to the arithmetic applications. But here also there are results of interest to researchers outside number theory. This part starts with a chapter on Galois cohomology, and involves some discussion of Brauer groups and Milnor K-theory in which topologists have also been very interested, following work of Voevodsky and others on motivic homotopy categories and the Milnor conjecture. Later, in Chapter 9, a proof is given of Shafarevich's theorem on the positive solution to the inverse Galois problem for finite soluble groups, and Chapter 10 contains a proof of the Golod–Shafarevich theorem.

Of course, much of the second half of the book is aimed largely at specialists. The heart of the book lies in Chapters 7 and 8, when the abstract results from the first part are applied to the arithmetic situation, and results such as Tate local duality, the Poitou–Tate 9-term exact sequence and the global Euler–Poincaré formula are proven. Iwasawa theory (as far as the statement of the Main Conjecture) is also treated. In the final chapter, one finds an introduction to Grothendieck's anabelian geometry. Its inclusion in the text is justified by the Neukirch–Uchida theorem which states that a global field is characterised up to isomorphism by its absolute Galois group. For once, the treatment here is slightly sketchy; anabelian geometry is still in its infancy, and will, in time, no doubt give rise to textbooks of its own.

The fact that the authors have tried (successfully) to make the book self-contained has the usual disadvantages: the book is lengthy, and the quantity of necessary detail occasionally threatens to become too much. But there are easily enough highlights in the book to compensate, and one rarely feels the need to look at any of the sources given in the excellent bibliography.

In summary, the book successfully attempts to survey the current state of knowledge regarding the cohomology of number fields. It is rather more specialised than the first volume [**2**], and is thus likely to have a smaller readership. On the other hand, it is a marvellous compendium of results in the cohomology theory of number fields.

References

1. E. Artin and J. Tate, *Class field theory* (W. A. Benjamin, New York, 1967).
2. J. Neukirch, translated by N. Schappacher, *Algebraic number theory*, Grundlehren Math. Wiss. 322 (Springer, Berlin, 1999).
3. J.-P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Math. 5 (2nd edn, Springer, Berlin, 1964).

The University of Sheffield Frazier Jarvis

FRACTAL GEOMETRY AND NUMBER THEORY: COMPLEX DIMENSIONS OF FRACTAL STRINGS AND ZEROS OF ZETA FUNCTIONS

By Michel L. Lapidus and Machiel van Frankenhuysen 268 pp., CHF 98.00, isbn 0-8176-4098-3 (Birkhäuser, Boston, 2000).

This monograph presents the work on zeta functions and complex dimensions of so-called fractal strings developed by the first author and various co-authors over the past decade; compare [**1**, **2**, **3**] and the references therein. The authors' main philosophy is that the notion of complex dimensions describes the oscillations in the fractal geometry of the string and the spectrum of the string.

A fractal string $\mathscr{L}$ is an open bounded subset Ω of the real line $\mathbb{R}$. Such a set is the disjoint union of open intervals with lengths $\mathscr{L} = (l_1, l_2, l_3, \ldots)$. The authors define the geometric zeta function $\zeta_{\mathscr{L}}$ of $\mathscr{L}$ by

$$\zeta_{\mathscr{L}}(s) = \sum_j l_j^s,$$

for s in a suitable region of the complex plane. The complex dimensions of $\mathscr{L}$ are by definition the poles of the meromorphic extension of $\zeta_{\mathscr{L}}$.

Let us illustrate how the complex dimensions describe the oscillatory behaviour in the geometry of $\mathscr{L}$. The fractal geometry of $\mathscr{L}$ is described by the Minkowski dimension and the Minkowski content of the boundary $\partial\Omega$ of Ω. For $\varepsilon > 0$, let

$$V(\varepsilon) = \mathrm{vol}_1\{x \in \mathbb{R} \mid \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$$

denote the (1-dimensional) volume of the ε neighbourhood of $\partial\Omega$. The Minkowski dimension, D, of $\partial\Omega$ is defined by

$$D = 1 - \liminf_{\varepsilon \searrow 0} \frac{\log V(\varepsilon)}{\log \varepsilon},$$

and the lower and upper Minkowski contents of $\partial\Omega$ are defined by

$$\mathscr{M}_* = \liminf_{\varepsilon \searrow 0} \varepsilon^{-(1-D)} V(\varepsilon), \quad \mathscr{M}^* = \limsup_{\varepsilon \searrow 0} \varepsilon^{-(1-D)} V(\varepsilon).$$

One of the key results in the book states that (under suitable conditions on the string $\mathscr{L}$) we have the following explicit formula for $V(\varepsilon)$:

$$V(\varepsilon) = \sum_{\omega} c_\omega \frac{(2\varepsilon)^{1-\omega}}{\omega(1-\omega)} + R(\varepsilon), \tag{0.1}$$

where the sum is over all the complex dimensions ω of $\mathscr{L}$, c_ω is a constant, and $R(\varepsilon)$ is an error term of lower order. It follows from (1) that $\varepsilon^{-(1-D)}V(\varepsilon) = g(\varepsilon) + \varepsilon^{-(1-D)}R(\varepsilon)$, where $g(\varepsilon)$ is a function defined explicitly in terms of the complex dimensions and whose oscillatory behaviour determines the values of the Minkowski contents $\mathscr{M}_*$ and $\mathscr{M}^*$ of $\mathscr{L}$, and hence the Minkowski measurability of $\mathscr{L}$. (A set is called Minkowski measurable if its lower and upper Minkowski contents coincide.)

The second main topic in the book is spectral analysis of the string. Spectral analysis refers to the study of the asymptotic behaviour of the eigenvalues of the Laplacian Δ on Ω, and one of the main goals is to give accurate estimates of the so-called counting function $N(x) = \#\{\lambda \leqslant x \mid \lambda \text{ is an eigenvalue of } \Delta\}$ for large values of x. This is a classical problem. Indeed, the detailed study of the counting function of a linear differential operator on a connected open subset of Euclidean space with piecewise smooth boundary was initiated by Weyl [**4**] at the beginning of the last century, and is still flourishing. Using the framework of complex dimensions, the authors obtain explicit formulas for the counting function in terms of the complex dimensions of $\mathscr{L}$, thus relating the geometric and spectral properties of the string.

Of course, the book treats classes of strings and zeta functions that are more general than those described above, and explores various connections and analogies with number theory, including connections with Dedekind and Epstein zeta functions and a geometric reformulation of the Riemann Hypothesis: the Hypothesis holds if and only if for every fractal string with Minkowski dimension $D \in (0, 1) \setminus \{1/2\}$, the absence of oscillations of order D in its spectrum implies that it is Minkowski measurable.

It is the reviewer's opinion that the authors have succeeded in showing that complex dimensions provide a very natural and unifying mathematical framework for investigating the oscillations in the geometry and the spectrum of a fractal string. The book is well-written. The exposition is self-contained, intelligent and well-paced. The book will appeal to people working in fractal geometry as well as people working in number theory.

References

1. C. He and M. Lapidus, 'Generalized Minkowski content, spectrum of fractal drums, fractal strings and the Riemann zeta function', *Mem. Amer. Math. Soc.* 127 (1997).

2. M. Lapidus and M. van Frankenhuysen, 'Complex dimensions of fractal strings and oscillatory phenomena in fractal geometry and arithmetic', *Spectral problems in geometry and arithmetic*, Contemp. Math. 237 (ed. T. Branson, Amer. Math. Soc., Providence, RI, 1999) 87–105.

3. M. Lapidus and H. Maier, 'The Riemann hypothesis and inverse spectral problems for fractal strings', *J. London Math. Soc.* 52 (1995) 15–34.

4. H. Weyl, 'Das asymptotische Veteilungsgesetz der Eigenwerte linearer partieller Diffentialgleichungen', *Math. Ann.* 71 (1912) 441–479.

University of St. Andrews L. Olsen

INDEX OF BOOK REVIEWS