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A FAMILY OF TREES WITH NO UNCOUNTABLE BRANCHES

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ABSTRACT. We construct a family of $2^{\aleph_1}$ trees of size $\aleph_1$ and no uncountable branches that in a certain way codes all ω_1-sequences of infinite subsets of ω. This coding allows us to conclude that in the presence of the club guessing between $\aleph_1$ and $\aleph_0$, these trees are pairwise very different. In such circumstances we can also conclude that the universality number of the ordered class of trees of size $\aleph_1$ with no uncountable branches under "metric-preserving" reductions must be at least the continuum. From the topological point of view, the above results show that under the same assumptions there are $2^{\aleph_1}$ pairwise non-isometrically embeddable first countable ω_1-metric spaces with a strong non-ccc property, and that their universality number under isometric embeddings is at least the continuum. Without the non-ccc requirement, a family of $2^{\aleph_1}$ pairwise non-isometrically embeddable first countable ω_1-metric spaces exists in ZFC by an earlier result of S. Todorčević. The set-theoretic assumptions mentioned above are satisfied in many natural models of set theory (such as the ones obtained after forcing by a ccc forcing over a model of $\diamondsuit$).

We use a similar method to discuss trees of size κ with no uncountable branches, for any regular uncountable κ.

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Introduction

The main objects of interest in this paper are trees of size $\aleph_1$ with no uncountable branches. We shall also consider some other classes of trees. The motivation for our study stems from several sources. Apart from the intrinsic interest in the combinatorial properties of such trees they also appear in infinitary model theory as clocks of generalized Ehrenfeucht-Fräissé games. This is explained in some detail in [10] where one can also find further references. Trees of height ω_1 are also of interest in topology, because they give rise to examples of the so-called ω_1-metric spaces. We recall the definition in §3. Such spaces were introduced and studied in some detail by R. Sikorski in a series of papers on ω_μ-additive spaces, an example of which is [8]. More recently, it was shown by J. Väänänen in [11] that from the consistency of a measurable cardinal it follows that the ω_1-metric space $\omega_1^{\omega_1}$, as well as any other ω_1-metric space satisfying a certain compactness condition, admits a natural generalization of the Cantor-Bendixson theorem. If we consider trees of size $\aleph_1$ with no uncountable branches, then they give rise to first countable ω_1-metric spaces. We obtain a result which says that under certain commonly present set-theoretic assumptions there is a large family of pairwise non-isomorphic first countable "large" ω_1-metric spaces (Theorem 3.8), *large* meaning a strong non-ccc condition, defined in Definition 3.7, and such that they cannot be jointly embedded into any family of $< 2^{\aleph_0}$ among these spaces. Without the largeness assumption, the first part of the conclusion is just true in ZFC as follows from a recent construction by S. Todorčević.

We hope that the family of trees we construct will find further applications in topology and note in passing that although we have not yet been able to apply this observation, there is a natural way to assign a Valdivia compact space to a tree with no uncountable branches. If we want such a space not to be Corson then it is important that the tree we start with has a lot of uncountable levels, which is the case with the trees we construct and not the case with the trees traditionally studied in this subject.

Before continuing to present our construction we shall pause to give some general background and recall some interesting results that are already present in the literature.

One of the main notions in the study of trees is that of a reduction.

Definition 0.1. For trees T and T' we say that $T \leq T'$ if there is a function $f : T \to T'$ satisfying $x <_T y \implies f(x) <_{T'} f(y)$. A function f as in this definition will be referred to as a *reduction of* T to T'.

Notice that the reduction f is not required to be injective or surjective. The main preservation property of reductions is that they map branches of T into branches of T'. Also note that any reduction of T to T' has the property that $\mathrm{ht}_{T'}(f(x)) \geq \mathrm{ht}_T(x)$ for all $x \in T$. Let $\mathbb{T}$ be the class of trees of size $\aleph_1$ with no uncountable branches. One is particularly interested in the structure of this class under reductions. Using powerful methods from the combinatorics of trees, in particular that of Aronszajn trees [10] and [9] give a number of structural theorems about this class, of which we quote some:

Theorem 0.2 ([10], [9])**.** *(1) For every countable ordinal α, there is a sequence of Aronszajn trees $\langle T_\beta : \beta < \alpha \rangle$ such that*

$$\beta < \gamma < \alpha \implies T_\beta \leq T_\gamma \;\&\; \neg T_\gamma \leq T_\beta.$$

(2) Assume $\diamondsuit$. Then there are Souslin trees T and T' that are incomparable in the $\leq$ order.

(3) There are $2^{\aleph_1}$ Aronszajn trees that are pairwise incomparable in the $\leq$ order.

Consistency results about $(\mathbb{T}, \leq)$ in models where CH holds were obtained by A. Mekler and Väänänen in [4], see, for example, Theorem 0.3 below.

Some other notions of reduction appear in the literature, such as the injective reduction $\leq_1$ and the homomorphic reduction $\preceq$, which is an injective reduction f satisfying $x \leq_T y \iff f(x) \leq_{T'} f(y)$. The Mekler-Väänänen result mentioned above is in fact about $(\mathbb{T}, \leq_1)$. Specifically, they prove:

Theorem 0.3 ([4])**.** *Assume CH holds and κ is a regular cardinal satisfying $\aleph_2 \leq \kappa$ and $\kappa \leq 2^{\aleph_1}$. Then there is a forcing notion that preserves cofinalities (hence cardinalities) and the value of 2^λ for all λ, which forces the universality number of both $(\mathbb{T}, \leq_1)$ and $(\mathbb{T}, \leq)$ to be κ.*

Here we study a very specific kind of reductions which have the property of preserving the "distance" between the nodes; i.e., thinking of trees in $\mathcal{T}$ as subtrees of ${}^{\omega_1 >}\omega_1$ and denoting by $\Delta(x, y)$ the first ordinal α where $x(\alpha) \neq y(\alpha)$, we study the reductions that preserve the value of $\Delta(x, y)$ and refer to this notion as "preserving Δ." This corresponds to isometries of ω_1-metric spaces. We summarize our main results in the following theorem.

Theorem 0.4. *Suppose that*

- : *(a) there is a ladder system $\bar{C} = \langle c_\delta : \delta < \omega_1 \rangle$ which guesses clubs, i.e. satisfies that for any club $E \subseteq \omega_1$ there are stationarily many δ such that $c_\delta \subseteq E$;*
- : *(b) $\aleph_1 < 2^{\aleph_0}$.*

Then no family of size $< 2^{\aleph_0}$ of trees of size $\aleph_1$, even if we allow uncountable branches, can $\leq$-embed all members of $\mathbb{T}$ in a way that preserves Δ.

Under the same assumptions there is a family of $2^{\aleph_1}$ pairwise non-isometric first countable large ω_1-metric spaces and these spaces cannot be jointly isometrically embedded into any family of $< 2^{\aleph_0}$ many among them.

Notice that the assumptions of Theorem 0.4 hold in particular in any model in which CH is violated by a ccc forcing over a model of $\diamondsuit$. As any club of ω_1 added by a ccc forcing will contain a ground model club, it suffices to have condition (a) hold in the ground model. In particular, the assumptions (hence, the conclusion) of the theorem are indestructible by a further ccc forcing. However, there is a model due to S. Shelah which was obtained by proper forcing over L ([6]) where (a) fails.

The techniques used for the proof of Theorem 0.4 apply also to the analogue of $\mathbb{T}$ at other uncountable regular cardinals κ, e.g., the class $\mathbb{T}_\kappa$ of trees of size κ with no uncountable branches, as we discuss in §2. In particular, we prove:

Theorem 0.5. *Suppose that κ is a regular cardinal satisfying $\aleph_1 < \kappa < 2^{\aleph_0}$. Then no family of trees of size κ, even if we allow uncountable branches, can $\leq$-embed all members of $\mathbb{T}_\kappa$ in a way that preserves Δ.*

Notice that this result is rather close to a ZFC result because even though it is an independent result it is not a result about a particular universe of set theory, but rather it simply follows from the assumptions on the cardinal arithmetic. This result also has an analogue in the class of ω_1-metric spaces.

1. Main Proofs

The aim of this section is to prove the first part of Theorem 0.4. Toward the main proof we shall merge techniques from M. Kojman and Shelah [3] with the classical construction of an Aronszajn tree (see [1]). The idea is to attach invariants to elements of $\mathbb{T}$ and then construct a large family of trees that all have different invariants (Construction Lemma). Then one can show that under certain circumstances the invariants are preserved under Δ-preserving $\leq$-reductions (Preservation Lemma), so that no small family from $\mathbb{T}$ can embed all trees constructed in the Construction Lemma. The idea of using invariants in this manner comes from [3], and it has subsequently been used in a number of contexts. The main difference here is that we are discussing reductions that are not necessarily homomorphic and that in the Construction Lemma we are producing a rather specific kind of tree.

We proceed towards the Construction Lemma. We need to construct a family of trees with no uncountable branches and to guarantee this property, we shall use bounded increasing sequences of rationals as one would when constructing an Aronszajn tree. For this we shall need an auxiliary notion, which is the family $\mathbb{T}'$ of all partial orders satisfying the properties from Definition 1.1. As one can easily observe, every element of $\mathbb{T}'$ is in fact a tree of size $\aleph_1$ with no uncountable chains; so $\mathbb{T}' \subseteq \mathbb{T}$.

Definition 1.1. (1) $\mathbb{T}'$ is the class of all partial orders T with the following properties:

- **:** (a) elements of T are pairs $(\bar{x}, \alpha)$ where $\bar{x}$ is an increasing bounded sequence of rationals and $\alpha < \omega_1$;
- **:** (b) $(\bar{x}, \alpha) <_T (\bar{y}, \beta)$ implies that $\bar{x}$ is a proper initial segment of $\bar{y}$ and $\alpha < \beta$; and
- **:** (c) for every $\alpha < \omega_1$ there is exactly one $\bar{x}$ such that $(\bar{x}, \alpha) \in T$.

(2) For $T \in \mathbb{T}'$ and $\alpha < \omega_1$ we let $\bar{x}_\alpha$ be the unique $\bar{x}$ such that $(\bar{x}, \alpha) \in T$.

Observation 1.2. *As mentioned above, every element of* $\mathbb{T}'$ *is a tree of size* $\aleph_1$ *without uncountable chains. It is not true that every every element of* $\mathbb{T}'$ *is an Aronszajn tree. Namely, choose a set* $\{r_\alpha : \alpha < \omega_1\}$ *of distinct reals and for each* $\alpha < \omega_1$ *an* ω*-sequence* $\bar{s}_\alpha$ *of rationals increasing to* r_α. *Then the antichain* $\{(\bar{s}_\alpha, \alpha) : \alpha < \omega_1\}$ *is an element of* $\mathbb{T}'$.

We shall deal with filtrations of the elements of $\mathbb{T}$, as defined in Definition 1.3 and to each attach a sequence of subsets of ω dependent on an additional parameter $\bar{C}$ which will be introduced below.

Definition 1.3. (1) For a tree T of size $\aleph_1$, we say that $\bar{T} = \langle T_\alpha : \alpha < \omega_1\rangle$ is a *filtration* of T if $\bar{T}$ is a continuous increasing sequence of countable partial orders whose union is T, such that for $\alpha < \beta < \omega_1$ the order of T_α is the restriction of that of T_β to the universe of T_α.

(2) Suppose that $\bar{T}$ is a filtration of T, where $T \in \mathbb{T}'$. An ordinal δ is *good* for $\bar{T}$ if $0 < \delta < \omega_1$ is a limit ordinal, T_δ is a subtree of T, and

$$\{\alpha : (\exists \beta < \delta)(\exists \bar{x})\, (\bar{x}, \alpha) \in T_\beta\} = \delta.$$

Observation 1.4. *If* $T \in \mathbb{T}$ *then* T *has height* $\leq \omega_1$ *and hence, for any* $\bar{T} = \langle T_\alpha : \alpha < \omega_1\rangle$ *which is a filtration of* T, *there is a club of* δ *such that* T_δ *is a subtree of* T. *If, in addition,* T *is an element of* $\mathbb{T}'$, *then there is a club of* $\delta < \omega_1$ *that are good for* $\bar{T}$.

Recall the following well-known notion:

Definition 1.5. A *ladder system* (on ω_1) is a sequence

$$\bar{C} = \langle c_\delta = \langle \alpha^\delta_n : n < \omega\rangle : 0 < \delta < \omega_1 \text{ a limit } \rangle$$

where each c_δ is an unbounded subset of δ and $\langle \alpha^\delta_n : n < \omega\rangle$ is its increasing enumeration.

The main definition we shall use is that of an invariant, which associates to every element of $\mathbb{T}'$ a certain sequence of subsets of ω.

Definition 1.6. Suppose that $T \in \mathbb{T}'$ and $\bar{T}$ is a filtration of T, while $\bar{C}$ is a ladder system and δ is a good point for $\bar{T}$. We define $\mathrm{inv}_{\bar{T},\bar{C}}(\delta)$ as the set

$$\{n < \omega : (\exists (\bar{y}, \beta) \in T_{\alpha^\delta_{n+1}} \setminus T_{\alpha^\delta_n})(\forall z \in T_{\alpha^\delta_n})$$
$$[z <_T (\bar{y}, \beta) \iff z <_T (\bar{x}_\delta, \delta)]\}.$$

Lemma 1.7 (Auxiliary Construction Lemma). *Suppose that $\bar{C}$ is a fixed ladder system and $\bar{A} = \langle A_\delta : \delta < \omega_1 \rangle$ a sequence of infinite subsets of ω. Then there is a member $T \stackrel{\text{def}}{=} T[\bar{A}]$ of $\mathbb{T}'$ and a filtration $\bar{T}$ of T such that for a club E of ω_1 we have*

$$\delta \in E \ \&\ c_\delta \subseteq E \implies \mathrm{ht}_T(\delta) = \delta \text{ and } \mathrm{inv}_{\bar{T},\bar{C}}(\delta) = A_\delta.$$

Proof: We shall give the filtration $\bar{T} = \langle T_\xi : \xi < \omega_1 \rangle$ and then let $T = \bigcup_{\xi<\omega_1} T_\xi$. By induction on $\xi < \omega_1$ we construct T_ξ so that for countable ordinals α, β, γ, and δ the following requirements hold:

- **:** (1) the universe of T_α is of the form $\{(\bar{x}_i, i) : i < o_\alpha\}$ for some $o_\alpha < \omega_1$, and if $\alpha < \beta$ then T_α is a subtree of T_β. The height of T_α is $\omega + \alpha$. Also, $(\bar{x}, i) <_T (\bar{y}, j)$ implies that $i < j$;
- **:** (2) if $\beta = \alpha + 1$ and α is a successor, then for any $s = (\bar{x}, i)$ in T_α and $\varepsilon > 0$ there is $s_\varepsilon >_T s$ such that $s_\varepsilon = (\bar{x}_\varepsilon, i_\varepsilon) \in T_\beta \setminus T_\alpha$, $t <_T s_\varepsilon \implies t \leq_T s$ and $\sup(\bar{x}_\varepsilon) < \sup(\bar{x}) + \varepsilon$;
- **:** (3) if $\beta = \alpha + 1 < \omega_1$ and $\gamma \leq \alpha$ is a limit (possibly 0) with $s \in T_\gamma$, then letting $\eta = \mathrm{ht}(T_\gamma) - \mathrm{ht}_T(s)$, for every $\varepsilon > 0$ there is a $<_T$-increasing η-sequence $\langle s_\rho = (\bar{y}_\rho, i_\rho) : \rho < \eta \rangle$ in T_γ, such that $s_0 = s$, $\{t \in T_\alpha : (\forall \rho < \eta) s_\rho <_T t\} = \emptyset$, but there is $t = (\bar{y}, i) \in T_\beta$ such that for all $\rho < \eta$ we have $s_\rho <_T t$, while $\sup_{\rho<\eta} \sup(\bar{y}_\rho) < \sup(\bar{y}_0) + \varepsilon$ and $\bar{y} = \bigcup_{\rho<\eta} \bar{y}_\rho$;
- **:** (4) if $0 < \delta < \omega_1$ is a limit ordinal then $T_\delta = \bigcup_{\alpha<\delta} T_\alpha$; and
- **:** (5) if δ is a good point satisfying that c_δ consists of limit ordinals, then $\mathrm{ht}_T((\bar{x}_\delta, \delta)) = \delta$ and $\mathrm{inv}_{\bar{T},\bar{C}}(\delta) = A_\delta$.

Note that (1) above guarantees that for every $\xi < \omega_1$ we have $<_{T_\xi} = <_T \restriction T_\xi$, so we are justified in using the single notation $<_T$ for all tree orders mentioned above, as well as a unique computation of the height of nodes in the tree. Regarding (5), notice that determining if δ is a good point of a filtration $\bar{T}$ and calculating its height and invariant depends only on $\langle T_\xi : \xi < \delta \rangle$ (and $\bar{C}$). Requirements (2) and (3) are reminiscent of the classical construction of an Aronszajn tree and will be used to satisfy (5) at relevant δ. The main difference is that in (3) we allow $\gamma < \alpha$, which is also responsible for the tree not necessarily having countable levels at the end. We need this to have a strong enough control of the required inv sets. In (1), the last requirement is simply an easy way to guarantee that the partial order we obtain is in fact a tree. Let us proceed with the inductive construction of T_ξ, taking as the inductive hypothesis that (1)-(5) hold for $\alpha, \beta, \gamma, \delta < \xi$.

• $\xi = 0$. We injectively enumerate all increasing finite sequences of rationals as $\langle \bar{x}_i : i < \omega \times \omega \rangle$ and therefore, if $\bar{x}$ is an initial segment of $\bar{y}$ then $\bar{x}$ is enumerated with an index smaller than that of $\bar{y}$; (this is why we use ω^2 rather than ω as the indexing ordinal). We let $T_0 = \{(\bar{x}_i, i) : i < \omega \times \omega\}$ and order it so that $(\bar{x}_i, i) <_T (\bar{x}_j, j)$ iff $\bar{x}_i$ is an initial segment of $\bar{x}_j$. This satisfies (1), while the other requirements are vacuous at this stage.

• $\xi = \zeta + 1$ *and* ζ *is not a good point or* ζ *is a good point but* c_ζ *does not consist of limit ordinals.* Requirement (2) is only relevant if when $\alpha = \zeta$ is a successor and $\beta = \xi$. Then we can satisfy (2) by adding countably many new nodes to the tree, one for each relevant $(\bar{x}, i)$ and rational ε, taking care that (1) continues to hold.

Suppose now that $\gamma \leq \zeta$ is a limit and we shall satisfy (3) with ξ in place of β; hence, $\alpha = \zeta$. Suppose first that $\gamma = 0$. Using the construction at $\xi = 0$, we can see that for any $\varepsilon > 0$ and $t \in T_0$ there are $2^{\aleph_0}$ $<_T$-increasing sequences $\langle t_n = (\bar{z}_n, j_n) : n < \omega \rangle$ in T_0 with $t_0 = t$ and $\sup_{n<\omega} \sup(\bar{z}_n) < \sup(\bar{z}_0) + \varepsilon$. Only countably many of these sequences may have an upper bound in T_ζ; hence, we can choose a sequence $\langle s_n : n < \omega \rangle$ among those of these sequences that do not have an upper bound in T_ζ, and make sure that it does have an upper bound of the required kind in T_ξ. Since it suffices to deal with rational ε we only have to make countably many choices in this step, and hence can successfully make them.

Suppose now that $\gamma > 0$. Let $\alpha_0 = \mathrm{ht}_T(s)$ and let $s = (\bar{y}_0, i_0)$. First suppose that γ is a limit of limits and let $\gamma_0 < \gamma$ be a limit > 0 such that $s \in T_{\gamma_0}$. Let $\langle \gamma_n : n < \omega \rangle$ be an increasing sequence of limits with supremum γ where γ_0 is given. Let $\langle \varepsilon_n : n < \omega \rangle$ be a sequence of positive rationals whose sum is less that ε. By induction on $n < \omega$ we define s_n as follows. Let $s_0 = s$. Applying the inductive hypothesis (3) to s_0 in place of s and and γ_0 in place of γ and α, we can find a sequence $\langle s^1_\rho = (\bar{y}^1_\rho, i^1_\rho) : \rho < \gamma_0 - \alpha_0 \rangle$ in T_{γ_0} which is unbounded in T_{γ_0}, yet there is $s_1 = (\bar{y}_1, i_1) \in T_{\gamma_0+1}$ such that for all ρ we have $s^1_\rho <_T s$ and $\sup(\bar{y_1}) < \sup(\bar{y}_0) + \varepsilon_1$ while $\bar{y}_1 = \bigcup_{\rho<\gamma_0-\alpha_0} \bar{y}^1_\rho$. Repeating this procedure inductively we obtain a $<_T$-sequence $\langle s_n = (\bar{y}_n, i_n) : n < \omega \rangle$ such that $\bigcup_{n<\omega} \bar{y}_n$ is bounded by $\sup(\bar{y}_0) + \varepsilon$. The construction of the sequence in fact shows that there are $2^{\aleph_0}$ sequences with the same properties, so there is such a sequence that does not have an upper bound in T_ζ. We can then add a node to T_ξ that will be a required upper bound for the sequence in T_ξ.

If γ is not a limit of limits, then we have $\gamma = \gamma' + \omega$ for some γ' a limit (possibly 0), and if $s \in T'_\gamma$, we can use the inductive hypothesis (3) for $\gamma' + 1$ to find $s_1 >_T s$ with the height of s_1 being $\omega + \gamma' + 1$ and $s_1 \in T_{\gamma'+1}$. Then we can build an $\omega + 1$-sequence starting with s_1 like in the above proof in the case of $\zeta = 0$. If $s \notin T'_\gamma$, we then simply build such an $\omega + 1$ sequence.

The other requirements are vacuous in this case.

• $\xi > 0$ *is a limit ordinal.* Let T_ξ be the union of T_ζ for $\zeta < \xi$.

• $\xi = \delta + 1$, δ *is a good point and* c_δ *consists of limits.* We shall first satisfy (5). Using the inductive assumption (3), we shall choose by induction on $n \in A_\delta$ a $<_T$-increasing sequence $\langle s_{n+1} = (\bar{y}_n, \beta_n) : n \in A_\delta \rangle$ of nodes with $s_0 \in T_0$ of height 0 and $s_{n+1} \in T_{\alpha^\delta_{n+1}} \setminus T_{\alpha^\delta_n}$ of height at least α^δ_n, for all $n \in A_\delta$ (and exactly for these n). In addition, we shall require that for all m in A_δ, if $n = \min(A_\delta \setminus (m+1))$ then

$$\sup(\bar{y}_n) < 2^{-(m+1)} + \sup(\bar{y}_m).$$

Together with the sequence of s_ns, we shall choose for each $m \in A_\delta$ and for $m = 0$, letting $n = \min(A_\delta \setminus (m+1))$, a $<_T$-increasing sequence $\bar{t}_m = \langle t^m_k = (\bar{z}^m_k, i^m_k) : k < \alpha^\delta_n - \alpha^\delta_{n-1} \rangle$ in $T_{\alpha^\delta_n} \setminus T_{\alpha^\delta_{n-1}}$ such that $s_{m+1} <_T t^m_0$ and

$$\sup_{k < \alpha^\delta_n - \alpha^\delta_{n-1}} \sup(\bar{z}^m_k) < \sup(\bar{y}_m) + 2^{-(m+1)}.$$

This will be done so that s_{n+1} is a $<_T$-minimal t such that $t^m_k <_T t$ holds for all k. This choice is the main difference between our construction and the classical construction of an Aronszajn tree, since an important point for us is to assure that for $n \notin A_\delta$, we do not obtain $n \in \mathrm{inv}_{\bar{T},\bar{C}}(\delta)$.

Let n be the least element of A_δ. We should distinguish the case when $n = 0$ and the one when $n > 0$. Suppose first that $n = 0$. To start the induction we choose $s_0 = (\bar{y}_0 = \langle\rangle, i_0) \in T_0$ of height 0. Applying (3) to $\beta = \alpha^\delta_0 + 1$, $\varepsilon = 1$, and s_0, we can find a sequence $\langle t^0_k = (\bar{y}^0_k, i^0_k) : k < \alpha^\delta_n \rangle$ of elements of $T_{\alpha^\delta_0}$ with the property that t^0_0 is an immediate successor of s_0 (which is possible by (2)), $\sup_{k<\alpha^\delta_0}\sup(\bar{y}^0_k) < 2^{-1}$ and such that $\{t \in T_{\alpha^\delta_0} : (\forall k < \alpha^\delta_0) t^0_k <_T t\} = \emptyset$, but for some $s_{n+1} = (\bar{y}_1, i_1) \in T_{\alpha^\delta_0+1}$ we have $(\forall k < \alpha^\delta_0) t^0_k <_T s_{n+1}$ and $\bar{y}_1 = \bigcup_{k<\alpha^\delta_0} \bar{y}^0_k$. If $n > 0$ then we first choose a sequence $\langle t^{-1}_m : m < \omega\rangle$ of elements of T_0 that does not have an upper bound in $T_{\alpha^\delta_n+1}$ but obtains an upper bound $s_0 \in T_{\alpha^\delta_{n-1}+1} \setminus T_{\alpha^\delta_{n-1}}$. This is possible by (3) applied to $\gamma = 0$ and $\alpha = \alpha^\delta_{n-1}$. Then we also make sure that $t^0_0 \in T_{\alpha^\delta_{n-1}+2} \setminus T_{\alpha^\delta_{n-1}+1}$, which is possible by requirement (2). The choice of $\bar{t}_0$ in both cases is possible because α^δ_n is a limit.

We continue this process inductively over $n \in A_\delta$ taking care that when we have chosen s_{m+1} for some $m \in A_\delta$, if $n = \min(A_\delta \setminus (m+1))$ then $t^m_0 \in T_{\alpha^\delta_{n-1}+2} \setminus T_{\alpha^\delta_{n-1}+1}$ is a least upper bound of a $<_T$-increasing sequence of length $\alpha^\delta_{m+1} - \mathrm{ht}(s_{m+1})$ of elements of $T_{\alpha^\delta_{m+1}}$ starting with s_{m+1}, which does not have a bound in $T_{\alpha^\delta_{n-1}+1}$. We choose t^m_k similarly. At the end of the induction we have a sequence $\langle s_n = (\bar{y}_n, i_n) : n \in A_\delta\rangle$ such that $\sup_{n<\omega}\sup(\bar{y}_n) \le 1$. Since A_δ is infinite the length of $\bigcup_{n\in A_\delta} \bar{y}_n$ is δ. Letting $\bar{x}_\delta = \bigcup_{n\in A_\delta} \bar{y}_n$ allows us to put $(\bar{x}_\delta, \delta)$ in T_ξ and to require $s_{n+1} <_T (\bar{x}_\delta, \delta)$ for all $n \in A_\delta$. It follows that the height of $(\bar{x}_\delta, \delta) = \delta$.

Let us also show that $\mathrm{inv}_{\bar{T},\bar{C}}(\delta) = A_\delta$. If $n \in A_\delta$ then $s_{n+1} \in T_{\alpha^\delta_{n+1}} \setminus T_{\alpha^\delta_n}$ shows that $n \in \mathrm{inv}_{\bar{T},\bar{C}}(\delta)$. If $l \notin A_\delta$ then either $l < \min(A_\delta)$ or there are successive elements $m < n$ of A_δ such that $m < l < n$. Let us suppose that the first case happened and let $n = \min(A_\delta)$. Hence, $n > 0$. Then $\{t \in T_{\alpha^\delta_l} : t <_T (\bar{x}_\delta, \delta)\} = \{t^{-1}_k : k < \omega\}$ and this set does not have an upper bound in $T_{\alpha^\delta_{l+1}}$; hence, $l \notin \mathrm{inv}_{\bar{T},\bar{C}}(\delta)$. In the other case, suppose that $m < n$ are successive elements of A_δ such that $m < l < n$. Then $\{t \in T_{\alpha^\delta_l} : t <_T (\bar{x}_\delta, \delta)\} = \{t \in T_{\alpha^\delta_l} : (\exists k)\, t \le_T t^m_k\}$ and this set does not have an upper bound in $T_{\alpha^\delta_{l+1}}$. Hence, $l \notin \mathrm{inv}_{\bar{T},\bar{C}}(\delta)$.

Having satisfied (5), we satisfy (2) and (3) as in the previous case.

Once the inductive construction is over, we let E be a club of good points of this filtration that happen to be limits of limits. Then E exemplifies that T and $\bar{T}$ are as required. $\square$

Once we have used the idea of increasing sequences of rationals to construct the trees $T[\bar{A}]$ to not have uncountable branches, we now can get rid of this through a simple translation into trees whose universe is ω_1.

Definition 1.8. We define the projection functor $\pi : \mathbb{T}' \to \mathbb{T}$ by letting $O \stackrel{\text{def}}{=} \pi(T)$ be the tree on ω_1 defined by letting $\alpha <_O \beta$ if $(\bar{x}_\alpha, \alpha) <_T (\bar{x}_\beta, \beta)$.

Given any filtration $\bar{T}$ of a $T \in \mathbb{T}'$ we similarly define the translation $\pi(\bar{T})$ of it to a filtration of $\pi(T)$.

We now extend the notion of an invariant to any tree whose universe is ω_1. At the same time we introduce a slight generalization that will be used in the main proof. The generalization allows us to compute an invariant of a $\delta < \omega_1$ using the entry $c_{\delta'}$ in the ladder system whose index is some δ' not necessarily equal to δ.

Definition 1.9. If T is a tree with universe ω_1 and $\bar{T} = \langle T_\alpha : \alpha < \omega_1 \rangle$ is a filtration of T, while $\bar{C}$ is a given ladder system and δ is such that the universe of T_δ is δ, while δ' is a limit, we define $\mathrm{inv}_{\bar{T},\bar{C},\delta'}(\delta)$ as the set

$$\{n < \omega : (\exists \beta \in T_{\alpha^{\delta'}_{n+1}} \setminus T_{\alpha^{\delta'}_n}) \{\gamma \in T_{\alpha^{\delta'}_n} : \gamma <_T \beta\} = \{\gamma \in T_{\alpha^{\delta'}_n} : \gamma <_T \delta'\}\}.$$

If $\delta' = \delta$, we write $\mathrm{inv}_{\bar{T},\bar{C}}(\delta)$ in place of $\mathrm{inv}_{\bar{T},\bar{C},\delta}(\delta)$.

Observation 1.10. *Let $\bar{C}$ be a given ladder system. If $T \in \mathbb{T}'$ and $\bar{T}$ is a filtration of T, then there is a club E of ω_1 such that for all $\delta \in E$ we have $\mathrm{inv}_{\bar{T},\bar{C}}(\delta) = \mathrm{inv}_{\pi(\bar{T}),\bar{C}}(\delta)$.*

This gives rise to the formulation of the Construction Lemma we need, which is the Auxiliary Construction Lemma in terms of $\mathbb{T}$ rather than $\mathbb{T}'$.

Lemma 1.11 (Construction Lemma)**.** *Suppose that $\bar{C}$ is a fixed ladder system and $\bar{A} = \langle A_\delta : \delta < \omega_1 \rangle$ a sequence of infinite subsets of ω. Then there is a member $T \stackrel{\text{def}}{=} T[\bar{A}]$ of $\mathbb{T}$ and a filtration $\bar{T}$ of T such that for a club E of ω_1 we have*

$$\delta \in E \ \&\ c_\delta \subseteq E \implies \mathrm{ht}_T(\delta) = \delta \text{ and } \mathrm{inv}_{\bar{T},\bar{C}}(\delta) = A_\delta.$$

Let us now proceed to the Preservation Lemma.

Notation 1.12. (1) For a tree T and $x \in T$, α an ordinal less than $\mathrm{ht}_T(x)$, we let $x(\alpha)$ be the unique $y \in T$ with $y \leq_T x$ and $\mathrm{ht}_T(y) = \alpha + 1$.

(2) For a tree T and $x \neq y \in T$, we let $\Delta(x, y) = \alpha$ if

$$\alpha = \min\{\beta : x(\beta) \neq y(\beta)\}.$$

Lemma 1.13 (Preservation Lemma)**.** *Suppose that $\bar{C}$ is a given ladder system and that $h : T_1 \to T_2$ is a $\leq$-reduction between trees with universe ω_1 satisfying that for all x, y in T_1*

$$\Delta(x, y) = \Delta(h(x), h(y)).$$

Then for any filtrations $\bar{T}^l$ for $l \in \{1, 2\}$ of T_1 and T_2, respectively, there is a club E of ω_1 such that whenever $\delta \in E$ and $c_\delta \subseteq E$ then for any $\delta \in E$ whose height in the tree T_1 is δ, we have

$$\mathrm{inv}_{\bar{T}^1, \bar{C}}(\delta) = \mathrm{inv}_{\bar{T}^2, \bar{C}, \delta}(h(\delta)).$$

Remark 1.14. The additional requirement on the reductions we needed for the Preservation Lemma is, unfortunately, rather strong. In particular, it implies that h is 1-1. Namely suppose that $x \neq y$; hence, $\Delta(x, y) < \mathrm{ht}(x)$ or $\Delta(x, y) < \mathrm{ht}(y)$. Let us say that the first case happens; then there is $z \leq_1 x$ with $\neg z \leq_1 y$. Then we have that $h(z) \leq_2 h(x)$ by the definition of a reduction and also that $\mathrm{ht}(h(z)) \geq \mathrm{ht}(z) > \Delta(z, y) = \Delta(h(x), h(y))$. In particular, $h(x) \neq h(y)$.

Proof: Let us write $\bar{T}^l = \langle T^l_j : j < \omega_1 \rangle$ for $l \in \{1, 2\}$ and simplify the notation by writing $<_1 \stackrel{\mathrm{def}}{=} <_{T_1}$ and $<_2 \stackrel{\mathrm{def}}{=} <_{T_2}$ and let M be the model $\langle \bar{T}^1, \bar{T}^2, \omega_1, <_1, <_2, h, \in \rangle$. There is a club E of ω_1 such that for any $\delta \in E$:

- **:** (i) the universe of T^1_δ and T^2_δ is δ and the height of both T^1_δ and T^2_δ is at most δ;
- **:** (ii) T^1_δ and T^2_δ are subtrees of T_1 and T_2, respectively;
- **:** (iii) $M \restriction \delta \prec M$ (so if $\beta < \delta$ then $h(\beta) < \delta$, and if β is in the image of h then $h^{-1}(\beta) < \delta$).

Suppose that $\delta \in E$ has height δ in T_1 and that $c_\delta \subseteq E$; we shall show that δ is as required. For simplicity in notation we shall omit $\bar{C}$ from the notation for invariant and subscripts T_1 and T_2 from the notation for height. Note that $\mathrm{ht}(h(\delta)) \geq \delta$ so $h(\delta) \geq \delta$ by (i) of the assumptions.

In the forward direction, suppose $n \in \mathrm{inv}_{\bar{T}^1}(\delta)$ and let $y \in T^1_{\alpha^\delta_{n+1}}$ demonstrate this. This means

$$\{w \in T^1_{\alpha^\delta_n} : w <_1 y\} = \{w \in T^1_{\alpha^\delta_n} : w <_1 \delta\}.$$

We may also assume that y is a $<_1$-minimal node satisfying this requirement. In particular, for all $z <_1 y$ we have $z \in T^1_{\alpha^\delta_n}$ by (ii) above. By (iii) above, we have that $h(y) \in T^2_{\alpha^\delta_{n+1}}$. We would like to use $h(y)$ as a witness that $n \in \mathrm{inv}_{\bar{T}^2,\bar{C},\delta}(h(\delta))$, for which we need to know that $h(y) \notin T^2_{\alpha^\delta_n}$. This follows from (iii) above since $h^{-1}(h(y)) = y$.

(We note that by a different proof this part of the argument in fact goes through even if h is assumed to be any reduction, not necessarily 1-1, but it is in the next part of the argument that we need the additional assumption about the preservation of Δ.)

For the other inclusion suppose that $n \in \mathrm{inv}_{\bar{T}^2,\delta}(h(\delta))$ as exemplified by z. This z might not itself be useful in showing that $n \in \mathrm{inv}_{\bar{T}^1,\delta}(\delta)$ as it may not be in the image of h. However we do have that

$$M \models \text{“}(\exists w > \alpha^\delta_n)(\forall \alpha < \alpha^\delta_n)\,[\alpha <_2 h(w) \iff \alpha <_2 z]\text{”}$$

(as exemplified by δ), so there is $w < \alpha^\delta_{n+1}$ satisfying the above property. We may without loss of generality assume that w is a $<_1$-minimal element of $T^1_{\alpha^\delta_{n+1}} \setminus T^1_{\alpha^\delta_n}$. In particular for every $o <_1 w$ we have $o \in T^1_{\alpha^\delta_n}$, and so $h(o) \in T^2_{\alpha^\delta_n}$. By the choice of w this implies $h(o) <_2 z$ and then by the choice of z we have $h(o) <_2 h(\delta)$. Hence, by the preservation of Δ, either $w \leq_1 \delta$ or $\mathrm{ht}(w) = \Delta(w, \delta) + 1$. In the first case, we have that w exemplifies that $n \in \mathrm{inv}_{\bar{T}^1,\delta}(\delta)$. In the second case, since $\delta \notin T^1_{\alpha^\delta_n}$, we may find $s \leq_1 \delta$ of the same height as w. If $s \notin T^1_{\alpha^\delta_n}$, then by the choice of E, we have that w exemplifies that $n \in \mathrm{inv}_{\bar{T}^1,\delta}(\delta)$, so let us show that this is the case. If $s \in T^1_{\alpha^\delta_n}$, then $h(s) \in T^2_{\alpha^\delta_n}$, but since $\Delta(h(w), h(s)) = \Delta(w, s)$, we may not have $h(s) <_2 h(w)$, contradicting the choice of w. $\square$

Now we can complete the proof of the main theorem:

Proof of Theorem: Suppose for contradiction that $\{T_i : i < i^* < 2^{\aleph_0}\}$ is a family of trees of size $\aleph_1$ such that every element of $\mathbb{T}$ is $\leq$-embeddable in some T_i by a Δ-preserving embedding. By passing to isomorphic copies if necessary, we may assume that the universe of each T_i is ω_1. Let $\bar{C}$ be given by (a) in the statement of the theorem with $c_\delta = \langle \alpha_n^\delta : n < \omega \rangle$ an increasing enumeration of c_δ for δ limit $< \omega_1$.

Choose a family $\mathcal{B}$ of $2^{\aleph_0}$ distinct infinite subsets B of ω. For $B \in \mathcal{B}$ and δ a limit ordinal, let $A_B^\delta = B$, and let $\bar{A}_B = \langle A_B^\delta : \delta \text{ limit } < \omega_1 \rangle$. Let $T_B = T[\bar{A}_B]$ be obtained as in the Construction Lemma and let $\bar{T}_B$ be the associated filtration. Note that T_B satisfies that its universe is ω_1.

Let $\bar{T}_i$ be any filtration of T_i, for $i < i^*$. Note that trivially for any i and δ we have

$$|\{\mathrm{inv}_{\bar{T}_i,\bar{C},\delta}(\alpha) : \alpha \in T_i\}| \leq |T_i| = \aleph_1.$$

Hence, we can choose $B \in \mathcal{B}$ such that for no $i < i^*$ and $\alpha, \delta < \omega_1$ do we have that $B = \mathrm{inv}_{\bar{T}_i,\bar{C},\delta}(\alpha)$. Let us suppose that for some $i < i^*$ and a Δ-preserving $\leq$-reduction h we have $h : T_B \to T_i$. Let E be a club, as in the Preservation Lemma, which we can without loss of generality assume consists of limits of limits which, by the Construction Lemma, satisfy that for $\delta \in E$ we have $\mathrm{ht}_{T_B}(\delta) = \delta$, and let $\delta < \omega_1$ be such that $c_\delta \subseteq E$. Then

$$B = \mathrm{inv}_{\bar{T}_B,\bar{C}}(\delta) = \mathrm{inv}_{\bar{T}_i,\bar{C},\delta}(h(\delta)),$$

which is a contradiction. $\square$

As a final remark we observe that it is clear that the trees constructed as in the Construction Lemma have the property that for $\bar{A}$ and $\bar{B}$ satisfying that for a club many δ we have $A_\delta \neq B_\delta$ the trees $T[\bar{A}]$ and $T[\bar{B}]$ are not $\leq$-reducible into each other by a Δ-preserving reduction. Hence, assuming that CH fails, for any fixed club guessing ladder system $\bar{C}$, this gives rise to a family of $2^{\aleph_1}$ pairwise $\leq$-incomparable by Δ-preserving reductions, elements of $\mathbb{T}$. A ZFC example of such a family where the elements are actually not $\leq$-reducible to each other is already provided by Theorem 0.2(3). The two families are necessarily distinct because of the following

Claim 1.15. *Suppose that $\bar{A} = \langle A_\delta : \delta < \omega_1 \rangle$ is a sequence of infinite subsets of ω, $\bar{C}$ is a fixed ladder system on and $T = T[\bar{A}]$ is as constructed in the Construction Lemma. Then for every limit $\gamma < \omega_1$ the level γ of T is uncountable.*

Proof: Let $\gamma < \omega_1$ be a limit ordinal. Requirement (3) in the Construction Lemma applied to s being the root of the tree gives for each $\alpha \geq \gamma$ a γ-sequence $\bar{s}_\alpha$ that is unbounded in T_α but bounded in $T_{\alpha+1}$. Hence, the $\bar{s}_\alpha$ are pairwise distinct and letting t_α be the least upper bound of $\bar{s}_\alpha$, we obtain a family of $\aleph_1$ many elements of T of height γ. $\square$

2. Other Classes of Trees

In the previous section we were mainly interested in the class $\mathbb{T}$, but in fact the proof used in that section can be used to give similar results in other classes of trees. We shall consider generalizations in two different directions: varying the size of the trees in question and leaving out the requirement that the trees do not have an uncountable branch. Let us begin with the first of them.

Definition 2.1. (1) For any cardinal κ let $\mathbb{T}_\kappa$ be the class of trees of size κ with no uncountable branches.

(2) If $\kappa > \aleph_0$ is regular and T is a tree of size κ, then the sequence $\langle T_\alpha : \alpha < \kappa \rangle$ is a *filtration of* T whenever it is a continuous sequence of subtrees of T each of size $< \kappa$, whose union is T.

Since the Auxiliary Construction Lemma did not require different nodes of the tree being constructed to have different rational sequences attached to them, we can use the same construction and the translation through the projection functor to obtain the following lemma. In it we use the obvious generalization of the notion of a ladder system to the set $S^\kappa_{\aleph_0}$, where S^κ_λ in general denotes the set of ordinals in κ whose cofinality is λ.

Lemma 2.2 (Construction Lemma for $\mathbb{T}_\kappa$)**.** *Suppose that κ is a regular uncountable cardinal and $\bar{C} = \langle c_\delta : \delta \in S^\kappa_{\aleph_0} \rangle$ is a fixed ladder system, while $\bar{A} = \langle A_\delta : \delta < \kappa \rangle$ is a sequence of infinite subsets of ω. Then there is a member $T \stackrel{\text{def}}{=} T[\bar{A}]^\kappa$ of $\mathbb{T}_\kappa$ and a filtration $\bar{T}$ of T such that for a club E of κ we have*

$$\delta \in E \ \& \ c_\delta \subseteq E \implies \mathrm{inv}_{\bar{T},\bar{C}}(\delta) = A_\delta.$$

Notice that of course we shall not be able to guarantee that the height of δ in T for $\delta \in E$ is δ, as the height of the tree is just ω_1. This causes some difficulties when trying to generalize the Preservation Lemma. In the first paragraph of its proof, after the choice of E we claim that for $\delta \in E$ we have $h(\delta) \geq \delta$ for δ whose height in T_1 is δ, and this is useful later when calculating invariants. However, here we still have the following:

Observation 2.3. *Suppose that h is a $< \kappa$-to-1 function from κ into itself. Then there is a club C of κ such that for all $\delta \in C$ we have $h(\delta) \geq \delta$.*

Proof: Otherwise there would be a stationary set S of κ such that h is regressive on S so there would be a stationary set on which h is constant, contradicting the assumption of h being $< \kappa$-to-1. $\square$

The Preservation Lemma now becomes:

Lemma 2.4 (Preservation Lemma for $\mathbb{T}_\kappa$)**.** *Suppose κ is a regular uncountable cardinal and $\bar{C} = \langle c_\delta : \delta \in S^\kappa_{\aleph_0} \rangle$ is a fixed ladder system while $h : T_1 \to T_2$ is a Δ-preserving $\leq$-reduction between trees with universe κ. Then for any filtrations $\bar{T}^l$ for $l \in \{1, 2\}$ of T_1 and T_2 respectively, there is a club E of κ such that whenever $\delta \in E$ and $c_\delta \subseteq E$ then for any $\delta \in E$ we have*

$$\mathrm{inv}_{\bar{T}^1,\bar{C}}(\delta) = \mathrm{inv}_{\bar{T}^2,\bar{C},\delta}(h(\delta)).$$

Recalling that if κ is regular $\geq \aleph_2$ then there is a ladder system $\langle c_\delta : \delta \in S^\kappa_0 \rangle$ which guesses clubs, as proved by Shelah [5], [7] (or see [3] for a proof); as a conclusion, we get Theorem 0.5.

Of course a very natural class to study here is that of the trees of size $2^{\aleph_0}$ with no uncountable branches, as they appear as the result of a number of natural constructions (see [10]); clearly our method does not apply to this class as it stands. Another generalization here is the replacement of $\aleph_0$ by a λ such that $\lambda^{<\lambda} = \lambda$, where one has the saturated dense linear order of size λ with no first or last element; denote it by $\mathbb{Q}_\lambda$. Then in the proof of Theorem 0.5, one can replace the increasing bounded sequences of rationals by increasing bounded sequences of elements of $\mathbb{Q}_\lambda$ and hence obtain the following analogue of Theorem 0.5 (using Shelah's club guessing between κ and λ).

Theorem 2.5. *Suppose that κ is a regular cardinal and $\lambda = \lambda^{<\lambda}$ satisfy $\lambda^+ < \kappa < 2^\lambda$. Let $\leq^*$ stand for the Δ-preserving reductions and $\mathbb{T}^\kappa_\lambda$ for trees of size κ with no branches of length λ^+. Then the universality number of $(\mathbb{T}^\kappa_\lambda, \leq^*)$ is at least 2^λ. In fact, no family of* *trees of size κ, even if we allow branches of length λ^+, can $\le^*$-embed all members of $\mathbb{T}^\kappa_\lambda$.*

Of course all we needed here in place of $\lambda^+ < \kappa$ is the existence of a club guessing sequence between κ and λ.

Note of course that the last sentence in each of the theorems shows that the universality number of the class of trees of size $\aleph_1$ (κ), forbidding uncountable (length λ^+) branches under the appropriate notion of reduction, is at least $2^{\aleph_0}$ (or 2^λ).

3. A Topological Application

Topologies constructed from tree orderings are common in general topology. In this section we consider the families of trees constructed in the previous sections of the paper from this point of view. For simplicity we shall concentrate on the trees considered in §1, i.e., the family $\mathbb{T}$ of trees of size $\aleph_1$ with no uncountable branches. But it is clear that a similar method can be applied to trees of size κ with no branches of size κ for κ regular.

Definition 3.1. Suppose that T is a tree. Let $[T]$ be set of its maximal branches. For every $b \in [T]$ and $\alpha \le \mathrm{ht}_T(b)$ we define

$$O_\alpha(b) = \{c \in [T] : \Delta(b,c) > \alpha\}.$$

We consider that $\Delta(b,b) = \infty$ for every $b \in [T]$.

Observation 3.2. *Suppose that $T \in \mathbb{T}$. Then*

$$\{\{O_\alpha(b) : \alpha \le \mathrm{ht}_T(b)\} : b \in [T]\}$$

gives a point base for a first countable Hausdorff topology on $[T]$.

In fact the topological spaces considered in Observation 3.2 come from a class of spaces called ω_1-metric spaces in which Δ is a quasi-distance taking values in ω_1. Such spaces were introduced in [8]. Some further references were given in the Introduction.

Definition 3.3. An *ordered group* is a structure $(G, +, \le)$ in which $(G,+)$ is a group satisfying that for all $a,b,c \in G$ we have $a + c \le b + c \iff a \le c$. Such a group is of *character* ω_1 if it has a decreasing ω_1-sequence $\langle s_\xi : \xi < \omega_1\rangle$ satisfying $s_\xi > 0$ and for every $\varepsilon > 0$ there is ξ such that $\zeta \ge \xi \implies s_\zeta < \varepsilon$.

Sikorski [8] notes that a general method for constructing ordered groups of character ω_1 follows from the work of F. Hausdorff in [2] and involves considering powers of ordered sets. The following is a definition of an ω_1-metric space from [8].

Definition 3.4. (1) Suppose that $(G, +, \leq)$ is an ordered group of character ω_1, as exemplified by a sequence $\langle s_\xi : \xi < \omega_1 \rangle$. Let G^+ denote the positive elements of G. A topological space X is called a $G-\omega_1$*-metric space* if there is a function $\delta : X^2 \to G^+$ satisfying the axioms of a metric in a metric space and such that the topology of X is induced by δ.

(2) A function $f : X \to Y$ between two G-ω_1-metric spaces is called an $\aleph_1$*-isometry* if for all $x, y \in X$ we have $\delta_X(x,y) = \delta_Y(x,y)$.

From this point on we fix an ordered group $(G, +, \leq)$ of character ω_1, as exemplified by a sequence $\langle s_\xi : \xi < \omega_1 \rangle$, and use G^+ to denote the positive elements of G, and only study ω_1-metric spaces with respect to G. These spaces are of interest to us because of the following:

Claim 3.5. *Suppose that $T \in \mathbb{T}$. Then $[T]$ with the topology introduced in Observation 3.2 is an ω_1-metric space.*

Proof: Let $\delta(x,y) = s_\xi$ if $x \neq y$ and $\Delta(x,y) = \xi$, for $\xi < \omega_1$. Let $\delta(x,y) = 0$ if $x = y$. Then δ is clearly a symmetric function satisfying $\delta(x,y) = 0$ iff $x = y$. If $x, y, z \in [T]$ then $\Delta(x,z) = \min\{\Delta(x,y), \Delta(y,z)\}$ so $\delta(x,z) \leq \delta(x,y) + \delta(y,z)$. Any set of the form $O_\alpha(b)$ can be written as $O_\alpha(b) = \{c \in [T] : \delta(b,c) < s_\alpha\}$, which shows that the topology is induced by the ω_1-metric. $\square$

It follows that for $T_1, T_2 \in \mathbb{T}$ the spaces $[T_1], [T_2]$ are isometric as ω_1-metric spaces iff there is a function $f : [T_1] \to [T_2]$ with $\Delta_{T_1}(b,c) = \Delta_{T_2}(f(b), f(c))$ for all $b, c \in [T_1]$. Then we can prove the following:

Claim 3.6. *Suppose that $T_1, T_2 \in \mathbb{T}$ and there is an $\aleph_1$- isometry from $[T_1]$ into $[T_2]$. Then T_1 is reducible to T_2 by a reduction h that satisfies*

$$\Delta_{T_1}(x,y) = \Delta_{T_2}(h(x), h(y)).$$

Proof: Let $f : [T_1] \to [T_2]$ be an $\aleph_1$-isometry, so by the above observation we may assume $\Delta_{T_1}(b,c) = \Delta_{T_2}(f(b), f(c))$ for all $b, c \in$

$[T_1]$. For $x \in [T_1]$ define

$$h(x) = f(b) \restriction \mathrm{ht}_{T_1}(x),$$

for any maximal branch b of T_1 containing x. Because f is an isometry, h is well-defined. It also follows that it is Δ-preserving. If $x \leq_{T_1} y$ then let b be a maximal branch of T_1 containing y, hence also x. Then

$$h(x) = f(b) \restriction \mathrm{ht}_{T_1}(x) \leq_{T_1} h(y) = f(b) \restriction \mathrm{ht}_{T_1}(y). \qquad \square$$

Definition 3.7. We say that an ω_1-metric space X is *large* if for uncountably many $\alpha < \omega_1$ the family $\{O_\alpha(b) : b \in X\}$ contains an uncountable disjoint subfamily.

Hence, being large means failing the ccc condition strongly. If $T \in \mathbb{T}$ then being large for $[T]$ is equivalent to having uncountably many uncountable levels. Namely suppose that level α of T is uncountable. Then choosing a family $\mathcal{F}$ of $\aleph_1$-many distinct nodes x on that level of T and picking for each such x a maximal branch b_x of T containing x, we obtain that the sets in $\{O_\alpha(b_x) : x \in \mathcal{F}\}$ are pairwise disjoint. The converse is proved similarly. It follows that the trees constructed in §1 give rise to large spaces (see Claim, page 14), while an Aronszajn tree cannot give rise to such a space.

Theorem 3.8. *Suppose that club guessing between $\aleph_0$ and $\aleph_1$ holds and that CH fails. Then there is a family $\mathcal{F}$ of $2^{\aleph_1}$ large ω_1-metric spaces (over the same G) that are not ω_1-isometrically embeddable into each other, and such that for every family $\mathcal{G}$ of $< 2^{\aleph_0}$ many elements of $\mathcal{F}$ there is an element of $\mathcal{F}$ that does not ω_1-isometrically embed into any element of $\mathcal{F}$.*

There is just in ZFC a family $\mathcal{F}$ of $2^{\aleph_1}$ ω_1-metric spaces that are not ω_1-isometrically embeddable into each other (and that are not large).

Proof: The first part follows from the proof of Theorem 0.4 and Claim 3.6. The second part follows from Todorčević's result in Theorem 0.2(3). $\square$

The theorems mentioned in the introduction clearly have similar translations to the class of ω_1-metric spaces.

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