INTRODUCING A NONTRIVIAL $\square_\omega$

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Abstract. We define a nontrivial version of the square principle $\square_\omega$, which we then show consistent by means of forcing with finite conditions.

1. Introduction

Square principle $\square_\kappa$ was introduced by Ronald Jensen in order to consider higher cardinal analogues of a Souslin tree. In his study [6] of constructible universe L he proved that the principle holds in L and therefore its existence is relatively consistent with the theory ZFC. It is known that $\square_\kappa$ holds in many other natural inner models, such as $L[E]$ (see [8]). Many versions of $\square_\kappa$ are considered in the literature, especially so for their connection with the pcf theory, see [3]). It is possible to add $\square_\kappa$ by forcing, as was done also by Jensen and later developed by Magidor and others (see [3] for more about the history of this).

The natural versions of $\square_\kappa$ are interesting for uncountable κ and trivial for $\kappa = \omega$. Here we introduce a nontrivial version of $\square_\omega$, and we show how to force its existence using finite conditions. The extent of its nontriviality, on the other hand, is not something we completely understand. In particular we do not know if the principle holds in L, or indeed if it just holds in ZFC.

Our interest in this principle also stems from the fact that to obtain it we use forcing with conditions with finite domain to obtain a nontrivial object on ω_1. Such forcing goes back to Baumgartner [2] but the examples of known forcing with these properties are not particularly numerous, and in particular our example has some new points from the forcing point of view.

2. Preliminaries and Notation

Most of the notation is standard. Relation $A \subseteq B$ means that A is either a proper subset of B or equal to B, and $A \subset B$ means that A is a proper subset of B. $|X|$ is the cardinality of set X. For a set of ordinals X, a limit point of X is an ordinal α such that $\alpha = \sup(Y)$ for some $Y \subseteq X$ or, equivalently, if $\alpha = \sup(X \cap \alpha)$. $\mathrm{Lim}(X)$ is a set of limit points of X. For a function f, $\mathcal{D}_f$ denotes the domain of f, and $f|_A$ denotes the restriction of f to the set $A \cap \mathcal{D}_f$. For a regular cardinal θ, H_θ is the set of all sets x with hereditary cardinality less than θ (i.e. the transitive closure of x has cardinality less than θ), and $[H_\theta]^\omega := \{x \subseteq H_\theta \mid |x| = \omega\}$. If α and β are ordinals then the interval (α, β) denotes the set $\{\mu \mid \mu$ is an ordinal, $\alpha < \mu < \beta\} = \beta \setminus (\alpha + 1)$. Closed and half open intervals are defined similarly.

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A model M is an *elementary submodel* of a model N, $M \prec N$, if $M \subseteq N$ and for every formula φ with parameters $a_1, \ldots, a_n \in M$, φ is true in M if and only if it is true in N. Here we assume that both M and N are models in the same language and the same logic, and for our application we shall only work with elementary submodels of $(H_\theta, \in)$ for some large enough θ. By "large enough" we mean so large that every set which will be considered in our construction is contained in H_θ. A *continuous elementary $\in$-chain* is a sequence $\langle M_i \mid i < i^* \rangle$ of elementary submodels of H_θ such that $M_i \prec M_{i+1}$ and $M_i \in M_{i+1}$ for every i, and $M_i = \bigcup_{j<i} M_j$ if i is a limit ordinal. If M is a countable elementary submodel of H_θ for $\theta \geq \omega_1$ then $M \cap \omega_1$ is an ordinal denoted by δ_M.

By a universe of set theory we mean a countable transitive model of (a sufficiently large finite fragment of) ZFC. If p, q are forcing conditions, we use the notation $q \geq p$ to say that q extends p.

A standard reference for set-theoretic notions and facts is [5]. Basic facts about forcing are well presented in [7] or the first chapter of [9]. A concise reference for elementary models is [4]. The following important theorem is essential in the use of elementary submodels.

Theorem 2.1 (Löwenheim-Skolem (see [4])). *For any H_θ and $x \subseteq H_\theta$ there exists an elementary submodel $M \prec H_\theta$ such that $x \subseteq M$ and $|M| \leq |x| \cdot \aleph_0$. In particular, if x is countable then so is M.*

Definition 2.2. *Suppose α is an ordinal. A set $C \subseteq \alpha$ is a* closed unbounded *set or a* club *in α if:*
(1) C contains all its limit points $< \alpha$ (closed);
(2) for every $\gamma < \alpha$ there exists some $\beta \in C$ such that $\beta > \gamma$ (unbounded).

The concept of a club can be extended from ordinals to various families of sets. We shall define such an extension only for $[H_\theta]^\omega$.

Definition 2.3. *Suppose $\theta \geq \omega_1$. A set $\mathfrak{C} \subseteq [H_\theta]^\omega$ is a* club *in $[H_\theta]^\omega$ if:*
(1) for every chain $x_0 \subseteq x_1 \subseteq x_2 \subseteq \ldots$ of elements from $\mathfrak{C}$, of length $\alpha < \omega_1$, the union $\bigcup_{i<\alpha} x_i \in \mathfrak{C}$ (closed);
(2) for every $x \in [H_\theta]^\omega$ there exists some set $y \in \mathfrak{C}$ such that $y \supseteq x$ (unbounded).

3. Squares

Definition 3.1. *Suppose κ is a regular cardinal.* Square principle $\square_\kappa$ *(square kappa) states that there is a sequence $\langle C_\alpha \mid \alpha$ is a limit ordinal in $\kappa^+ \rangle$ such that:*
(1) C_α is a club in α for every α;
(2) if $\alpha \in \mathrm{Lim}(C_\beta)$ then $C_\alpha = C_\beta \cap \alpha$ (coherence);
(3) if $\mathrm{cf}(\alpha) < \kappa$ then $|C_\alpha| < \kappa$ (nontriviality).

The nontriviality condition excludes the possibility of a trivial *full* square ($C_\alpha = \alpha$ for every α) for $\kappa > \omega$. Notice though that this condition in the case of ω does not rule out the full square, as the condition itself is vacuous.

We will be interested to formulate a nontrivial version of $\square_\kappa$ for $\kappa = \omega$. To preserve the spirit of the definition of a square, we shall construct a nontrivial $\square_\omega$ in the sense of following definition.

Definition 3.2. *We call a sequence $\langle C_\alpha \mid \alpha$ limit $< \omega_1 \rangle$ a* nontrivial square on ω *if it satisfies the properties (1) and (2) of Definition 3.1, and:*

(a) for every $\alpha < \omega_1$ there exists some limit β, $\alpha < \beta < \omega_1$, such that $C_\beta \cap \alpha = \emptyset$;
(b) for every $\omega < \alpha < \omega_1$ there exists some limit β, $\alpha < \beta < \omega_1$, such that $\mathrm{Lim}(C_\beta) \cap \alpha \neq \emptyset$.

Part (a) of Definition 3.2 prevents our $\Box_\omega$-sequence from consisting of $C_\alpha = \alpha$ for all α. Part (b) prevents the $\Box_\omega$-sequence from having each C_α a cofinal ω-sequence in α, which would make the coherence requirement (2) from 3.1 trivial. Note that there is a tension between (a) and (2); to satisfy just (a) we could simply decompose the set of limit ordinals in ω_1 into a disjoint union $\bigcup_{\alpha<\omega_1} A_\alpha$ satisfying that $\gamma \in A_\alpha \implies \gamma > \alpha$, and let $C_\gamma = \gamma \setminus \alpha$ for $\gamma \in A_\alpha$. But we could end up having $\alpha_0 < \alpha_1 < \beta < \gamma$ such that $\gamma \in A_{\alpha_0}$ and $\beta \in A_{\alpha_1}$. Then $\beta \in \mathrm{Lim}(C_\gamma)$ but $\alpha_1 \in (C_\gamma \cap \beta) \setminus C_\beta$.

Note that sequences $\langle C_\alpha \mid \alpha \text{ limit } < \omega_1\rangle$ where C_α is an unbounded subset of α are usually called *ladder systems* and their study forms an important topic in set theory (see [10]). We are hence interested in special ladder systems which have strong coherence properties making them into a $\Box_\omega$-sequence.

Let us make an easy observation, which simply uses the fact that notions involved in the definition of $\Box_\omega$ are absolute for models with the same ω_1:

Observation 3.3. *Suppose that $V \subseteq W$ are two universes of set theory with the same ordinals and the same ω_1 and that $\Box_\omega$ holds in V. Then $\Box_\omega$ holds in W.*

As mentioned above, it is well known that $\Box_\kappa$ for $\kappa > \omega$ holds in L. However, the proof showing that this is the case, when applied to $\kappa = \omega$ simply yields the full square – there is no reason to believe that our nontriviality conditions are satisfied. In particular we do not know if the principle we are considering is true in L. For all we know, it may be that the principle is just true in ZFC.

4. Forcing $\Box_\omega$

In this section we show how to force a nontrivial $\Box_\omega$-sequence by a forcing which preserves cofinalities and cardinalities. We force with conditions that are functions with finite domain, and in this sense the forcing notion is inspired by Baumgartner's forcing to add a club on ω_1 using finite conditions [2], as well as by Abraham's rendition of that forcing [1]. The range of our conditions consists however of infinite sets.

Fix a continuous elementary $\in$-chain $\overline{N} := \langle N_i \mid i < \omega_1\rangle$ of elementary submodels of H_θ for a large enough cardinal θ with $N_i \in N_{i+1}$ for each i. Then the sequence $\langle \delta_{N_i} \mid i < \omega_1\rangle$ is strictly increasing with supremum ω_1. Hence, we can define $i_\alpha := \min\{i < \omega_1 \mid \alpha \in N_i\}$ for every $\alpha < \omega_1$. Notice that $i_\alpha \leq \alpha$.

First we prove a lemma about $\overline{N}$ which will be used later.

Lemma 4.1. $\mathfrak{C} := \{N \prec H_\theta \mid N$ *is countable*, $\overline{N} \in N$, $N_i \subseteq N$ *for all* $i < N \cap \omega_1\}$ *is a club in* $[H_\theta]^\omega$.

Proof. To see that $\mathfrak{C}$ is unbounded, start with an arbitrary $A \in [H_\theta]^\omega$. By the downward Löwenheim-Skolem theorem find a countable elementary submodel $M_0 \prec H_\theta$ such that $A \cup \{\overline{N}\} \subseteq M_0$. By induction on $j < \omega$ find a countable elementary submodel $M_{j+1} \prec H_\theta$ such that $A \cup \{\overline{N}, M_j\} \cup \left(\bigcup_{i<M_j\cap\omega_1} N_i\right) \subseteq M_{j+1}$. Define $M_\omega := \bigcup_{j<\omega} M_j$. Obviously $\overline{N} \in M_\omega$. Now, $M_\omega \cap \omega_1 = \bigcup_{j<\omega}(M_j \cap \omega_1)$, hence for every $i < M_\omega \cap \omega_1$ there exists a $j < \omega$ such that $i < M_j \cap \omega_1$. Therefore, $N_i \subseteq M_{j+1} \subseteq M_\omega$, so $M_\omega \in \mathfrak{C}$ and $A \subseteq M_\omega$.

To see that $\mathfrak{C}$ is closed argue similarly. If $\langle M_j \mid j < \omega \rangle$ is a sequence in $\mathfrak{C}$ such that $M_j \subseteq M_{j+1}$ for all $j < \omega$ then $\bigcup_{j<\omega} M_j = M_\omega$ is in $\mathfrak{C}$, just as above. $\surd_{4.1}$

Definition 4.2. *The forcing notion P is the set of conditions of the form $(p, \mathcal{S}_p)$, where:*

(1) p is a function from some finite subset D_p of $\mathrm{Lim}(\omega_1) \cap \omega_1$ to $\mathcal{P}(\omega_1)$ such that $C_\alpha := p(\alpha) \subseteq \alpha$ is a club for each $\alpha \in \mathcal{D}_p$;

(2) for every $\alpha, \beta \in \mathcal{D}_p$, if $\mu \in \mathrm{Lim}(C_\alpha) \cap \mathrm{Lim}(C_\beta)$ then $C_\alpha \cap \mu = C_\beta \cap \mu$;

(3) $\mathcal{S}_p \subset \mathrm{Lim}(\omega_1) \cap \omega_1$, $|\mathcal{S}_p| < \omega$;

(4) if $\sigma \in \mathcal{S}_p$ and $\alpha \in \mathcal{D}_p$, $\sigma < \alpha$, then $\mathrm{Lim}(C_\alpha \cap \sigma) \subseteq \bigcup_{\beta \in \mathcal{D}_p \cap \sigma} \mathrm{Lim}(C_\beta)$;

(5) for every $\alpha \in \mathcal{D}_p$, if $\delta \in \mathrm{Lim}(C_\alpha)$ then $C_\alpha \cap \delta \in N_{i_\delta}$.

For $(p, \mathcal{S}_p), (q, \mathcal{S}_q) \in P$ define $(p, \mathcal{S}_p) \leq (q, \mathcal{S}_q)$, which means that $(q, \mathcal{S}_q)$ is an extension of $(p, \mathcal{S}_p)$, if

(i) $p \subseteq q$ and $\mathcal{S}_p \subseteq \mathcal{S}_q$;

(ii) if $\sigma \in \mathcal{S}_q \setminus \mathcal{S}_p$ then either $\sigma > \max \mathcal{D}_p$ or there exists some $\sigma' \in \mathcal{S}_p \setminus \sigma$ such that $[\sigma, \sigma') \cap \mathcal{D}_p = \emptyset$.

Elements of $\mathcal{S}_p$ are called *safeguards.* Their purpose is to prevent the forced clubs from mixing with each other. In particular, we would like to prevent new clubs from an extension $q \geq p$ from interfering with the clubs from p in an inappropriate manner, i.e. in a contradiction with the coherency condition of the definition 3.1. Clause (2) in Definition 4.2 is somewhat stronger than the coherency condition. This will help us when we wish to extend a condition p to a q with $\mathcal{D}_q$ containing some $\alpha \notin \mathcal{D}_p$.

Proposition 4.3. *Relation $\leq$ is reflexive and transitive.*

Proof. Reflexivity is trivial. For transitivity, suppose that $(p, \mathcal{S}_p) \leq (q, \mathcal{S}_q) \leq (r, \mathcal{S}_r)$ are conditions from P, and $\sigma \in \mathcal{S}_r \setminus \mathcal{S}_p$. If $\sigma \in \mathcal{S}_q$ then, by first inequality, either $\sigma > \max \mathcal{D}_p$ or there exists some $\sigma' \in \mathcal{S}_p \setminus \sigma$ such that $[\sigma, \sigma') \cap \mathcal{D}_p = \emptyset$, and we are done. Assume now that $\sigma \in \mathcal{S}_r \setminus \mathcal{S}_q$. If $\sigma > \max \mathcal{D}_q$ then $\sigma > \max \mathcal{D}_p$. On the other hand, if there exists some $\sigma' \in \mathcal{S}_q \setminus \sigma$ such that $[\sigma, \sigma') \cap \mathcal{D}_q = \emptyset$ then $[\sigma, \sigma') \cap \mathcal{D}_p \subseteq [\sigma, \sigma') \cap \mathcal{D}_q = \emptyset$. If $\sigma' \in \mathcal{S}_p$ then we are done. If $\sigma' \in \mathcal{S}_q \setminus \mathcal{S}_p$ then we have to consider two cases.

Case 1. If σ' is greater than every $\alpha \in \mathcal{D}_p$ then so is σ, and we are done.

Case 2. If there exists some $\sigma'' \in \mathcal{S}_p \setminus \sigma'$ such that $[\sigma', \sigma'') \cap \mathcal{D}_p = \emptyset$ then $[\sigma, \sigma'') \cap \mathcal{D}_p = \Big([\sigma, \sigma') \cap \mathcal{D}_p\Big) \cup \Big([\sigma', \sigma'') \cap \mathcal{D}_p\Big) = \emptyset$, and we are finished. $\surd_{4.3}$

Clearly P has the weakest element, namely $(\emptyset, \emptyset)$, so P is really a forcing notion. We note:

Proposition 4.4. *The forcing notion P is separative.*

Proof. Let $p \in P$ be an arbitrary condition and $\beta := \max(\mathcal{D}_p \cup \mathcal{S}_p) + 1$. Define $\alpha := \beta + \omega$. Now let $C_\alpha := [\beta, \alpha)$ and $C'_\alpha := (\beta, \alpha)$. By the definition of i_α, $\alpha \in N_{i_\alpha}$. Hence $\alpha < \delta_{N_{i_\alpha}}$. Then also $\beta \in N_{i_\alpha}$, so $C_\alpha \in N_{i_\alpha}$ and $C'_\alpha \in N_{i_\alpha}$.

Define $q := p \cup \{(\alpha, C_\alpha)\}$ and $q' := p \cup \{(\alpha, C'_\alpha)\}$. It is easily seen that $(q, \mathcal{S}_p)$ and $(q', \mathcal{S}_p)$ are both conditions, that they both extend $(p, \mathcal{S}_p)$ and that they are incompatible. $\surd_{4.4}$

Definition 4.5. *For a generic $G \subseteq P$, define $\mathfrak{S} := \bigcup \{p \mid$ there exists some $\mathcal{S}_p$ such that $(p, \mathcal{S}_p) \in G\}$.*

We now claim:

Theorem 4.6. *$\mathfrak{S}$ is a nontrivial $\square_\omega$-sequence in $V[G]$.*

To prove this theorem, we isolate the required density lemmas.

Lemma 4.7. *For each $\alpha < \omega_1$ the following sets are dense in P:*
(1) $\mathfrak{D}_\alpha := \{(q, \mathcal{S}_q) \in P \mid$ there exists some $\beta \in \mathcal{D}_q$ such that $\beta > \alpha$ and $C_\beta \cap \alpha = \emptyset\}$;
(2) (if α is a limit > 0), $\mathfrak{D}'_\alpha := \{(q, \mathcal{S}_q) \in P \mid \alpha \in \mathcal{D}_q\}$;
(3) (if α is a limit $> \omega$), $\mathfrak{E}_\alpha := \{(q, \mathcal{S}_q) \in P \mid$ there exists some $\beta \in \mathcal{D}_q$ such that $\beta > \alpha$ and $\mathrm{Lim}(C_\beta) \cap \alpha \neq \emptyset\}$.

Proof. (1) Consider an arbitrary condition $(p, \mathcal{S}_p) \in P$ such that $(p, \mathcal{S}_p) \notin \mathfrak{D}_\alpha$. Define $\beta' := \max(\mathcal{D}_p \cup \mathcal{S}_p \cup \{\alpha\}) + 1$ and $\beta := \beta' + \omega$. Let $C_\beta := (\beta', \beta)$. Just as in the proof of Proposition 4.4, $(p \cup \{(\beta, C_\beta)\}, \mathcal{S}_p)$ is a condition in $P \cap \mathfrak{D}_\alpha$ which extends $(p, \mathcal{S}_p)$.

(2) Fix a limit ordinal $\alpha > 0$ and let $(p, \mathcal{S}_p) \in P$ be such that $\alpha \notin \mathcal{D}_p$. Consider two cases.

Case 1. If there exists $\beta \in \mathcal{D}_p$ such that $\alpha \in \mathrm{Lim}(C_\beta)$ then let $C_\alpha := C_\beta \cap \alpha$. Define $q := p \cup \{(\alpha, C_\alpha)\}$ and $\mathcal{S}_q := \mathcal{S}_p$. Clauses (1) and (3) of Definition 4.2 for $(q, \mathcal{S}_q)$ are trivially true, while Clauses (2), (4) and (5) follow from respective clauses for $(p, \mathcal{S}_p)$.

Case 2. If $\alpha \notin \bigcup_{\beta \in \mathcal{D}_p} \mathrm{Lim}(C_\beta)$ then let $\alpha' := \max\big((\mathcal{D}_p \cup \mathcal{S}_p \cup \bigcup_{\beta \in \mathcal{D}_p} \mathrm{Lim}(C_\beta)) \cap \alpha\big)$ and let $C_\alpha := (\alpha', \alpha)$. Define $q := p \cup \{(\alpha, C_\alpha)\}$ and $\mathcal{S}_q := \mathcal{S}_p$. All the clauses of Definition 4.2 are either trivial or easy to check.

In both cases $(q, \mathcal{S}_q)$ is a condition from $\mathfrak{D}'_\alpha$ which extends $(p, \mathcal{S}_p)$.

(3) If $(p, \mathcal{S}_p) \in P \setminus \mathfrak{E}_\alpha$, first note that by (2) we can assume that there is some $\delta \in \mathcal{D}_p$ such that $\delta < \alpha$. Define, as above, $\beta' := \max(\mathcal{D}_p \cup \mathcal{S}_p)$ and $\beta := \beta' + \omega$. Let $C_\beta := C_\delta \cup \{\delta\} \cup (\beta', \beta) \in N_{i_\beta}$. Since C_δ is a club in δ, we have that δ is a limit point of C_β. If $\gamma \neq \delta$ is in $\mathcal{D}_p$ and μ is a limit point of both C_β and C_γ, then by the choice of β we must have that $\mu \leq \delta$ and so μ is a limit point of C_δ. By the coherence in p we have that $C_\gamma = C_\delta \cap \gamma = C_\beta \cap \gamma$. It is easy to check the other requisites of being a condition, so $(p \cup \{(\beta, C_\beta)\}, \mathcal{S}_p)$ is a condition in $P \cap \mathfrak{E}_\alpha$ which extends p. $\surd_{4.7}$

Now for the proof of Theorem 4.6:

Proof. Since conditions from G are pairwise compatible, every pair has to agree on the intersection of their domains. Hence, glued together into $\mathfrak{S}$, conditions from G form a function, whose domain we shall denote by $\mathcal{D}_{\mathfrak{S}}$. Obviously, C_α is a club in α for every $\alpha \in \mathcal{D}_{\mathfrak{S}}$.

First let us prove that $\mathcal{D}_{\mathfrak{S}} = \mathrm{Lim}(\omega_1) \cap \omega_1$. Choose a limit ordinal $\alpha < \omega_1$ and let $\mathfrak{D}'_\alpha$ be the set defined in clause (2) of Lemma 4.7. Since G is generic it contains some condition $(q, \mathcal{S}_q) \in \mathfrak{D}'_\alpha$. Hence, there exists some $(q, \mathcal{S}_q) \in G$ such that $\alpha \in \mathcal{D}_q \subset \mathcal{D}_{\mathfrak{S}}$.

Let $\alpha, \beta \in \mathcal{D}_{\mathfrak{S}}$ be such that $\alpha \in \mathrm{Lim}(C_\beta)$. Then there exists some $r \in G$ such that $\alpha, \beta \in \mathcal{D}_r$. Since $\alpha \in \mathrm{Lim}(C_\alpha) \cap \mathrm{Lim}(C_\beta)$, we have, by clause (2) of definition 4.2, $C_\alpha = C_\alpha \cap \alpha = C_\beta \cap \alpha$, hence C_α and C_β are coherent.

The nontriviality of the proposed $\square$-sequence now follows easily. For the first clause choose an arbitrary $\alpha < \omega_1$ and let $\mathfrak{D}_\alpha$ be the set from Clause (1) of

Lemma 4.7. By genericity, there exists some condition $(q, \mathcal{D}_q) \in G$ such that for some $\beta \in \mathcal{D}_q \subset \mathcal{D}_{\mathfrak{S}}$, $\beta > \alpha$, we have $C_\beta \cap \alpha = \emptyset$.

For the second nontriviality condition work with $\mathfrak{E}_\alpha$ instead of $\mathfrak{D}_\alpha$. $\surd_{4.6}$

5. Preservation of cardinals

The set $\mathfrak{S}$ from Theorem 4.6 is certainly a sequence of clubs on $\mathrm{Lim}(\omega_1^N)$. As with many other forcing constructions, the crucial question here is whether ω_1 is preserved as a cardinal in $M[G]$. The fact that higher cardinals are preserved is not important for this construction, and is actually trivial since P—being of cardinality ω_1 (assuming that $2^\omega = \omega_1$ in M)— has the ω_2-chain condition. We shall prove that P is proper.

Theorem 5.1. *Forcing P is proper.*

Proof. Fix a countable elementary submodel $N \in \mathfrak{C} := \{N \prec H_\theta \mid N \text{ countable}, \overline{N} \in N,\ N_i \subseteq N \text{ for all } i < N \cap \omega_1\}$ such that $P \in N$ (recall that $\mathfrak{C}$ is a club in $[H_\theta]^\omega$). Consider an arbitrary $(p, \mathcal{S}_p) \in P \cap N$, and let $\delta := N \cap \omega_1 (= \delta_N)$. Let C_δ be some cofinal ω-sequence in δ such that $C_\delta \in N_{i_\delta}$. Define $\mathcal{S}_q := \mathcal{S}_p \cup \{\delta\}$ and $q := p \cup \{(\delta, C_\delta)\}$. Note that $\max(\mathcal{D}_p \cup \mathcal{S}_p) < \delta$, since p and $\mathcal{S}_p$ are both in N. It is easily seen that $(q, \mathcal{S}_q)$ is a condition and that it extends $(p, \mathcal{S}_p)$. We shall prove that $(q, \mathcal{S}_q)$ is an N-generic extension of $(p, \mathcal{S}_p)$.

Intuitively speaking, placing δ into the set of safeguards will ensure that in any extention of $(p, \mathcal{S}_q)$, the part which will extend to α's outside of N and the part which will extend to α's inside of N will have nothing in common. This is very important because when we restrict that extension to N, the whole information about the part that hung out of N will have been lost. For $\alpha > \delta$ only (some part of) $C_\alpha \cap \delta$ remains (see Clause (4) of Definition 4.2). It will then be crucial to know that the lost part has no influence on what is going on in N. Let us proceed to the details.

Let $D \subseteq P$ be a dense set with $D \in N$ and $(r, \mathcal{S}_r) \geq (q, \mathcal{S}_q)$ an arbitrary extension. Let us first prove that $(r_N, \mathcal{S}_{r_N}) := (r|_\delta, \mathcal{S}_r \cap N)$ is in $P \cap N$. To check that it is in P is trivial. Proving that $(r_N, \mathcal{S}_{r_N}) \in N$ is slightly trickier, because it is not obvious that $C_\alpha \in N$ for $\alpha \in \mathcal{D}_r \cap N$. This is where we make use of the models from $\overline{N}$. Fix an $\alpha \in \mathcal{D}_{r_N}$. Then $i_\alpha \leq \alpha < \delta$. Now, since $C_\alpha \in N_{i_\alpha} \subseteq N$, we are done.

By the density of N and elementarity there is an extension $(s, \mathcal{S}_s)$ of $(r_N, \mathcal{S}_{r_N})$ in $D \cap N$. We would like to see that $(s, \mathcal{S}_s)$ and $(r, \mathcal{S}_r)$ are compatible. Informally, this is true because $\mathcal{D}_s \cup \mathcal{S}_s \subset \delta$, hence, by definition of $(q, \mathcal{S}_q)$, it has no effect on the part of r above δ, while on the other hand $\delta \in \mathcal{S}_r$ prevents the part of $(r, \mathcal{S}_r)$ above δ from having too big an effect on what is going on in $N \cap \omega_1$.

Technically we argue as follows. We will prove that $(t, \mathcal{S}_t) := (r \cup s, \mathcal{S}_r \cup \mathcal{S}_s)$ is a condition which extends both $(r, \mathcal{S}_r)$ and $(s, \mathcal{S}_s)$. To see that $(t, \mathcal{S}_t)$ is a condition, note that clauses (1), (3) and (5) of Definition 4.2 are trivially true for t. For Clause (2) we may restrict our attention to $\alpha \in \mathcal{D}_t \setminus \mathcal{D}_s = \mathcal{D}_r \setminus \mathcal{D}_s$ and $\beta \in \mathcal{D}_t \setminus \mathcal{D}_r = \mathcal{D}_s \setminus \mathcal{D}_r$. If $\mu \in \mathrm{Lim}(C_\alpha) \cap \mathrm{Lim}(C_\beta)$ then $\mu < \delta$ as $\beta \in D_s$. Notice that $\alpha > \delta$ as C_δ has no limit points below δ. By Clause (4) for r applied to δ in place of σ and to α, there exists $\beta' \in \mathcal{D}_r \cap \delta$ such that $\mu \in \mathrm{Lim}(C_{\beta'})$. Then, by Clause (2) for r, we have $C_\alpha \cap \mu = C_{\beta'} \cap \mu$. But by Clause (2) for s we have $C_{\beta'} \cap \mu = C_\beta \cap \mu$.

For Clause (4) we only have to consider some $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $\sigma \in \mathcal{S}_s \setminus \mathcal{S}_r$. Notice that $\sigma < \delta$. If $\alpha = \delta$ then $\mathrm{Lim}(C_\alpha \cap \sigma) = \emptyset$ and there is nothing to prove. Hence, we can assume that $\alpha > \delta$. We therefore have $\mathrm{Lim}(C_\alpha \cap \sigma) \subseteq \mathrm{Lim}(C_\alpha \cap \delta)$. By Clause (4) for r applied to δ and α we have $\mathrm{Lim}(C_\alpha \cap \delta) \subseteq \bigcup_{\beta \in \mathcal{D}_r \cap \delta} \mathrm{Lim}(C_\beta) = \bigcup_{\beta \in \mathcal{D}_{r_N}} \mathrm{Lim}(C_\beta)$. If $\sigma > \max \mathcal{D}_{r_N}$ then the latter is equal to $\bigcup_{\beta \in \mathcal{D}_{r_N} \cap \sigma} \mathrm{Lim}(C_\beta) \subseteq \bigcup_{\beta \in \mathcal{D}_t \cap \sigma} \mathrm{Lim}(C_\beta)$ and we are done. On the other hand, if $\sigma < \max \mathcal{D}_{r_N}$ then, by Definition 4.2(ii) and the fact that $s \geq r_N$, there exists some $\sigma' \in \mathcal{S}_{r_N}$, $\sigma < \sigma'$, such that $[\sigma, \sigma') \cap \mathcal{D}_{r_N} = \emptyset$. By Clause (4) for r we have $\mathrm{Lim}(C_\alpha \cap \sigma) \subseteq \mathrm{Lim}(C_\alpha \cap \sigma') \subseteq \bigcup_{\beta \in \mathcal{D}_r \cap \sigma'} \mathrm{Lim}(C_\beta) = \bigcup_{\beta \in \mathcal{D}_r \cap \sigma} \mathrm{Lim}(C_\beta) \subseteq \bigcup_{\beta \in \mathcal{D}_t \cap \sigma} \mathrm{Lim}(C_\beta)$. The case $\sigma = \max \mathcal{D}_{r_N}$ is prohibited by Definition 4.2 because $(s, \mathcal{S}_s) \geq (r_N, \mathcal{S}_{r_N})$.

To prove that $(t, \mathcal{S}_t)$ is an extension of $(r, \mathcal{S}_r)$, consider an arbitrary $\sigma \in \mathcal{S}_t \setminus \mathcal{S}_r = \mathcal{S}_s \setminus \mathcal{S}_{r_N}$. If $\sigma > \max \mathcal{D}_{r_N}$ then $[\sigma, \delta) \cap \mathcal{D}_r = [\sigma, \delta) \cap \mathcal{D}_{r_N} = \emptyset$. However, if $\sigma < \max \mathcal{D}_{r_N}$ then there exists some $\sigma' \in \mathcal{S}_{r_N}$, $\sigma < \sigma'$, such that $[\sigma, \sigma') \cap \mathcal{D}_{r_N} = \emptyset$. Since $\sigma' \in \mathcal{S}_r$, $\sigma' < \delta$ and $[\sigma, \sigma') \cap \mathcal{D}_r = [\sigma, \sigma') \cap \mathcal{D}_{r_N} = \emptyset$, we are done.

The fact that $(t, \mathcal{S}_t)$ is an extension of $(s, \mathcal{S}_s)$ is trivial because every $\sigma \in \mathcal{S}_t \setminus \mathcal{S}_s$ is obviously greater than every $\alpha \in \mathcal{D}_s$. $\surd_{5.1}$

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