FORCING $\square_{\omega_1}$ WITH FINITE CONDITIONS

GREGOR DOLINAR AND MIRNA DŽAMONJA

Abstract. We give a construction of the square principle $\square_{\omega_1}$ by means of forcing with finite conditions.

1. Introduction

The square principle on a cardinal κ states that there is a sequence $\langle C_\alpha \rangle_\alpha$ indexed by the limit ordinals in $[\kappa, \kappa^+)$ such that each C_α is a club subset of α of order type $\leq \kappa$ and the sequence is coherent in the sense that if β is a limit point of α then $C_\beta = C_\alpha \cap \beta$. This principle is a feature of the constructible universe $\mathbf{L}$ which was discovered by Jensen and used by him to show the existence of an ω_2-Souslin tree in $\mathbf{L}$ [7]. The related principle $\diamondsuit$, which was used to construct an ω_1-Souslin tree in $\mathbf{L}$ by Jensen, may be added or destroyed by forcing as wished (see [10] for examples and discussion). Also, by recent work of Shelah ([12]), at $\kappa \geq \omega_2$ which are successor cardinals of the form $\kappa = \theta^+ = 2^\theta$, $\diamondsuit_\kappa$ simply holds, i.e. it is equivalent to the cardinal arithmetic assumption $\theta^+ = 2^\theta$. However, $\square$ is connected to large cardinals. For example, by a well known result of Solovay [13], square cannot hold above a supercompact cardinal, and on smaller cardinals, it cannot hold in the presence of forcing axioms, e.g. Todorčević [14] proved that PFA implies that for all $\kappa \geq \omega_2$, $\square_\kappa$ fails. Therefore $\square$ can be seen as a reflection principle inimical to large cardinals, and in fact by varying the definition of square by allowing a cardinal parameter which measures how many guesses to C_α we are allowed at each α, we obtain a hierarchy of principles of decreasing strength which can be used to test consistency strength of various principles (see more on this in [3]). In the light of these facts it is natural that the question of how to add or destroy a square principle by forcing has been a central theme. See [3] for a description of some of the many known results including versions of an older result of Jensen and Magidor in which a square sequence is added by forcing.

One way to add a square, due to Jensen, is to force by initial segments along a closed unbounded subset of the domain, and to use the existence of the "top" point

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in the domain of a forcing condition to show that the forcing is strategically closed. Note that the principle $\square_\omega$ is trivially true, by taking C_α to be any club of α of order type ω, so the first non-trivial instance of square is $\square_{\omega_1}$. The method of forcing by initial segments means that to get $\square_{\omega_1}$ we need to force with conditions whose domain has size ω_1. The referee has kindly informed us that in an unpublished work Foreman and Magidor added square by a countably closed forcing using countable conditions. A condition p in their forcing prescribes C_α for α of countable cofinality in $\mathrm{dom}(p)$, and for $\alpha \in \mathrm{dom}(p)$ of uncountable cofinality, p prescribes an initial segment of C_α which goes past $\sup(\mathrm{dom}(p) \cap \alpha)$. Assuming CH this poset has the ω_2-c.c. In this work we have been interested in another way of adding a square, using conditions whose domain is a finite set. The interest in doing this stems from a need to understand how one can control a one cardinal gap in forcing notions, which is a subject that has been of interest for various combinatorial issues for a long time. A glaring example of the need to develop this subject is the combinatorics of the structure $(\omega_1^{\omega_1}, \leq_{\mathrm{Fin}})$, which in contrast with the vast body of knowledge about $(\omega^\omega, \leq_{\mathrm{Fin}})$, remains a mysterious object. An important development on the subject of $(\omega_1^{\omega_1}, \leq_{\mathrm{Fin}})$ is Koszmider's paper [9] in which he shows that it is consistent to have an increasing chain of length ω_2 in this structure. Koszmider's paper also gives an overview of the difficulties that there are in forcing one gap results.

Koszmider's method is to force with conditions where a morass is used as a side condition. Our method is more directly connected to a different approach, which was used to force a club on ω_2 using finite conditions. This was done in two different but similar ways by Friedman in [5] and Mitchell in [11]. Both approaches are built upon a version of adding a club subset of ω_1 using finite conditions, as discovered by Baumgartner [2] and modified by Abraham in [1]. The main idea in Baumgartner's approach is that to force a club in ω_1 and avoid problems at the limit stages, one needs to specify by each condition not only what will go in the club, but also whole intervals that need to stay out of it. At ω_2 one can do the same, but now one needs to add side conditions in the form of coherent systems of models in order to make sure that cardinals are preserved, as was first done by Todorčević in [15]. This already is technically rather involved. What we have done is add to this the coherent partial square sequence. Namely, we actually force a square indexed by a club set. The existence of such a square implies the existence of an actual square sequence. This club set is like the one added by Friedman and Mitchell. The actual forcing notion needs to take into account the coherence of the square sequence, and this is reflected in the complexity of the coherence conditions between the models which form part of the forcing conditions. An advantage of this type of approach over the morass-based approach is that it requires less from the ground model—for example Friedman's forcing only needs a weakening of CH in the ground model. We use the full CH together with $2^{\omega_1} = \omega_2$. The main difficulties of both approaches of course are the same, and they stem from the fact that combinatorics at ω_2 is much less prone to independence than the combinatorics at ω_1, as exemplified by the above mentioned result of Shelah on $\diamondsuit$ ([12]). It is both in developing combinatorics and fine forcing techniques that we can better understand the truth about ω_2. An interesting unified approach to adding objects to ω_2 is being developed by Neeman as well as Veličković and Venturi, in works in progress.

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2. Preliminaries

Most of the notation is standard. The relation $A \subset B$ means that A is either a proper subset of B or equal to B. $|X|$ is the cardinality of the set X. For a set of ordinals X, a limit point of X is an ordinal α such that $\alpha = \sup(Y)$ for some $Y \subset X$ or, equivalently, if $\alpha = \sup(X \cap \alpha)$. $\mathrm{Lim}(X)$ is the set of limit points of X. For a function f, $\mathcal{D}_f$ denotes the domain of f, and $f \restriction A$ denotes the restriction of f to the set $A \cap \mathcal{D}_f$. If α and β are ordinals then the interval (α, β) denotes the set $\{\mu \mid \mu \text{ is an ordinal}, \alpha < \mu < \beta\} = \beta \setminus (\alpha + 1)$. Closed and half open intervals are defined similarly. $[A]^\kappa$ is the set of all subsets of A of cardinality κ. The set $[A]^{\leq\kappa}$ is defined analogously.

For a regular cardinal θ, H_θ is the set of all sets x with hereditary cardinality less than θ (i.e. the transitive closure of x has cardinality less than θ). For $\theta > \omega_2$ we consider H_θ to be a model with the standard relation $\in$ and a fixed well-ordering $\leq^*$ and we write H_θ for the structure $(H_\theta, \in, \leq^*)$. We will primarily work with H_{ω_2} which we view as a model with $\in$ and $\leq^* \restriction H_{\omega_2}$. A cardinal θ is said to be *large enough* if every set in consideration is an element of H_θ.

Definition 2.1. *Suppose κ is a regular cardinal. A set $C \subset \kappa$ is called a* closed unbounded *set or a* club *in κ if:*

(1) for every $\lambda < \kappa$ and an increasing sequence $\langle \alpha_i \mid i < \lambda \rangle$ of elements from C, we have that $\bigcup_{i<\lambda} \alpha_i \in C$ (closed)*;*

(2) for every $\alpha < \kappa$ there exists some $\beta \in C$ such that $\beta > \alpha$ (unbounded)*.*

The assumption that κ is a regular cardinal can be replaced by a singular cardinal or even an ordinal, which to avoid trivialities we usually take to be of uncountable cofinality. In that case, λ from clause (1) has to be below $\mathrm{cf}(\kappa)$. In fact, clause (1) can be replaced by an equivalent notion, that $\mathrm{Lim}(C) \cap \kappa \subset C$.

Definition 2.2. *Suppose that κ is a regular cardinal. A* square sequence on κ *is a sequence of the form $\langle C_\alpha \mid \alpha$ is a limit ordinal in $\kappa^+\rangle$ such that:*

(1) C_α is a club in α for every α;

(2) if $\alpha \in \mathrm{Lim}(C_\beta)$ then $C_\alpha = C_\beta \cap \alpha$ (coherence)*;*

(3) if $\mathrm{cf}(\alpha) < \kappa$ then $|C_\alpha| < \kappa$ (nontriviality)*.*

$\square_\kappa$ (*square kappa*) is the statement that there is a square sequence on κ. In the case $\kappa = \omega_1$, the nontriviality clause simply stipulates that if $\mathrm{cf}(\alpha) = \omega$ then $|C_\alpha| = \omega$.

3. Background on elementary submodels

A model M is an *elementary submodel* of a model N, $M \prec N$, if for every formula φ with parameters $a_1, \ldots, a_n \in M$, φ is true in M if and only if it is true in N. If M is a countable elementary submodel of H_θ for $\theta \geq \omega_1$ then $M \cap \omega_1$ is an ordinal denoted by δ_M. Also, if $|x| \leq \omega$ and $x \in M$ then $x \subset M$.

We begin by listing a few lemmas about elementary submodels which will be useful later. We add proofs for completeness. A useful tool when dealing with elementary submodels is the Tarski-Vaught test [8]:

Theorem 3.1 (Tarski-Vaught test)**.** *Let M be a submodel of N. Then M is an elementary submodel of N if and only if for every formula $\phi(x, a_1, \dots, a_n)$ and $a_1 \dots, a_n \in M$, if $N \models \exists x \phi(x, a_1, \dots, a_n)$ then there exists $b \in M$ such that $N \models \phi(b, a_1, \dots, a_n)$.*

Lemma 3.2. *Suppose $N \prec H_\theta$ for some large enough θ. Then $N \cap H_{\omega_2} \prec H_{\omega_2}$.*

Proof. Let $a_1, \dots, a_n \in N \cap H_{\omega_2}$ and suppose that $H_{\omega_2} \models \psi(a_1, \dots, a_n)$ where ψ is the formula $\exists x \phi(x, a_1, \dots, a_n)$. Then $\psi^{H_{\omega_2}}$—the relativization of ψ to H_{ω_2}—is true. Formula $\psi^{H_{\omega_2}}$ is equivalent to the formula ψ^* obtained by replacing every occurrence of $\exists y \in H_{\omega_2}\ \chi(y, \dots)$ with $\exists y(\chi(y, \dots) \wedge |\operatorname{tr}\operatorname{cl}(y)| \leq \omega_1)$, and similarly for the universal quantifier. We get ϕ^* from ϕ in the same way. Now, $H_\theta \models \psi^*(a_1, \dots, a_n)$, or in other words, $H_\theta \models \exists x(\phi^*(x, a_1, \dots, a_n) \wedge |\operatorname{tr}\operatorname{cl}(x)| \leq \omega_1)$.

Since $\omega_1 \in N$, by Tarski-Vaught test there exists some $b \in N$ such that $H_\theta \models \phi^*(b, a_1, \dots, a_n) \wedge |\operatorname{tr}\operatorname{cl}(b)| \leq \omega_1$. Hence, there exists $b \in N \cap H_{\omega_2}$ such that $H_\theta \models \phi^{H_{\omega_2}}(b, a_1, \dots, a_n)$, and as a consequence, $H_{\omega_2} \models \phi(b, a_1 \dots, a_n)$, which by Tarski-Vaught test means that $N \cap H_{\omega_2} \prec H_{\omega_2}$. $\surd$

Lemma 3.3. *Suppose $N, M \prec H_{\omega_2}$. Then $N \cap M \prec H_{\omega_2}$.*

Proof. Let $a_1, \dots, a_n \in N \cap M$ and suppose that $H_{\omega_2} \models \exists x \phi(x, a_1, \dots, a_n)$. Let $\psi(x, a_1, \dots, , a_n)$ be the formula $\phi(x, a_1, \dots, a_n) \wedge \forall y(\phi(y, a_1, \dots, a_n) \to x \leq^* y)$. Then $H_{\omega_2} \models \exists x \psi(x, a_1, \dots, a_n)$. By the Tarski-Vaught test there exist $x_1 \in M$ and $x_2 \in N$ such that $H_{\omega_2} \models \psi(x_1, a_1, \dots, a_n)$ and $H_{\omega_2} \models \psi(x_2, a_1, \dots, a_n)$. But then $x_1 = x_2 =: x^* \in M \cap N$, and $H_{\omega_2} \models \phi(x^*, a_1, \dots, a_n)$. By the Tarski-Vaught test, $M \cap N \prec H_{\omega_2}$. $\surd$

Lemma 3.4. *If $M \prec H_\kappa$ for some $\kappa > \omega_1$, and $\sup(M \cap \alpha) < \alpha$ for some ordinal $\alpha \in M$, then $\operatorname{cf}(\alpha) > \omega$.*

Proof. If $\operatorname{cf}(\alpha) = \omega$ then there is a cofinal function $f : \omega \to \alpha$ in M, hence $\sup(M \cap \alpha) = \alpha$, a contradiction. $\surd$

Lemma 3.5. *Let $M, N \prec H_\kappa$ be countable for some $\kappa > \omega_1$ and suppose that $M \in N$. If $\alpha \notin N$ then $\sup(M \cap \alpha) \in N$ and $\sup(M \cap \alpha) < \sup(N \cap \alpha)$.*

Proof. If $\alpha \geq \sup(N \cap \kappa)$ then $\sup(M \cap \alpha) = \sup(M \cap \kappa) < \sup(N \cap \kappa) = \sup(N \cap \alpha)$. Suppose now that $\alpha < \sup(N \cap \kappa)$ and let $\beta := \sup(M \cap \alpha)$ and $\beta' := \min((N \cap \kappa) \setminus \alpha) \in N$. Since $M \subset N$, $\beta = \sup(M \cap \beta')$. Hence, by elementarity, $\beta \in N$, and therefore $\beta < \sup(N \cap \alpha)$. $\surd$

The standard reference for basic set-theoretic notions and facts is [6]. Additional source for results on elementary models in a very concise form is [4], as well as [8].

In our application of elementary submodels we will basically only be interested in the ordinals that lie inside them. To simplify the notation we will write $\mathscr{M}$ for a model and M for its set of ordinals $\mathscr{M} \cap Ord$. In addition, we shall be making the *assumption* that $2^{\omega_1} = \omega_2$. Therefore $|H_{\omega_2}| = \omega_2$ and we may assume that the well ordering $\leq^* \restriction H_{\omega_2}$ actually well orders H_{ω_2} in order type ω_2. As the referee points out, this is useful because of the following:

Lemma 3.6. *Suppose that $\leq^*$ is a well ordering of H_{ω_2} in order type ω_2 and $\mathscr{M} \prec (H_{\omega_2}, \in, \leq^*)$. Then $\mathscr{M}$ is uniquely determined by $M = \mathscr{M} \cap Ord$.*

Proof. For $\alpha < \omega_2$ let x_α be the object in H_{ω_2} enumerated at place α. Then $x_\alpha \in \mathscr{M}$ iff $\alpha \in M$. $\surd$

This justifies the notation $\mathscr{M}[M]$ for the unique model $\mathscr{M} \prec (H_{\omega_2}, \in, \leq^*)$, if there is such a model for a given $M \subset \omega_2$. If $\mathscr{M}[M]$ is well defined we shall say that M is the *trace of a model*.

4. Forcing a square

Let V be some countable transitive model of (a sufficiently large finite fragment of) ZFC together with CH and the assumption that the well ordering $\leq^* \restriction H_{\omega_2}$ well orders H_{ω_2} in the order type ω_2 (so in particular $2^{\omega_1} = \omega_2$ holds in V). Throughout the rest of the paper everything is carried out inside V.

Since we want to force the existence of a square sequence, the working part of forcing notion P will consist of finite partial square sequences. We will add *safeguards* which will help us separate clubs from a condition q and clubs from a restriction $p \leq q$. This will be instrumental in the proof of properness.

It should be noted once again that we do not have to build a square sequence on the whole $\mathrm{Lim}(\omega_2)$. Instead, it is enough for the domain of the built sequence to be a club in ω_2, because we can always extend a square sequence from a club to the full $\mathrm{Lim}(\omega_2)$ (see Lemma 5.13). This is the reason why we add intervals as a part of conditions. These intervals will serve as gaps in what will ultimately be the desired club in $\mathrm{Lim}(\omega_2)$. This way of forcing a club was introduced by Baumgartner in [2] in the context of ω_1.

Before we are ready to present the definition of forcing we have to define a few auxiliary notions. For $\alpha < \omega_2$, $\mathrm{cf}(\alpha) = \omega_1$, let E_α denote some fixed club in α of order type ω_1, and let $\mathscr{E} := \langle E_\alpha \mid \alpha < \omega_2 \rangle$. Define $\mathfrak{M}_0 := \{\mathscr{M} \prec H_{\omega_2} \mid \mathscr{M}$ is countable and $\mathscr{E} \in \mathscr{M}\}$. The set $\mathfrak{M}_0$ will act as a pool of possible side conditions.

For a large enough cardinal θ let $\mathfrak{M}_1 := \{\mathscr{M} \prec H_\theta \mid \mathscr{M}$ is countable, $\mathscr{E} \in \mathscr{M}\}$. Then $\mathfrak{M}_1$ is a club set in $[H_\theta]^\omega$. Also, if $\mathscr{N} \in \mathfrak{M}_1$ and $\alpha \in \mathscr{N}$ has cofinality ω_1, then, by elementarity, $E_\alpha \in \mathscr{N}$. Also note that $\mathscr{N} \cap H_{\omega_2} \in \mathfrak{M}_0$, by Lemma 3.2.

Definition 4.1. *Suppose that $\mathscr{M}_1, \mathscr{M}_2 \prec H_{\omega_2}$ are countable and let $\delta := \sup(M_1 \cap M_2)$. Then:*
(1) the set $\{\min(M_1 \setminus \lambda) \mid \lambda \in M_2,\ \delta < \lambda < \sup(M_1)\} \cup \{\min(M_1 \setminus \delta)\}$ is called the set of M_1-fences for M_2*;*
(2) we say that M_1 and M_2 are compatible *if the following two clauses hold as stated and with M_1 and M_2 switched:*
  (a) either $\delta \in M_1$ and $M_1 \cap M_2 \in \mathscr{M}_1$, or $\delta \notin M_1$ and $M_1 \cap M_2 = M_1 \cap \delta$, and
  (b) the set of M_1-fences for M_2 is finite.

The most trivial case of two compatible models is if $\mathscr{M}_1 \in \mathscr{M}_2$. Then $\delta = \sup(M_1) \in M_2$, $M_1 \cap M_2 = M_1 \in \mathscr{M}_2$, and $M_1 \cap M_2 = M_1 \cap \delta = M_1$. The set of M_1-fences for M_2 is the empty set and the set of M_2-fences for M_1 is the set $\{\delta\}$.

We are particularly interested in the following consequence of compatibility and the assumption $2^{\omega_1} = \omega_2$.

Lemma 4.2. *Suppose that $\mathscr{M}_1$ and $\mathscr{M}_1$ are models compatible in the sense of Definition 4.1 and let δ be as defined there. Then if $\delta \notin M_1$, then $[\delta]^{\leq\omega} \cap \mathscr{M}_1 \subset \mathscr{M}_2$.*

Proof. Assume $\delta \notin M_1$ and consider the two possible cases:
 (a) $\delta \in M_2$. In this case $M_1 \cap M_2 = M_1 \cap \delta \subsetneq M_2 \cap \delta$,
 (b) $\delta \notin M_1 \cup M_2$. In this case $M_1 \cap M_2 = M_1 \cap \delta = M_2 \cap \delta$.

In any case we have $M_1 \cap \delta \subset M_2 \cap \delta$. Let $x \in \mathscr{M}_1$ be a countable subset of δ, so that if $\gamma := \sup(x)$ then $\gamma \in M_1$ and hence $\gamma < \delta$ and $\gamma \in M_2$. Let η be the least ordinal such that every countable subset of γ appears before stage η in the well-ordering $\leq^* \restriction H_{\omega_2}$. Then η is definable from γ so that $\eta \in M_1 \cap M_2$ and hence $\eta < \delta$. Let x appear at stage ζ in the well-ordering; then $\zeta \in M_1$ because $x \in \mathscr{M}_1$, so $\zeta \in M_1 \cap \eta \subset M_1 \cap \delta \subset M_2 \cap \delta$ and hence $x \in \mathscr{M}_2$. $\surd$

We thank the referee for noticing Lemma 4.2 and providing us with its proof. In its absence, the previous version of this paper used the conclusion of Lemma 4.2 as part of the definition of compatibility, in place of Mitchell's condition in (b) of that definition. Together with the following simple lemma, Lemma 4.2 shows that under our assumptions the two definitions of compatibility are actually equivalent.

Lemma 4.3. *With the notation of Definition 4.1, if* $[\delta]^{\leq\omega} \cap \mathscr{M}_1 \subset \mathscr{M}_2$ *then* $M_1 \cap M_2 = M_1 \cap \delta$.

Proof. Consider $\alpha \in M_1 \cap \delta$. Then $\{\alpha\} \in [\delta]^{\leq\omega} \cap \mathscr{M}_1$, hence $\{\alpha\} \in \mathscr{M}_2$ and $\alpha = \max(\{\alpha\}) \in M_2$. $\surd$

We include another comment by the referee, which sheds more light on the advantages of working with compatible models.

Lemma 4.4. *Suppose that* $\mathscr{M}_1$ *and* $\mathscr{M}_2$ *are compatible models in the sense of Definition 4.1 and let* δ *be as defined there. Then* $M_1 \cap \delta = M_2 \cap \delta$ *iff* $M_1 \cap \omega_1 = M_2 \cap \omega_1$, *and* $M_1 \cap \delta \subsetneq M_2 \cap \delta$ *iff* $M_1 \cap \omega_1 < M_2 \cap \omega_1$.

Proof. Let $\gamma \in M_1$, so $\gamma < \omega_2$. If f is the $\leq^*$-least injection from γ to ω_1 then $M_1 \cap \gamma = f^{-1}[M_1 \cap \omega_1]$, and so if $M_2 \cap \omega_1 \geq M_1 \cap \omega_1$ then also $M_2 \cap \gamma \supset M_1 \cap \gamma$. $\surd$

Lemma 4.5. *Suppose that* $\mathscr{M}_1$ *and* $\mathscr{M}_2$ *are compatible models in the sense of Definition 4.1 and let* δ *be as defined there. Further suppose that for some* $\gamma > \delta$ *we have that* $\delta < \sup(M_1 \cap \gamma) = \alpha \notin M_1$. *Then* $\sup(M_2 \cap \alpha) < \alpha$.

Proof. Suppose otherwise. Since $\alpha \notin M_1$, certainly α is a limit ordinal. Since $\sup(M_2 \cap \alpha) = \alpha$, we can find $\beta_0 \in (\delta, \alpha)$ with $\beta_0 \in M_2$. Hence $\alpha_0 := \min(M_1 \setminus \beta_0) \in M_1 \cap \alpha$, and so $\beta_1 := \min(M_2 \setminus \alpha_0) \in M_2 \cap \alpha$, etc., continuing for ω steps. But each β_n is in the M_2-fence for M_1, and there are only finitely many ordinals in that fence, by compatibility, a contradiction. $\surd$

We are now ready to define the forcing notion we shall use. To motivate it, let us recall Baumgartner's idea of adding a club of ω_1 using finite conditions. Each condition p gives finitely many elements $\mathcal{I}_p$ of the future club. However, since we know that the added set is a club, we know that some points should be forced to be in implicitly, that is the ones that are limit points of the explicitly added ones. As we only have finite conditions at our disposal, our control of this requirement must come not from what we put in but from what we leave out. So each condition specifies also some points to leave out, and once we have decided to leave a point out of the future club, we have to make sure that it does not get in accidentally. This is achieved by having the condition specify a half-open interval of points below the given one, which will also be excluded. Hence each condition comes with finitely many intervals $\mathcal{O}_p$ of that form. This works well at ω_1 and it preserves cardinals, but at ω_2 it would collapse cardinals if we do not do anything else to prevent that. That is where the models as side conditions come in, used by both Friedman and Mitchell. Hence each condition has finitely many models ($\mathcal{M}_p$) and it is their interaction with the club added that is used to preserve ω_1. Here, a Friedman-Mitchell club is added as the domain of the square sequence (using $\mathcal{D}_p$), so we have to have similar concerns about preserving ω_1.

The interaction between the models and the club is achieved through the notion of safeguards $\mathcal{S}_p$ and fences, as in both Friedman's and Mitchell's work (although our notation and presentation corresponds more to Mitchell's). Clause (6b) below tells us that a gap in a model M has to be closed from above by a safeguard if there is something (i.e. an ordinal $\alpha \in \mathcal{D}_p$) inside that gap. This safeguard is an echo of α resonating in M, warning everybody in M to stay away from that gap. Fences from clause (9) serve exactly the same purpose.

Definition 4.6. *The forcing notion P is the set of conditions of the form $p := (\mathcal{F}_p,\ \mathcal{S}_p,\ \mathcal{O}_p,\ \mathcal{M}_p)$, where*

(1) $\mathcal{F}_p : \mathrm{Lim}(\omega_2) \to \mathcal{P}(\omega_2)$, $|\mathcal{F}_p| < \omega$ and for all $\alpha \in \mathcal{D}_p := \mathrm{dom}(\mathcal{F}_p)$, $\mathcal{F}_p(\alpha)$ is a club $C_\alpha \subset \alpha$ whose order type is $< \omega_1$ if $\mathrm{cf}(\alpha) = \omega$ and which satisfies $C_\alpha \in \{E_\alpha \setminus \beta \mid \beta \in \mathcal{D}_p \cap \alpha\}$ if $\mathrm{cf}(\alpha) = \omega_1$;

(2) $\mathcal{S}_p \subset \mathcal{D}_p$ and $\alpha \in \mathcal{S}_p$ for every $\alpha \in \mathcal{D}_p$ with $\mathrm{cf}(\alpha) = \omega_1$;

(3) $\mathcal{M}_p$ is a finite set of countable traces of models from $\mathfrak{M}_0$ and $\sup(M) \in \mathcal{S}_p$ for every $M \in \mathcal{M}_p$;

(4) for every $\alpha \neq \beta \in \mathcal{D}_p$, if $\mu \in \mathrm{Lim}(C_\alpha) \cap \mathrm{Lim}(C_\beta)$ then $C_\alpha \cap \mu = C_\beta \cap \mu$;

(5) if $\alpha \in \mathcal{D}_p$ and $\sigma \in \mathcal{S}_p \cap \alpha$, then $C_\alpha \cap \sigma$ is a finite set;

(6) for all $\alpha \in \mathcal{D}_p$ and $M \in \mathcal{M}_p$:

(a) if $\alpha \in M$ then $C_\alpha \in \mathscr{M}[M]$,

(b) if $\alpha \notin M$ is such that $\alpha < \sup(M)$, or if $\alpha \in M$ is such that $\sup(M \cap \alpha) < \alpha$, then $\min(M \setminus \alpha) \in \mathcal{S}_p$ and $\sup(M \cap \alpha) \in \mathcal{D}_p$[1]*,*

(c) if $\alpha \notin M$, $\sup(M \cap \alpha) < \alpha < \sup(M)$ and there is no $\beta \in \mathcal{D}_p \setminus (\alpha + 1)$, such that $\alpha \in \mathrm{Lim}(C_\beta)$, then $C_\alpha \cap \sup(M \cap \alpha)$ is a finite set,

(d) if $\alpha \notin M$, $\sup(M \cap \alpha) = \alpha$ and there is no $\beta \in \mathcal{D}_p \setminus (\alpha + 1)$, such that $\alpha \in \mathrm{Lim}(C_\beta)$, then C_α is some cofinal sequence in α of length ω;

(7) $\mathcal{O}_p$ is a finite set of half open nonempty intervals $(\beta', \beta] \subset \omega_2$ such that $\mathcal{D}_p \cap \bigcup \mathcal{O}_p = \emptyset$;

(8) if $(\beta', \beta] \in \mathcal{O}_p$ and $M \in \mathcal{M}_p$ then either $(\beta', \beta] \in \mathscr{M}$ or $(\beta', \beta] \cap \mathscr{M} = \emptyset$;

(9) if $M_1, M_2 \in \mathcal{M}_p$ then they are compatible, and the M_1-fence for M_2 is a subset of $\mathcal{S}_p$.

For $p, q \in P$ define $p \leq q \overset{def}{\iff} \mathcal{F}_p \subset \mathcal{F}_q,\ \mathcal{S}_p \subset \mathcal{S}_q,\ \mathcal{O}_p \subset \mathcal{O}_q,\ \mathcal{M}_p \subset \mathcal{M}_q.$

Notice that in clause (8), the interval $(\beta', \beta]$ is an element of the model $\mathscr{M}$ if and only if both β' and β are in M.

We will occasionally have to work with quadruples $p = (\mathcal{F}_p,\ \mathcal{S}_p,\ \mathcal{O}_p,\ \mathcal{M}_p)$ which do not satisfy all of the clauses of Definition 4.6.

Definition 4.7. *Let $p = (\mathcal{F}_p,\ \mathcal{S}_p,\ \mathcal{O}_p,\ \mathcal{M}_p)$ be a quadruple with the sets $\mathcal{F}_p$, $\mathcal{S}_p$, $\mathcal{O}_p$ and $\mathcal{M}_p$ defined as in Definition 4.6.*

(1) Define sets

$$A_p := \{(N, \gamma) \mid N \in \mathcal{M}_p \text{ and } \gamma \in \mathcal{D}_p \cap N \text{ such that } \sup(N \cap \gamma) \notin \mathcal{D}_p\},$$

$$B_p := \{\alpha \in \mathcal{D}_p \mid \mathrm{cf}(\alpha) = \omega \text{ and there exists } (N, \gamma) \in A_p \text{ such that } \alpha \notin N,$$
$$\sup(N \cap \alpha) < \alpha \text{ and } \gamma = \min(N \setminus \alpha)\},$$

[1]Note that if $\alpha \in M$ then $\sup(M \cap \alpha) < \alpha$ iff $\mathrm{cf}(\alpha) = \omega_1$.

$$J_p := \{\delta' \in \mathcal{D}_p \setminus \mathcal{S}_p \mid \textit{there exist } M, M' \in \mathcal{M}_p \textit{ and } \delta \in \mathcal{S}_p \cap M \textit{ such that } \delta' = \sup(M \cap M') \in M \textit{ and } \delta' < \delta < \min(M' \setminus \delta')\}.$$

(2) We call p a semi-condition *if it satisfies all of the clauses of Definition 4.6 except clauses (6b) and (9), and it violates clause (6b) only in such a way that $A_p \neq \emptyset$ or $B_p \neq \emptyset$, while violating clause (9) only in such a way that $J_p \neq \emptyset$.*

(3) Quadruple p is a precondition *if it is a semi-condition satisfying clause (9).*

Remark 4.8. (1) Instead of clause (6b) a precondition (or a semi-condition) satisfies the following weaker version:

(6b*) if $\alpha \notin M$ is such that $\alpha < \sup(M)$ and $\mathrm{cf}(\alpha) = \omega_1$, then $\min(M \setminus \alpha) \in \mathcal{S}_p$ and $\sup(M \cap \alpha) \in \mathcal{D}_p$.

(2) $\delta' \in J_p$ means that δ' should be in the M-fence for M' (hence in $\mathcal{S}_p$) but is not, which is the reason why clause (9) fails.

Lemma 4.9. *$(P, \leq)$ is a non-trivial forcing notion.*

Proof. Transitivity is trivial. The minimal element is $(\emptyset, \emptyset, \emptyset, \emptyset)$. For non-triviality, consider an arbitrary condition $p \in P$: we will find two incompatible extensions of p. Let $\alpha := \sup(\mathcal{D}_p \cup \bigcup \mathcal{O}_p \cup \bigcup \mathcal{M}_p)$, and $\beta := \alpha + \omega < \omega_2$. Define $C_\beta := [\alpha, \beta)$ and $C'_\beta := (\alpha, \beta)$. It is easy to check that $q := (\mathcal{F}_p \cup \{(\beta, C_\beta)\}, \mathcal{S}_p, \mathcal{O}_p, \mathcal{M}_p)$ and $q' := (\mathcal{F}_p \cup \{(\beta, C'_\beta)\}, \mathcal{S}_p, \mathcal{O}_p, \mathcal{M}_p)$ are both conditions extending p, and that they are incompatible. Notice, that since $\mathrm{cf}(\beta) = \omega$, C_β and C'_β need not interact with $\mathscr{E}$. $\surd$

We now prove several lemmas that show us a little bit more about the structure of the conditions in P, and will be helpful in further proofs. Most notably, they will shed some light on the correspondence between models and clubs, and thus clarify clause (6).

Lemma 4.10. *Let p be a precondition, and suppose that $\alpha, \gamma \in \mathcal{D}_p$ and $M \in \mathcal{M}_p$ are such that $\alpha < \sup(M)$, $\alpha \notin M$, and $\alpha \in \mathrm{Lim}(C_\gamma)$. Then $\gamma \leq \min(M \setminus \alpha)$.*

Proof. Since $\alpha \notin M$, we have that $(M, \alpha) \notin A_p$. Therefore we can use clause (6b) to conclude that $\sigma := \min(M \setminus \alpha) \in \mathcal{S}_p$. Hence, if $\gamma > \sigma$ then, by (5), C_γ has no limit points below σ, a contradiction. $\surd$

Notice that if $\alpha \in \mathrm{Lim}(C_\gamma)$ then $\mathrm{cf}(\alpha) = \omega$, otherwise C_γ would have order type larger than ω_1.

Lemma 4.11. *Let p be a precondition, $\alpha \in \mathcal{D}_p$ and $M \in \mathcal{M}_p$ be such that $\alpha \notin M$. Suppose that either $\{\gamma \in \mathcal{D}_p \setminus (\alpha + 1) \mid \alpha \in \mathrm{Lim}(C_\gamma)\} \neq \emptyset$ and $\eta := \max\{\gamma \in \mathcal{D}_p \mid \alpha \in \mathrm{Lim}(C_\gamma)\} < \min(M \setminus \alpha)$, or $\alpha > \sup(M)$. Then $C_\alpha \cap \sup(M \cap \alpha)$ is finite (and therefore $\sup(M \cap \alpha) < \alpha$).*

Proof. If $\alpha > \sup(M)$ then the conclusion follows from clauses (3) and (5), as $\sup(M) \in \mathcal{S}_p$ by (3). If $\alpha = \sup(M)$ then $\alpha \in \mathcal{S}_p$ hence it cannot be a limit point of any C_γ for $\gamma \in \mathcal{D}_p \setminus (\alpha + 1)$. So assume that $\alpha < \sup(M)$ and $\alpha \in \mathrm{Lim}(C_\eta)$. If η is not a limit point of any $C_{\eta'}$ for $\eta' \in \mathcal{D}_p$ then, by (6c), $C_\eta \cap \sup(M \cap \eta)$ is finite. Here we use the fact that $\eta \notin M$ and $\sup(M \cap \eta) < \eta$. Since $C_\alpha \subset C_\eta$ and $\sup(M \cap \alpha) = \sup(M \cap \eta)$, $C_\alpha \cap \sup(M \cap \alpha)$ is also finite. If $\eta \in \mathrm{Lim}(C_{\eta'})$ for some $\eta' \in \mathcal{D}_p \setminus (\eta + 1)$ then $\alpha \in \mathrm{Lim}(C_{\eta'})$ which contradicts the assumption that η is the largest such ordinal. $\surd$

Lemma 4.12. *Let p be a precondition, $\alpha \notin \mathcal{D}_p$ and $M \in \mathcal{M}_p$ be such that $\alpha \notin M$, $\alpha < \sup(M)$ and $\alpha = \sup(M \cap \alpha)$. If there exists some $\varepsilon \in \mathcal{D}_p$, $\varepsilon \leq \min(M \setminus \alpha)$, such that $\alpha \in \operatorname{Lim}(C_\varepsilon)$ then $\max\{\varepsilon' \in \mathcal{D}_p \mid \alpha \in \operatorname{Lim}(C_{\varepsilon'})\} = \min(M \setminus \alpha)$.*

Proof. Let $\gamma := \max\{\varepsilon' \in \mathcal{D}_p \mid \alpha \in \operatorname{Lim}(C_{\varepsilon'})\} > \alpha$. By the definition of precondition, only $\sup(M \cap \varepsilon)$ may be missing from $\mathcal{D}_p$, hence $\min(M \setminus \alpha) = \min(M \setminus \varepsilon) \in \mathcal{S}_p$ by clause (6b) for ε and M. Therefore $\gamma \leq \min(M \setminus \alpha)$ by clause (5). Suppose that $\gamma < \min(M \setminus \alpha)$. Since there is no $\beta \in \mathcal{D}_p \setminus (\gamma + 1)$ such that $\gamma \in \operatorname{Lim}(C_\beta)$, because otherwise $\alpha \in \operatorname{Lim}(C_\gamma) \subset \operatorname{Lim}(C_\beta)$, we can apply clause (6c) for γ and M and we get that $C_\gamma \cap \sup(M \cap \gamma) = C_\gamma \cap \sup(M \cap \alpha)$ is finite and therefore α cannot be a limit point of C_γ, a contradiction. Therefore, $\gamma = \min(M \setminus \alpha)$. $\surd$

Lemma 4.13. *Let p be a precondition. If $M \in \mathcal{M}_p$ then $C_{\sup(M)}$ is an ω-sequence.*

Proof. By clauses (2) and (3), $\sup(M) \in \mathcal{S}_p \subset \mathcal{D}_p$. By (5), $\sup(M)$ cannot be a limit point of any C_γ for $\gamma \in \mathcal{D}_p$. Since M is countable, $\sup(M)$ has countable cofinality, so $C_{\sup(M)}$ is an ω-sequence by clause (6d). $\surd$

Recall the definitions of $\mathfrak{M}_0$ and $\mathfrak{M}_1$ from the beginning of this section.

Lemma 4.14. *Let $\mathscr{N}' \in \mathfrak{M}_1$. If p is a condition in $P \cap \mathscr{N}'$ then there exists an extension $q \geq p$ such that $\mathscr{N}' \cap H_{\omega_2} \in \mathcal{M}_q$.*

Proof. Let p be of the form $(\mathcal{F}_p, \mathcal{S}_p, \mathcal{O}_p, \mathcal{M}_p)$ and let $\mathscr{N} := \mathscr{N}' \cap H_{\omega_2} \in \mathfrak{M}_0$. By Lemma 3.2, $\mathscr{N} \prec H_{\omega_2}$. Note also that $p \in \mathscr{N}$.

We are now going to extend p by adding clubs C_α for certain α. The point is that we want our q to satisfy $\mathscr{N}' \cap H_{\omega_2} \in \mathcal{M}_q$, so in order to also satisfy (6b) we shall have to add various other things to q.

Suppose that $\alpha \notin N$ is such that $\alpha = \sup(N \cap \gamma)$ for some $\gamma \in \mathcal{D}_p$. Notice that then $\gamma > \alpha$ and $\sup(N \cap \alpha) = \alpha$, since $p \in \mathscr{N}$ implies that $\mathcal{D}_p \in \mathscr{N}$ and so $\gamma \in N$ and hence $\gamma = \min(N \setminus \alpha)$. By Lemma 3.4, $\operatorname{cf}(\gamma) = \omega_1$, therefore $\gamma \in \mathcal{S}_p$ by clause (2) in p. It is worth mentioning that $\operatorname{cf}(\alpha) = \omega$, hence C_α—once it is defined—is not required to interact with $\mathscr{E}$. In the case of $\alpha \in \operatorname{Lim}(C_\beta)$ for some $\beta \in \mathcal{D}_p$ let $C_\alpha = C_\beta \cap \alpha$. The choice for C_α is well-defined by clause (4) in p. If there is no such β then let C_α be the $\leq^*$-first ω-sequence cofinal in α. We will also have to add $\sup(N)$ to the set of safeguards. For the corresponding club $C_{\sup(N)}$ we pick the $\leq^*$-first cofinal ω-sequence in $\sup(N)$. Again, $\operatorname{cf}(\sup(N)) = \omega$, therefore $C_{\sup(N)}$ does not have to interact with $\mathscr{E}$.

Define $q := (\mathcal{F}_p \cup \{(\alpha, C_\alpha) \mid \alpha \notin N, \alpha = \sup(N \cap \gamma) \text{ for some } \gamma \in \mathcal{D}_p\} \cup \{(\sup(N), C_{\sup(N)})\}, \mathcal{S}_p \cup \{\sup(N)\}, \mathcal{O}_p, \mathcal{M}_p \cup \{N\})$. Clauses (1)-(4) of Definition 4.6 are trivially true. For clause (5), suppose that $\alpha \in \mathcal{D}_q$ and $\sigma \in \mathcal{S}_q \cap \alpha$. If both $\alpha \in \mathcal{D}_p$ and $\sigma \in \mathcal{S}_p$ then clause (5) holds by the fact that $p \in P$. Suppose $\alpha \notin \mathcal{D}_p$. The first case is that $\alpha \notin N$ and $\alpha = \sup(N \cap \gamma)$ for some $\gamma \in \mathcal{D}_p$. As mentioned above, in this case $\operatorname{cf}(\alpha) = \omega$, so if C_α has order type ω then certainly $C_\alpha \cap \sigma$ is finite. If not, then $C_\alpha = C_\beta \cap \alpha$ for some $\beta \in \mathcal{D}_p$. On the other hand, $\sigma < \sup(N)$ and so $\sigma \in \mathcal{S}_p$. Hence $C_\beta \cap \sigma$ is finite and so $C_\alpha \cap \sigma$ is also finite. Now suppose $\alpha \in \mathcal{D}_p$ but $\sigma \notin \mathcal{S}_p$. Hence $\sigma = \sup(N)$, but $\alpha \in N$ and $\alpha > \sigma$, a contradiction.

Clause (6a) is vacuous for every $\alpha \in \mathcal{D}_q \setminus \mathcal{D}_p$ and $M \in \mathcal{M}_q$ because $M \subset N$ and $\alpha \notin N$, and trivial for every $\alpha \in \mathcal{D}_p$ and N.

For clause (6b) first consider some $\alpha \in \mathcal{D}_q \setminus \mathcal{D}_p$ and $M \in \mathcal{M}_p$ such that $\alpha < \sup(M)$. Then $\alpha = \sup(N \cap \gamma)$ for some $\gamma \in \mathcal{D}_p$, and either $\gamma \notin M$ with $\gamma < \sup(M)$

or $\gamma \in M$ with $\sup(M \cap \gamma) < \gamma$. In both cases, by (6b) in p, $\sup(M \cap \alpha) = \sup(M \cap \gamma) \in \mathcal{D}_p \subset \mathcal{D}_q$ and $\min(M \setminus \alpha) = \min(M \setminus \gamma) \in \mathcal{S}_p \subset \mathcal{S}_q$. Similarly, if $\alpha \in \mathcal{D}_q \setminus \mathcal{D}_p$, $\alpha \neq \sup(N)$, and we consider the model N, then $\alpha = \sup(N \cap \alpha) \in \mathcal{D}_q$ and $\min(N \setminus \alpha) = \gamma \in \mathcal{S}_p \subset \mathcal{S}_q$. Now consider some $\eta \in \mathcal{D}_p$ and the model N such that $\sup(N \cap \eta) < \eta$. Then $\mathrm{cf}(\eta) = \omega_1$ by Lemma 3.4 hence $\min(N \setminus \eta) = \eta \in \mathcal{S}_p \subset \mathcal{S}_q$ by clause (2) in p. On the other hand, $\sup(N \cap \eta) \in \mathcal{D}_q$ by definition of q. Finally, if $\alpha \in \mathcal{D}_p$ and $M \in \mathcal{M}_p$ then (6b) in q follows from (6b) in p.

For (6c) first assume that $\alpha \in \mathcal{D}_q \setminus \mathcal{D}_p$ and $M \in \mathcal{M}_p$ are such that $\sup(M \cap \alpha) < \alpha < \sup(M)$ and there is no $\beta \in \mathcal{D}_q \setminus (\alpha + 1)$ such that $\alpha \in \mathrm{Lim}(C_\beta)$. Then C_α is an ω-sequence and (6c) is trivially true. If $\alpha \in \mathcal{D}_p$ and $M \in \mathcal{M}_p$ then (6c) is true in q because it is true in p. The case of $\alpha \in \mathcal{D}_q \setminus \mathcal{D}_p$ and N is irrelevant for (6c) because $\sup(N \cap \alpha) = \alpha$, as is the case of $\alpha \in \mathcal{D}_p$ and N since $\alpha \in N$. Clause (6d) is proved similarly.

As for clause (7), suppose that some newly added $\alpha < \sup(N)$ falls into some interval $(\beta', \beta]$. Then its corresponding $\gamma \in \mathcal{D}_p$ was already in this interval, since $\{\beta', \beta\} \subset N$. But that is in a contradiction with clause (7) in p. Condition (8) is easily seen to hold. Finally, for (9), notice, that for $M \in \mathcal{M}_p$ the M-fence for N is the empty set, while the N-fence for M is $\{\sup(M \cap N)\} = \{\sup(M)\}$ which is a subset of $\mathcal{S}_p \subset \mathcal{S}_q$ by clause (3).

Hence q is a condition extending p and having the desired property. $\surd$

Imitating the above proof gives us the following result.

Lemma 4.15. *The set of conditions $p \in P$ such that $\mathcal{M}_p \neq \emptyset$ is open and dense.*

Proof. Clearly, the set is open. Let us show that it is dense. Let $p \in P$ and assume $\mathcal{M}_p = \emptyset$. Let $\mathscr{N} \in \mathfrak{M}_0$ be such that $p \in \mathscr{N}$. Define q as in the proof of Lemma 4.14. Then $\mathcal{M}_q \neq \emptyset$ and $q \geq p$. $\surd$

Lemma 4.16. *Suppose $p = (\mathcal{F}_p, \mathcal{S}_p, \mathcal{O}_p, \mathcal{M}_p)$ is a semi-condition. Then $q := (\mathcal{F}_p, \mathcal{S}_p \cup J_p, \mathcal{O}_p, \mathcal{M}_p)$ is a precondition.*

Proof. Recall that $J_p = \{\delta' \in \mathcal{D}_p \setminus \mathcal{S}_p \mid \text{there exist } M, M' \in \mathcal{M}_p \text{ and } \delta \in \mathcal{S}_p \cap M$ such that $\delta' = \sup(M \cap M') \in M$ and $\delta' < \delta < \min(M' \setminus \delta')\}$. Pick some $\delta' \in J_p$. We have to show that clause (5) is true for δ' and every $\alpha \in \mathcal{D}_p \setminus (\delta' + 1)$. The other clauses follow trivially from the respective clauses for p. Clause (9) is also true for q because the M-fence for M' is now a subset of $\mathcal{S}_q$. Hence $J_q = \emptyset$, while $A_q = A_p$ and $B_q = B_p$.

Let $\mu := \min(M' \setminus \delta) = \min(M' \setminus \delta')$. First assume that $\alpha > \delta$. Since $\delta \in \mathcal{S}_p$, we can use (5) in p to deduce that $C_\alpha \cap \delta' \subset C_\alpha \cap \delta$ is finite. Suppose now that $\alpha \leq \delta$. Then $\sup(M' \cap \alpha) = \delta' < \alpha < \sup(M')$. Now we use (6c) in p. If there is no $\beta \in \mathcal{D}_p \setminus (\alpha + 1)$ such that $\alpha \in \mathrm{Lim}(C_\beta)$ then $C_\alpha \cap \delta'$ is finite. However, if $\alpha \in \mathrm{Lim}(C_\beta)$ for some $\beta \in \mathcal{D}_p \setminus (\alpha + 1)$ then we can invoke Lemma 4.10 to see that the maximal such β is $\leq \mu$. If it is $< \mu$ then, by Lemma 4.11, $C_\alpha \cap \delta'$ is finite. On the other hand, α cannot be a limit point of C_μ, since $\alpha < \delta \in \mathcal{S}_p$, hence the maximal β cannot be equal to μ. $\surd$

Lemma 4.17. *Suppose $p_0 = (\mathcal{F}_{p_0}, \mathcal{S}_{p_0}, \mathcal{O}_{p_0}, \mathcal{M}_{p_0})$ is a precondition. Then there exists a precondition $p_1 = (\mathcal{F}_{p_1}, \mathcal{S}_{p_0}, \mathcal{O}_{p_0}, \mathcal{M}_{p_0})$ such that $\mathcal{F}_{p_0} \subsetneq \mathcal{F}_{p_1}$ and $A_{p_1} \subsetneq A_{p_0}$.*

Proof. Pick a pair $N \in \mathcal{M}_{p_0}$ and $\gamma \in \mathcal{D}_{p_0} \cap N$ such that $\alpha := \sup(N \cap \gamma) \notin \mathcal{D}_{p_0}$. Note that in this case $\sup(N \cap \gamma) < \gamma$, hence $\mathrm{cf}(\gamma) = \omega_1$ and $\gamma \in \mathcal{S}_{p_0}$. Also, $\alpha \notin N$

and $\alpha = \sup(N \cap \alpha)$. Let C_α be as in the proof of Lemma 4.14. This is to say that if $\alpha \in \mathrm{Lim}(C_\beta)$ for some $\beta \in \mathcal{D}_{p_0} \setminus (\alpha+1)$ then $C_\alpha := C_\beta \cap \alpha$. By clause (5) we have that $\beta \leq \gamma$, since $\gamma \in \mathcal{S}_{p_0}$. Applying Lemma 4.12 to α and N, we see that we can assume without loss of generality that $\beta = \gamma$. This choice of C_α is well-defined because by clause (4) it does not depend on β anyway. If there is no such β then let C_α be the $\leq^*$-first ω-sequence cofinal in α, so it will end up being an element of all $\mathscr{M}$ relevant to (6a). Define $p_1 := (\mathcal{F}_{p_0} \cup \{(\alpha, C_\alpha)\}, \mathcal{S}_{p_0}, \mathcal{O}_{p_0}, \mathcal{M}_{p_0})$. We will prove that p_1 is a precondition and that A_{p_1} is a proper subset of A_{p_0}. We will do that by checking that α satisfies all the relevant clauses of Definition 4.6.

Clause (1) is trivial, while clauses (2) and (3) are irrelevant for α.

For clause (4), consider some $\beta \in \mathcal{D}_{p_0}$. Suppose first that $\beta > \alpha$. If $\beta > \gamma$ then by (5) C_α and C_β cannot have any common limit points, since $\gamma \in \mathcal{S}_{p_0}$. Assume now that $\beta \leq \gamma$ and there exists some $\mu \in \mathrm{Lim}(C_\alpha) \cap \mathrm{Lim}(C_\beta)$. If $C_\alpha = C_\gamma \cap \alpha$ then $\mu \in \mathrm{Lim}(C_\gamma) \cap \mathrm{Lim}(C_\beta)$, hence by (4) in p_0, $C_\alpha \cap \mu = C_\gamma \cap \mu = C_\beta \cap \mu$. If C_α is an ω-sequence then $\mu = \alpha$, hence $\alpha \in \mathrm{Lim}(C_\beta)$ for some $\beta \in \mathcal{D}_{p_0} \setminus (\alpha+1)$, therefore $C_\alpha = C_\gamma \cap \alpha$ and $C_\alpha \cap \mu = C_\beta \cap \mu$ was already shown.

Suppose now that $\beta < \alpha$. If α is an ω-sequence then C_α and C_β have no common limit points. If $C_\alpha = C_\gamma \cap \alpha$ and there is some $\mu \in \mathrm{Lim}(C_\alpha) \cap \mathrm{Lim}(C_\beta)$ then $\mu \in \mathrm{Lim}(C_\gamma) \cap \mathrm{Lim}(C_\beta)$, hence by (4) in p_0, $C_\alpha \cap \mu = C_\gamma \cap \mu = C_\beta \cap \mu$.

For clause (5), consider some $\sigma \in \mathcal{S}_{p_0}$, $\sigma < \alpha$. If α is an ω-sequence then $C_\alpha \cap \sigma$ is finite. If $C_\alpha = C_\gamma \cap \alpha$ then $C_\alpha \cap \sigma = C_\gamma \cap \sigma$, which is finite by (5) in p_0.

For clause (6), let $M \in \mathcal{M}_{p_0} \setminus \{N\}$. Note that all the instances of clause (6) for α and N are fulfilled by construction. Clause (6a) holds because of the way we defined C_α. Namely, $C_\alpha \in \mathscr{M}[M]$ for all models $M \in \mathcal{M}_{p_0}$ such that $\alpha \in M$, because C_α is either the $\leq^*$-first relevant ω-sequence or $C_\alpha = C_\gamma \cap \alpha$, and the latter is an intersection of two objects already in $\mathscr{M}$. To see that, we must prove that $\gamma \in M$ if $\alpha \in M$. Then by (6a) in p_0, $C_\gamma \in \mathscr{M}$. So assume that $\alpha \in M$. First suppose that $\alpha < \sup(M \cap N) =: \delta$. Since $\alpha \in M \setminus N$, we know that $M \cap N \notin N$, because otherwise $\alpha = \sup((M \cap N) \cap \gamma) \in N$ by elementarity. Hence, by compatibility of M and N, $M \cap N = N \cap \delta$. But then $\gamma \in M$, as $\gamma < \delta$. If $\delta = \alpha \in M$ then α is in the M-fence for N, hence it is in $\mathcal{S}_{p_0} \subset \mathcal{D}_{p_0}$ by (9), a contradiction. Suppose now that $\alpha > \delta$. Then, by Lemma 4.5, $\sup(M \cap \alpha) < \alpha$ and since $\alpha \in M$, we can conclude by applying Lemma 3.4 that $\mathrm{cf}(\alpha) = \omega_1$, which is in a contradiction with the fact that $\alpha = \sup(N \cap \gamma)$.

For (6b) assume that $\alpha < \sup(M)$ and $\alpha \notin M$. The situation $\alpha \in M$ and $\sup(M \cap \alpha) < \alpha$ cannot occur because that would mean that $\mathrm{cf}(\alpha) = \omega_1$. Suppose first that $\alpha \geq \sup(M \cap N) =: \delta$. If $\gamma' := \min(M \setminus \alpha) < \gamma$ then γ' is in the M-fence for N, hence it is in $\mathcal{S}_{p_0}$. The pair (N, γ') is not in A_{p_0} since $\gamma' \notin N$. Also, $\gamma' \notin B_{p_0}$ since $\mathrm{cf}(\gamma') = \omega_1$. But then by the part of (6b) that holds for p_0, we have that $\alpha = \sup(N \cap \gamma') \in \mathcal{D}_{p_0}$, a contradiction. Since $\alpha \geq \delta$, we know that $\gamma' \neq \gamma$. So suppose now that $\gamma' > \gamma$. In this case, $(M, \gamma) \notin A_{p_0}$ since $\gamma \notin M$, and $\gamma \notin B_{p_0}$ since $\mathrm{cf}(\gamma) = \omega_1$. Again we can use the part of (6b) that is true for p_0 and conclude that $\min(M \setminus \alpha) = \gamma' = \min(M \setminus \gamma) \in \mathcal{S}_{p_0}$ and $\sup(M \cap \alpha) = \sup(M \cap \gamma) \in \mathcal{D}_{p_0}$.

Suppose now that $\alpha < \delta$. We consider two cases. If $\alpha = \sup(M \cap \alpha)$ then $M \cap N$ cannot be an element of either M or N. We see that by applying Lemma 3.5 to the pair $M \cap N$, M or to the pair $M \cap N$, N, taking into account that $\sup(M \cap \alpha) = \sup((M \cap N) \cap \alpha) = \sup(N \cap \alpha)$. Hence $M \cap \delta = M \cap N = N \cap \delta$. Consequently $\min(M \setminus \alpha) = \min(N \setminus \alpha) = \gamma \in \mathcal{S}_{p_0}$, and $\sup(M \cap \alpha) = \alpha$ was just added to $\mathcal{D}_{p_0}$.

This means that (M,γ) is in A_{p_0} but it is not in A_{p_1}, and the reason for the latter is α. If $\sup(M\cap\alpha)<\alpha$ then $M\cap N\neq N\cap\delta$ hence $M\cap N=M\cap\delta$. Since $\gamma<\delta$ it follows that $\gamma\leq\min(M\setminus\alpha)$. If $\gamma<\min(M\setminus\alpha)$ then $(M,\gamma)\notin A_{p_0}$ and since $\gamma\notin B_{p_0}$ we can use (6b) to get that $\min(M\setminus\alpha)=\min(M\setminus\gamma)\in\mathcal{S}_{p_0}$ and $\sup(M\cap\alpha)=\sup(M\cap\gamma)\in\mathcal{D}_{p_0}$. However, if $\gamma=\min(M\setminus\alpha)$ then $\min(M\setminus\alpha)=\gamma\in\mathcal{S}_{p_0}$. On the other hand, if $\sup(M\cap\alpha)=\sup(M\cap\gamma)$ does not happen to be in $\mathcal{D}_{p_0}$ then α is in B_{p_1} and it corresponds to the pair (M,γ) which is in A_{p_0} and remains in A_{p_1}.

It is important to notice that whenever we used (6b) in p_0, we never called upon the (incorrect) assumption that it holds for some M and α such that $(M,\alpha)\in A_{p_0}$ or for some $\gamma\in B_{p_0}$.

For (6c) assume that $\alpha\notin M$, $\sup(M\cap\alpha)<\alpha<\sup(M)$ and there is no $\beta\in\mathcal{D}_{p_0}$ such that $\alpha\in\operatorname{Lim}(C_\beta)$. Then C_α is an ω-sequence, hence $C_\alpha\cap\sup(M\cap\alpha)$ is finite. Similarly, for (6d) assume that $\alpha\notin M$, $\sup(M\cap\alpha)=\alpha$ and there is no $\beta\in\mathcal{D}_{p_0}$ such that $\alpha\in\operatorname{Lim}(C_\beta)$. Then C_α is again an ω-sequence, hence (6d) for M and α holds automatically.

For clause (7) let $(\beta',\beta]\in\mathcal{O}_{p_0}$, and suppose for contradiction that $\alpha\in(\beta',\beta]$. Then $(\beta',\beta]\cap\mathscr{N}\neq\emptyset$, hence by (8) in p_0, $(\beta',\beta]\in\mathscr{N}$. Therefore $\beta\geq\gamma$ and $\gamma\in(\beta',\beta]$, which contradicts (7) in p_0. Finally, clauses (8) and (9) are irrelevant for α.

When we added α to $\mathcal{D}_{p_0}$ we did not produce any new pair to be added to A_{p_0}. Hence $A_{p_1}\subset A_{p_0}\setminus\{(N,\gamma)\}$, since one α may actually cause several pairs to disappear from A_{p_0}, as seen in the proof of (6b). $\surd$

Lemma 4.18. *Suppose $p_0=(\mathcal{F}_{p_0},\mathcal{S}_{p_0},\mathcal{O}_{p_0},\mathcal{M}_{p_0})$ is a precondition. Then there exists a condition $p^*=(\mathcal{F}_{p^*},\mathcal{S}_{p_0},\mathcal{O}_{p_0},\mathcal{M}_{p_0})\in P$ such that $\mathcal{F}_{p_0}\subsetneq\mathcal{F}_{p^*}$.*

Proof. Let p_1 be the precondition given by Lemma 4.17. Then $A_{p_1}\subsetneq A_{p_0}$. It is true that B_{p_1} may be larger than B_{p_0}, as seen at the end of proof of (6b), but that is of no consequence. Now we apply Lemma 4.17 to p_1 and repeat the procedure at most $|A_{p_0}|$ many times. Ultimately we get $\mathcal{F}_{p^*}=\mathcal{F}_{p_0}\cup\{(\alpha,C_\alpha)\mid(N,\gamma)\in A_{p_0},\,\alpha=\sup(N\cap\gamma)\}$. Notice that if there are (N,γ) and (N',γ') in A_{p_0} such that $\alpha=\sup(N\cap\gamma)=\sup(N'\cap\gamma')$ then α makes these both pairs satisfy clause (6b), and that happens at the same step of the procedure. Hence C_α is uniquely determined. Since $A_{p^*}=\emptyset$, we have that $B_{p^*}=\emptyset$, hence $p^*\in P$. $\surd$

Lemma 4.19. *Let $\mathscr{N}\in\mathfrak{M}_0$ and suppose that $r\in P$ is such that $N\in\mathcal{M}_r$. Then $r_{\mathscr{N}}:=(\mathcal{F}_r\cap\mathscr{N},\mathcal{S}_r\cap\mathscr{N},\mathcal{O}_r\cap\mathscr{N},(\mathcal{M}_r\cap\mathscr{N})\cup\{M\cap N\mid M\in\mathcal{M}_r,\,M\notin\mathscr{N},\,M\cap N\in\mathscr{N}\})$ is a condition in $P\cap\mathscr{N}$.*

Proof. Note that since $N\in\mathcal{M}_r$ we have $r\notin\mathscr{N}$, as otherwise $\mathscr{N}\in\mathscr{N}$. Clearly $r_{\mathscr{N}}\in\mathscr{N}$. Let us prove that $r_{\mathscr{N}}\in P$. First note that by (6a), $\mathcal{F}_{r_{\mathscr{N}}}=\mathcal{F}_r\restriction\mathscr{N}$ hence $\mathcal{D}_{r_{\mathscr{N}}}=\mathcal{D}_r\cap\mathscr{N}$. Also, by Lemma 3.3, $\mathscr{M}\cap\mathscr{N}\prec H_{\omega_2}$ for every $\mathscr{M}\in\mathcal{M}_r$ and clearly $\mathscr{M}\cap\mathscr{N}\in\mathfrak{M}_0$, hence $M\cap N$ can be added to $\mathcal{M}_{r_{\mathscr{N}}}$ for the relevant M. Notice that for such M since $M\cap N\in\mathscr{N}$ then $\delta_{M,N}:=\sup(M\cap N)\in\mathcal{S}_r\cap\mathscr{N}$ because it is in the N-fence for M, hence clause (3) is satisfied. Also note that then it follows that $M\cap N\notin\mathscr{M}$ [2] as otherwise $\mathscr{M}\cap\mathscr{N}\in\mathscr{M}\cap\mathscr{N}$. By the compatibility of M and N in r, it must be the case that $\delta_{M,N}\in N$ and $M\cap N=M\cap\delta_{M,N}$. To continue now with checking that $r_{\mathscr{N}}\in P$, clauses (4) and (5) follow from the same clauses for r as does (6a).

[2]Here we use that any model $M'\in\mathcal{M}_r$ uniquely determines $\mathscr{M}[M']$.

For clause (6b) consider $\alpha \in \mathcal{D}_{r_{\mathscr{N}}}$ and $M \cap N \in \mathcal{M}_{r_{\mathscr{N}}} \setminus \mathcal{M}_r$ such that $\alpha \notin M \cap N$. That means that $\alpha \notin M$. Since $M \cap N \in \mathscr{N}$, $M \cap N$ is an initial segment of M. If $\alpha < \delta_{M,N}$ then $\min((M \cap N) \setminus \alpha) = \min(M \setminus \alpha) \in \mathcal{S}_r \cap \mathscr{N}$ by clause (6b) in r, hence $\min((M \cap N) \setminus \alpha) \in \mathcal{S}_{r_{\mathscr{N}}}$. By the same argument, $\sup((M \cap N) \cap \alpha) \in \mathcal{D}_{r_{\mathscr{N}}}$. Now suppose that $\alpha \in M \cap N$ is such that $\sup((M \cap N) \cap \alpha) < \alpha$. Then $\mathrm{cf}(\alpha) = \omega_1$ and $\sup(M \cap \alpha) < \alpha$. Now, as above, use (6b) in r for α and M to get that $\sup((M \cap N) \cap \alpha) = \sup(M \cap \alpha) \in \mathcal{D}_r \cap \mathscr{N} = \mathcal{D}_{r_{\mathscr{N}}}$ and $\min((M \cap N) \setminus \alpha) = \alpha \in \mathcal{S}_{r_{\mathscr{N}}}$.

For clause (6c) suppose that $\alpha \notin M \cap N$ is such that $\sup((M \cap N) \cap \alpha) < \alpha < \delta_{M,N}$ and there is no $\beta \in \mathcal{D}_{r_{\mathscr{N}}} \setminus (\alpha+1)$ such that $\alpha \in \mathrm{Lim}(C_\beta)$. We again use the fact that $M \cap N$ is an initial segment of M. Then $\alpha \notin M$ and $\sup(M \cap \alpha) < \alpha < \sup(M)$, since $\alpha < \delta_{M,N} \leq \sup(M)$. If there is no $\beta \in \mathcal{D}_r \setminus (\alpha+1)$ such that $\alpha \in \mathrm{Lim}(C_\beta)$ then we can use (6c) in r for M and α to get that $C_\alpha \cap \sup((M \cap N) \cap \alpha) = C_\alpha \cap \sup(M \cap \alpha)$ is finite. By assumption, there is no such $\beta \in \mathcal{D}_r \cap \mathscr{N}$. If there exists such β in $\mathcal{D}_r \setminus \mathscr{N}$ then $\beta < \min(M \setminus \alpha)$, because $\min(M \setminus \alpha) \in \mathcal{S}_r \cap \mathscr{N}$ by (6b) in r. Then we can apply Lemma 4.11 and again conclude that $C_\alpha \cap \sup((M \cap N) \cap \alpha) = C_\alpha \cap \sup(M \cap \alpha)$ is finite.

For clause (6d) suppose that $\alpha \notin M \cap N$ is such that $\sup((M \cap N) \cap \alpha) = \alpha$ and there is no $\beta \in \mathcal{D}_{r_{\mathscr{N}}} \setminus (\alpha+1)$ such that $\alpha \in \mathrm{Lim}(C_\beta)$. Hence $\alpha \leq \delta_{M,N}$. If $\alpha = \delta_{M,N}$ then, as noted above, by (9) in r, $\delta_{M,N} \in \mathcal{S}_r$ hence by (5) cannot be a limit point of any C_β in r so certainly not in $r_{\mathscr{N}}$. If $\alpha < \delta_{M,N}$ then $\alpha = \sup(M \cap \alpha)$, and if there is no $\beta \in \mathcal{D}_r \setminus (\alpha + 1)$ such that $\alpha \in \mathrm{Lim}(C_\beta)$ then we can use the compatibility between α and M in r. Therefore by (6d) in r we have that (6d) is also satisfied in $r_{\mathscr{N}}$. Again we have to consider the possibility that such β exists in $\mathcal{D}_r \setminus \mathscr{N}$. A similar argument as with (6c) shows that Lemma 4.11 prohibits such β to exist.

Clauses (7) and (8) are clear.

To check (9) consider the compatibility between two models of the form $M \cap N \in \mathcal{M}_{r_{\mathscr{N}}} \setminus \mathcal{M}_r$. Suppose that $M_i \in \mathcal{M}_r \setminus \mathscr{N}$ for $i = 1, 2$ are such that $M_i' := M_i \cap N$ satisfy that $M_i' \in \mathscr{N}$. Let x_1 be the M_1-fence for M_2. Then $x_1 \cap N = x_1 \cap \sup(M_1 \cap N)$ is the M_1'-fence for M_2', so certainly finite and included in $\mathcal{S}_r \cap \mathscr{N}$. Here we have used the fact that $M_1 \cap N$ is an initial segment of M_1.

Now note that $M_1' \cap M_2' = M_1 \cap M_2 \cap N = (M_1 \cap N) \cap (M_2 \cap N)$. We shall consider two cases, denoting by $\delta_{M_1' \cap M_2'}$ the ordinal $\sup(M_1' \cap M_2')$:

Case 1: $\delta_{M_1,N} \leq \delta_{M_2,N}$.

Hence $\delta_{M_1' \cap M_2'} = \delta_{M_1,N} \notin M_1'$ and $M_1' \cap M_2' = M_1' \cap \delta_{M_1' \cap M_2'}$.

Case 2: $\delta_{M_1,N} > \delta_{M_2,N}$.

Hence $\delta_{M_1' \cap M_2'} = \delta_{M_2,N} \in M_1'$ and $M_1' \cap M_2' \in \mathscr{M}_1'$. $\surd$

We are now ready to prove the most important facet of forcing P, namely the fact that it preserves ω_1. We do that by proving that P is proper. There are several equivalent definitions of properness. We shall use the following one.

Definition 4.20. *Let Q be a forcing notion and θ a large enough cardinal.*

*(1) Suppose that $\mathscr{N} \prec H_\theta$. A condition $q \in Q$ is $\mathscr{N}$-*generic *if for every extension $r \geq q$ in Q, and every dense set $\mathscr{D} \subset Q$ with $\mathscr{D} \in \mathscr{N}$, there exists a condition $s \in \mathscr{D} \cap \mathscr{N}$ which is compatible with r.*

(2) Q is proper *if there is a club $\mathfrak{N}$ of $[H_\theta]^\omega$ consisting of countable elementary submodels of H_θ such that for every $\mathscr{N} \in \mathfrak{N}$ with $Q \in \mathscr{N}$, every condition in $Q \cap \mathscr{N}$ has an $\mathscr{N}$-generic extension.*

Proposition 4.21. *The forcing P is proper.*

Proof. Let θ be a large enough cardinal. The club witnessing the properness of P will be the collection $\mathfrak{M}_1$ defined at the beginning of this section. Fix an $\mathscr{N}' \in \mathfrak{M}_1$, such that $P \in \mathscr{N}'$, and consider an arbitrary $p = (\mathcal{F}_p, \mathcal{S}_p, \mathcal{O}_p, \mathcal{M}_p) \in P \cap \mathscr{N}'$. Define $\mathscr{N} := \mathscr{N}' \cap H_{\omega_2} \in \mathfrak{M}_0$ and let q be the extension of p given by Lemma 4.14. We will prove that q is an $\mathscr{N}'$-generic extension of p.

Suppose $r \in P$ is an arbitrary extension of q. Let $r_{\mathscr{N}}$ be the condition given by Lemma 4.19. Proceed by fixing a dense open subset $\mathscr{D} \subset P$, $\mathscr{D} \in \mathscr{N}'$, and extend $r_{\mathscr{N}}$ to $s \in \mathscr{D} \cap \mathscr{N}'$. Since we can find such $s \in H_{\omega_2}$, by elementarity we can assume that $s \in \mathscr{N}$. Let $t := (\mathcal{F}_r \cup \mathcal{F}_s, \mathcal{S}_r \cup \mathcal{S}_s, \mathcal{O}_r \cup \mathcal{O}_s, \mathcal{M}_r \cup \mathcal{M}_s)$. We shall prove that t is a semi-condition. In particular, following Remark 4.8 we prove that clause (6b*) holds for t instead of clause (6b). We then use Lemmas 4.16 and 4.18 to extend t to a condition $t^* \in P$. Since then clearly t^* extends both r and s, we will have proved that r and s are compatible.

Clauses (1), (2) and (3) are obviously true.

Clause (4): take arbitrary $\alpha \neq \beta \in \mathcal{D}_t$. We can assume without loss of generality that $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $\beta \in \mathcal{D}_s \setminus \mathcal{D}_r$. In particular, $\beta \in N$. We shall use (6) for r to discuss the possibilities for α and β.

If $\beta > \alpha$ there are two possibilities. If $\beta = \min(N \setminus \alpha)$ then $\alpha \notin N$ and $\alpha < \sup(N)$, hence by (6b) in r we have $\beta = \min(N \setminus \alpha) \in \mathcal{S}_r \subset \mathcal{D}_r$, which we assumed was not the case. If $\beta > \min(N \setminus \alpha)$ then $C_\beta \cap C_\alpha \subset C_\beta \cap \min(N \setminus \alpha)$ which is finite by (5) in s, because $\min(N \setminus \alpha) \in \mathcal{S}_r \cap \mathscr{N} \subset \mathcal{S}_s$ by (6b) in r. Hence, $\mathrm{Lim}(C_\beta) \cap \mathrm{Lim}(C_\alpha) = \emptyset$.

If $\beta < \sup(N) = \alpha$ then C_α is an ω-sequence by Lemma 4.13, so $C_\alpha \cap \beta$ is finite. If $\beta < \sup(N) < \alpha$ then since $\sup(N) \in \mathcal{S}_r$ we can apply clause (5) in r to get that $C_\alpha \cap \beta$ is finite. Hence in either of these two cases we have $\mathrm{Lim}(C_\alpha) \cap \mathrm{Lim}(C_\beta) = \emptyset$. Finally consider the case $\beta < \alpha < \sup(N)$. If there is no $\gamma \in \mathcal{D}_r \setminus (\alpha + 1)$ such that $\alpha \in \mathrm{Lim}(C_\gamma)$ then one of the cases (6c) or (6d) in r applies to α and N. Either C_α is an ω-sequence or $C_\alpha \cap \sup(N \cap \alpha)$ is finite. In any case $C_\alpha \cap C_\beta$ is finite so $\mathrm{Lim}(C_\alpha) \cap \mathrm{Lim}(C_\beta) = \emptyset$. So suppose that there is such γ and let η be the maximal such. By Lemma 4.10, $\eta \leq \min(N \setminus \alpha)$. If $\eta < \min(N \setminus \alpha)$ then $C_\alpha \cap \beta$ is finite by Lemma 4.11, because $\beta < \sup(N \cap \alpha)$. If $\eta = \min(N \setminus \alpha)$ then $\eta \in N$ and hence $\eta \in \mathcal{D}_{r_{\mathscr{N}}} \subset \mathcal{D}_s$. Suppose that C_α and C_β have a common limit point μ. Then $\mu \in \mathrm{Lim}(C_\eta)$ since $\alpha \in \mathrm{Lim}(C_\eta)$ and so by (4) in r we have $C_\alpha = C_\eta \cap \alpha$. Hence $C_\eta \cap \mu = C_\alpha \cap \mu$ by (4) in r and $C_\beta \cap \mu = C_\eta \cap \mu$ by (4) in s and hence we are done.

Clause (5): first consider the case of $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $\sigma \in \mathcal{S}_s \setminus \mathcal{S}_r$, $\sigma < \alpha$. In particular $\alpha \notin N$ and $\sigma \in N$. Suppose first $\alpha < \sup(N)$. We can apply (6c) or (6d) in r to α and N. The first possibility is that C_α is an ω-sequence or $C_\alpha \cap \sup(N \cap \alpha)$ is finite, in which case we are done. The second possibility is that there is $\beta' \in \mathcal{D}_r \setminus (\alpha + 1)$ with $\alpha \in \mathrm{Lim}(C_{\beta'})$. Let β be the largest such β'. In particular $C_\alpha = C_\beta \cap \alpha$ by (4) in r. By Lemma 4.10, $\beta \leq \min(N \setminus \alpha)$. If $\beta < \min(N \setminus \alpha)$ then by Lemma 4.11 we have that $C_\beta \cap \sup(N \cap \beta)$ is finite, and hence $C_\alpha \cap \sigma$ is finite, since $\sigma < \sup(N \cap \beta)$. If, on the other hand, $\beta = \min(N \setminus \alpha) \in \mathcal{S}_r \cap \mathscr{N} = \mathcal{S}_s$ then by (5) in s we have that $C_\beta \cap \sigma$ is finite, and hence $C_\alpha \cap \sigma$ is finite.

Suppose $\alpha = \sup(N)$. Then by Lemma 4.13, C_α is an ω-sequence, hence $C_\alpha \cap \sigma$ is certainly finite.

If $\alpha > \sup(N)$ then $C_\alpha \cap \sup(N)$ is finite since $\sup(N) \in \mathcal{S}_r$, so $C_\alpha \cap \sigma$ is finite.

Now consider the case $\alpha \in \mathcal{D}_s \setminus \mathcal{D}_r$ and $\sigma \in \mathcal{S}_r \setminus \mathcal{S}_s$, $\sigma < \alpha$. Then $\min(N \setminus \sigma) \in \mathcal{S}_r \cap \mathscr{N} \subset \mathcal{S}_s$. Also, $\alpha \geq \min(N \setminus \sigma)$, but $\alpha \neq \min(N \setminus \sigma)$, otherwise $\alpha \in \mathcal{S}_r \subset \mathcal{D}_r$.

Hence $\alpha > \min(N \setminus \sigma) \in \mathcal{S}_s$ and therefore $C_\alpha \cap \sigma \subset C_\alpha \cap \min(N \setminus \sigma)$ which is a finite set by (5) in s.

Clause (6): first consider an arbitrary $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $M \in \mathcal{M}_s \setminus \mathcal{M}_r$. Then $\alpha \notin N \supset M$ and $\sup(M \cap \alpha) < \alpha$ by Lemma 3.5. Clause (6a) does not apply. For (6b) since $\alpha \notin M$, the only relevant situation could be that $\alpha < \sup(M)$. Then $\alpha < \sup(N)$ and so by (6b) applied to r we have that $\beta := \min(N \setminus \alpha) \in \mathcal{S}_r \cap N = \mathcal{S}_{r_{\mathscr{N}}} \subset \mathcal{S}_s$. Note that $\min(M \setminus \alpha) \geq \beta$. If $\min(M \setminus \alpha) = \beta$ then $\min(M \setminus \alpha) \in \mathcal{S}_s \subset \mathcal{S}_t$. If $\beta < \min(M \setminus \alpha)$ then $M \setminus \alpha = M \setminus \beta$, $\beta \notin M$ and $\beta < \sup(M)$, hence $\min(M \setminus \beta) \in \mathcal{S}_s$ by (6b) in s. Also note that $\sup(M \cap \alpha) = \sup(M \cap \beta)$ so by the same clause, $\sup(M \cap \alpha) \in \mathcal{D}_s$.

For (6c), suppose that $\alpha < \sup(M)$ and there is no $\beta \in \mathcal{D}_t \setminus (\alpha + 1)$ such that $\alpha \in \operatorname{Lim}(C_\beta)$. Then in the case that $\sup(N \cap \alpha) < \alpha$, we can apply (6c) from r to conclude that $C_\alpha \cap \sup(N \cap \alpha)$ is finite, so certainly $C_\alpha \cap \sup(M \cap \alpha)$ is finite. If $\sup(N \cap \alpha) = \alpha$ we can apply clause (6d) from r to conclude that C_α is an ω-sequence cofinal in α and hence $C_\alpha \cap \sup(M \cap \alpha)$ is finite, since by Lemma 3.5, $\sup(M \cap \alpha) < \sup(N \cap \alpha)$.

Since $\sup(M \cap \alpha) < \alpha$ was shown above, case (6d) is irrelevant.

Now consider an arbitrary $\alpha \in \mathcal{D}_s \setminus \mathcal{D}_r$ and $M \in \mathcal{M}_r \setminus \mathcal{M}_s$. For (6a), if $\alpha \in M$ then note that then $\alpha \in M \cap N$. Note also that $\mathscr{M}$ and $\mathscr{N}$ are compatible, as they are both from $\mathcal{M}_r$. Let $\delta := \sup(M \cap N)$, hence $\alpha < \delta$. Suppose first that $\delta \notin N$. Because $s \in \mathscr{N}$ we have that $C_\alpha \in \mathscr{N}$. Hence if $\operatorname{cf}(\alpha) = \omega$, C_α is countable (by (1) for s) and we have that $C_\alpha \in [\delta]^{\leq\omega} \cap \mathscr{N}$. By Lemma 4.2, we conclude that $C_\alpha \in \mathscr{M}$. If $\operatorname{cf}(\alpha) = \omega_1$ we have that $C_\alpha = E_\alpha \setminus \beta$ for some $\beta \in \mathcal{D}_s \cap \alpha$, by clause (1) for s. Since $\alpha \in M$ and $M \in \mathfrak{M}_0$, we have that $E_\alpha \in \mathscr{M}$. Then $\beta < \alpha < \delta$ and $\beta \in N$, since $\beta \in \mathcal{D}_s$. Since $\delta \notin N$ then by compatibility in r, $M \cap N = N \cap \delta$ and so $\beta \in M$ and hence $C_\alpha \in \mathscr{M}$. If $\delta \in N$ then $M \cap N \in \mathscr{N}$ by compatibility in r, and hence $M \cap N \in \mathcal{D}_{r_{\mathscr{N}}} \subset \mathcal{D}_s$. Hence by (6a) in s we have $C_\alpha \in \mathscr{M} \cap \mathscr{N}$, so $C_\alpha \in \mathscr{M}$.

For (6b*), suppose that $\alpha \notin M$ and $\alpha < \sup(M)$. This will be enough since by Remark 4.8, the case $\alpha \in M$ and $\sup(M \cap \alpha) < \alpha$ is irrelevant for (6b*). We know that $\alpha \in N$. If $\delta < \alpha$ we have that $\alpha' := \min(M \setminus \alpha)$ is in the M-fence for N, and hence a member of $\mathcal{S}_r \subset \mathcal{S}_t$, by (9) in r. We have that $\sup(M \cap \alpha) = \sup(M \cap \alpha')$ and the latter is in $\mathcal{D}_r \subset \mathcal{D}_t$ by the second clause of (6b) applied to α' and M in r. In the case $\delta = \alpha$ we conclude similarly that $\min(M \setminus \alpha) \in \mathcal{S}_t$. In this case we have $\sup(M \cap \alpha) = \sup((M \cap N) \cap \alpha)$. We also know that $\delta = \alpha \in N$ and so $M \cap N \in \mathcal{M}_{r_{\mathscr{N}}} \subset \mathcal{M}_s$. Hence we have that $\sup((M \cap N) \cap \alpha)$ is in $\mathcal{D}_{r_{\mathscr{N}}} \subset \mathcal{D}_t$. Suppose then that $\alpha < \delta$. Hence $\alpha \in (N \cap \delta) \setminus M$ and therefore $M \cap N \neq N \cap \delta$. By the compatibility between M and N we conclude that it must the case that $\delta \in N$ and $M \cap N \in \mathscr{N}$. Then $\delta \notin M$, so $M \cap N = M \cap \delta$, and hence $\min(M \setminus \alpha) = \min((M \cap N) \setminus \alpha)$. But $M \cap N \in \mathcal{M}_{r_{\mathscr{N}}} \subset \mathcal{M}_s$ and hence $\min(M \setminus \alpha) \in \mathcal{S}_s \subset \mathcal{S}_t$. Also $\sup(M \cap \alpha) = \sup((M \cap N) \cap \alpha) \in \mathcal{D}_s \subset \mathcal{D}_t$.

For (6c), suppose that $\alpha \notin M$ and $\sup(M \cap \alpha) < \alpha < \sup(M)$, while there is no $\beta \in \mathcal{D}_t \setminus (\alpha + 1)$ with $\alpha \in \operatorname{Lim}(C_\beta)$. If $\alpha < \delta$ then $M \cap N \neq N \cap \delta$, hence $M \cap N \in \mathscr{N}$ and $M \cap N \in \mathcal{M}_{r_{\mathscr{N}}} \subset \mathcal{M}_s$. Also $M \cap N \notin \mathscr{M}$, hence $M \cap N = M \cap \delta$. Since $\sup((M \cap N) \cap \alpha) < \alpha < \sup(M \cap N)$, we can use (6c) for $M \cap N$ and α in s to deduce that $C_\alpha \cap \sup(M \cap \alpha) = C_\alpha \cap \sup((M \cap N) \cap \alpha)$ is finite. If $\alpha > \delta$ then there exists some σ in the N-fence for M such that $\sup(M \cap \alpha) \leq \sigma \leq \alpha$. Then $\sigma \in \mathcal{S}_r \cap \mathscr{N} \subset \mathcal{S}_s$. In fact, $\sigma < \alpha$, because otherwise $\alpha \in \mathcal{D}_r$ which we assumed is not the case. But then, by (5) in s, we have that $C_\alpha \cap \sigma$ is finite, hence $C_\alpha \cap \sup(M \cap \alpha)$ is finite as well. The option that $\alpha = \delta$ is not possible because we assumed that $\alpha > \sup(M \cap \alpha)$.

For (6d) assume that $\alpha \notin M$ and $\sup(M \cap \alpha) = \alpha$, while there is no $\beta \in \mathcal{D}_t \setminus (\alpha+1)$ with $\alpha \in \operatorname{Lim}(C_\beta)$. Since $\alpha \in \mathcal{D}_s$ we have $\alpha \in N$. Suppose first $\alpha \leq \delta$.

If $\delta \in N$ then $M \cap N \in \mathcal{M}_s$ and $M \cap \delta = M \cap N$. So $\sup((M \cap N) \cap \alpha) = \alpha$ and by (6d) in s we conclude that C_α is a cofinal ω-sequence in α. Suppose that $\delta \notin N$. In particular then $\alpha < \delta$. By compatibility of M and N in r we have that $N \cap \delta = M \cap N$. If $\alpha < \delta$ then $\alpha \in (N \cap \delta) \setminus M$, a contradiction.

Now suppose that $\delta < \alpha$. By Lemma 4.5 applied to M and α (so $\gamma = \alpha$) we have that $\sup(N \cap \alpha) < \alpha$, hence $\operatorname{cf}(\alpha) = \omega_1$ by Lemma 3.4. On the other hand, $\operatorname{cf}(\alpha) = \omega$ since $\sup(M \cap \alpha) = \alpha$, and we have a contradiction.

Clause (7): clearly, $\mathcal{O}_t$ is a finite set of intervals of the form $(\beta', \beta] \subset \omega_2$. Consider an arbitrary $(\beta', \beta] \in \mathcal{O}_r \setminus \mathcal{O}_s$ and $\alpha \in \mathcal{D}_s \setminus \mathcal{D}_r$. Use (8) in r. If $(\beta', \beta] \cap N = \emptyset$ then since $\alpha \in N$, we have that $\alpha \notin (\beta', \beta]$. If $(\beta', \beta] \in \mathscr{N}$ then $(\beta', \beta] \in \mathcal{O}_{r_{\mathscr{N}}} \subset \mathcal{O}_s$, a contradiction.

Suppose now that $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $(\beta', \beta] \in \mathcal{O}_s \setminus \mathcal{O}_r$. In particular $(\beta', \beta] \in \mathscr{N}$ and $\alpha \notin N$. By (6b) in r we have that $\min(N \setminus \alpha) \in \mathcal{S}_r \cap \mathscr{N} \subset \mathcal{S}_s$. If $\alpha \in (\beta', \beta]$ then $\min(N \setminus \alpha) \in (\beta', \beta]$, in contradiction with (7) in s.

Clause (8): suppose that $(\beta', \beta] \in \mathcal{O}_r \setminus \mathcal{O}_s$ and $M \in \mathcal{M}_s \setminus \mathcal{M}_r$. If $(\beta', \beta] \in \mathscr{N}$ then $(\beta', \beta] \in \mathcal{O}_{r_{\mathscr{N}}} \subset \mathcal{O}_s$, a contradiction. Hence the interval is disjoint from N by (8) in r, so it is disjoint from $M \subset N$.

Now consider an arbitrary $M \in \mathcal{M}_r \setminus \mathcal{M}_s$ and $(\beta', \beta] \in \mathcal{O}_s \setminus \mathcal{O}_r$. In particular $(\beta', \beta] \in \mathscr{N}$. Suppose for a contradiction that $(\beta', \beta] \cap M \neq \emptyset$ but $(\beta', \beta] \notin \mathscr{M}$. Let $\delta := \sup(M \cap N)$. If $\beta' \geq \delta$ then, by (9) in r, there is some γ from the N-fence for M in the interval $(\beta', \beta]$. But $\gamma \in \mathcal{S}_r \cap \mathscr{N} \subset \mathcal{S}_s$, a contradiction with (7) in s. On the other hand, if $\beta' < \delta$ and $\beta \geq \delta$, then $\min(N \setminus \delta) \in (\beta', \beta]$. But $\min(N \setminus \delta) \in \mathcal{S}_r \cap \mathscr{N} \subset \mathcal{S}_s$ since it is in the N-fence for M, and again we are in contradiction with (7) in s. Finally, suppose that $\beta < \delta$. Then $\{\beta', \beta\} \subset N \cap \delta$ but $\{\beta', \beta\} \not\subset M$, hence $M \cap N \neq N \cap \delta$. But then $M \cap N \in \mathscr{N}$, so $M \cap N \in \mathcal{M}_{r_{\mathscr{N}}} \subset \mathcal{M}_s$. Since $(\beta', \beta] \cap (M \cap N) \neq \emptyset$ but $(\beta', \beta] \notin \mathscr{M} \cap \mathscr{N}$, we get a contradiction with (8) in s.

Clause (9): consider arbitrary models $M \in \mathcal{M}_r \setminus \mathcal{M}_s$ and $M' \in \mathcal{M}_s \setminus \mathcal{M}_r$. Notice that $M' \in N$ and so $M' \subset N$ as M' is countable. Let $\delta := \sup(N \cap M)$ and $\delta' := \sup(M' \cap M) = \sup(M' \cap N \cap M) \leq \delta$. Let us consider the correspondence between δ' and M and M'.

Suppose first that $\delta \in M$, and hence $\delta \notin N$. In this case $N \cap \delta = N \cap M$. By Lemma 4.2 we know that $[\delta]^{\leq \omega} \cap \mathscr{N} \subset \mathscr{M}$. We have that $M' \cap \delta' \in [\delta]^{\leq \omega} \cap \mathscr{N}$, so $M' \cap \delta' \in \mathscr{M}$ and hence $M' \cap \delta' \subset M$ and $M \cap M' = M' \cap \delta'$. We also conclude that $\delta' = \sup(M' \cap \delta') \in M$, and hence $\delta' \notin M'$. This establishes (a) from the definition of compatibility for M and M'. Now assume that $\delta \notin M$. Therefore $M \cap \delta = M \cap N \in \mathscr{N}$ and so $M \cap \delta' = M \cap N \cap \delta'$. Also $M \cap \delta = M \cap N \in \mathscr{N}$ and hence $M \cap N \in \mathcal{M}_{r_{\mathscr{N}}} \subset \mathcal{M}_s$. In particular, M' and $M \cap N$ are compatible. If $\delta' \in M'$ then $M' \cap M \cap N \in \mathscr{M}'$ and so $M' \cap M \in \mathscr{M}'$. If $\delta' \notin M'$ then $M' \cap \delta' = M' \cap M \cap N = M' \cap M$. If $\delta' \in M$ and $\delta' \in N$ then $\delta' \in M \cap N$ and so $M' \cap M \cap N \in \mathscr{M} \cap \mathscr{N}$ and in particular $M' \cap M \in \mathscr{M}$. Finally suppose that $\delta' \in M$ but $\delta' \notin N$. Hence $\delta' \notin M'$ and the conclusion follows as before. This finishes the proof of the condition (a) from the compatibility.

Let us now establish the finiteness of fences. Consider the M'-fence for M. To see that it is a subset of $\mathcal{S}_t$, we need to establish that the set $T := \{\min(M' \setminus \lambda) \mid \lambda \in M,\ \delta' < \lambda < \sup(M')\} \cup \{\min(M' \setminus \delta')\}$ is a subset of $\mathcal{S}_t$. As $T \setminus \delta$ is a subset of the N-fence for M, which is a subset of $\mathcal{S}_r \subset S_t$ by the compatibility of M and N in r, it suffices to show that $T \cap \delta \subset \mathcal{S}_t$. If $M \cap N \notin \mathscr{N}$ then $\delta \notin N$ and $N \cap \delta = N \cap M$, so $M' \cap \delta \subset N \cap \delta$ and hence $M' \cap \delta \subset M$. Let $\varepsilon := \min(M' \setminus \delta')$. Then $\varepsilon \notin M$ so $\varepsilon > \delta$ and hence $T \cap \delta = \emptyset$. If $M \cap N \in \mathscr{N}$ then $M \cap N \in \mathcal{M}_s$. Also $\delta \in N$, so $\delta \notin M$ and hence $M \cap \delta = M \cap N$ and so $T \cap \delta$ is a subset of the M'-fence for $M \cap N$, which is a subset of $\mathcal{S}_s \subset \mathcal{S}_t$ by their compatibility in s.

For the M-fence for M', we need to see that the set $S := \{\min(M \setminus \lambda) \mid \lambda \in M',\ \delta' < \lambda < \sup(M)\} \cup \{\min(M \setminus \delta')\}$ is a subset of $\mathcal{S}_t$. As $S \setminus \delta$ is a subset of the M-fence for N, which is a subset of $\mathcal{S}_r \subset \mathcal{S}_t$ by the compatibility of M and N in r, it suffices to show that $S \cap \delta \subset \mathcal{S}_t$. If $\delta \notin M$ then as above $M \cap \delta = M \cap N \in \mathcal{M}_s$ and hence $S \cap \delta$ is a subset of the $M \cap N$-fence for M', which is a subset of $\mathcal{S}_s \subset \mathcal{S}_t$ by their compatibility in s. If $\delta \in M$ then as above $N \cap \delta = M \cap N$ and in particular $M' \cap \delta \subset M$ and hence $S \cap \delta$ is at most a singleton, namely $\{\delta'\}$. If $\delta' = \sup(M')$ then $\delta' \in \mathcal{S}_s$ by (3) in s. Otherwise let $\mu := \min(N \setminus \delta)$. Since δ is in the M-fence for N, we have that $\mu \in \mathcal{S}_r \cap \mathscr{N} \subset \mathcal{S}_s$. But then $\delta' = \sup(M' \cap \mu) \in \mathcal{D}_s$ by (6b) in s. However, we have no reason to believe that $\delta' \in \mathcal{S}_s$. If $\delta' \notin \mathcal{S}_s \cup \mathcal{S}_r$ then $\delta' \in J_t$, hence δ' need not be in $\mathcal{S}_t$ for t to be a semi-condition. $\surd$

5. Preservation of ω_2

We have thus far proved that forcing with P preserves ω_1. We also need ω_2 to be preserved. For that purpose we use a weak closure property of the forcing, which was also used in [11].

Definition 5.1. *Assume that the forcing notion P preserves cardinals $< \kappa$. P is said to be* κ-presaturated *if for every $A \subset V$, $A \in V[G]$, with $|A|^{V[G]} < \kappa$, there exists $A' \in V$ such that $|A'|^V < \kappa$ and $A' \supset A$.*

Notice that in the case of a κ-presaturated forcing P, since it preserves cardinals below κ, $|A'|^V = |A'|^{V[G]}$ as soon as $|A'|^V < \kappa$. Hence we can omit the superscript when dealing with this situation.

Proposition 5.2. *Suppose κ is a regular cardinal in V. If P is κ-presaturated then P preserves κ.*

Proof. Suppose for contradiction that $A \in V[G]$ is a cofinal subset of κ of cardinality $< \kappa$. Let $A' \in V$, $A' \supset A$, $|A'| < \kappa$, be the set guaranteed by κ-presaturatedness. But $A' \cap \kappa \in V$ is a cofinal subset of κ with cardinality $< \kappa$, and we get a contradiction. $\surd$

Lemma 5.3. *Let κ be a regular cardinal in V such that P preserves cardinals below κ. Suppose that for every collection $\mathcal{A}$ of fewer than κ antichains in P there exists a dense set $\mathscr{D} \subset P$ such that for every $p \in \mathscr{D}$, the set $\{q \in \bigcup \mathcal{A} \mid p$ and q are compatible$\}$ has size less than κ. Then P is κ-presaturated.*

Proof. Suppose $A \subset V$ and $|A|^{V[G]} < \kappa$. Let $p \in G$ be a condition such that $p \Vdash$ "$|\underset{\sim}{A}| < \kappa$". Therefore $p \Vdash$ "there exists $\mu < \kappa$ such that $|\underset{\sim}{A}| = \mu$". Let $p_0 \geq p$, $\underset{\sim}{g}$ and $\mu^* < \kappa$ be such that $p_0 \Vdash$ "$\underset{\sim}{g} : \mu^* \to \underset{\sim}{A}$ is a bijection". For each $\alpha < \mu^*$ let $\mathcal{A}_\alpha$ be a maximal antichain of conditions in the set $\{q \mid (q \geq p_0 \wedge q$ decides $\underset{\sim}{g}(\alpha)) \vee q \perp p_0\}$. Hence $\mathcal{A}_\alpha$ is a maximal antichain.

Define $\mathcal{A} := \{\mathcal{A}_\alpha \mid \alpha < \mu^*\}$. Let $\mathscr{D}$ be a dense set guaranteed by the assumption, and let $p_1 \in \mathscr{D}$, $p_1 \geq p_0$. Then the set $X := \{q \in \bigcup_{\alpha<\mu^*} \mathcal{A}_\alpha \mid q$ is compatible with $p_1\}$ has size $< \kappa$. Let $\Gamma := \{\beta \mid$ there exist $q \in X$ and $\alpha < \mu^*$ such that $q \Vdash$ "$\underset{\sim}{g}(\alpha) = \beta$"$\}$, so $|\Gamma| < \kappa$ by the regularity of κ. Consider an arbitrary $\alpha < \mu^*$. Since $\mathcal{A}_\alpha$ is a maximal antichain there exists some $q \in \mathcal{A}_\alpha$, compatible with p_1, such that q decides $\underset{\sim}{g}(\alpha)$. Hence there exists β such that $q \Vdash$ "$\underset{\sim}{g}(\alpha) = \beta$", and therefore $\beta \in \Gamma$. Let r be a common upper bound for q and p_1. Then $r \Vdash$ "$\underset{\sim}{g}(\alpha) = \beta$", and since $r \geq p_0$, $p_0 \Vdash$ "there exists $\beta \in \Gamma$ such that $\underset{\sim}{g}(\alpha) = \beta$". It follows that $p_0 \Vdash$ "$\underset{\sim}{g}(\alpha) \in \Gamma$", so $p_0 \Vdash$ "$\underset{\sim}{g}[\mu^*] = \underset{\sim}{A} \subset \Gamma$". Therefore $p \Vdash$ "there exists $A' \in V$, $\underset{\sim}{A} \subset A'$ and $|A'| < \kappa$". $\surd$

The next lemma shows that κ-presaturation is, in fact, a generalization of properness to cardinals above ω_1.

Lemma 5.4. *Let κ be a regular cardinal in V and suppose that P preserves cardinals below κ. Suppose that θ is a large enough cardinal, and that for stationarily many models $\mathscr{N}$ in $[H_\theta]^{<\kappa}$ with $P \in \mathscr{N}$, and for each $p \in P \cap \mathscr{N}$, there exists an $\mathscr{N}$-generic extension $q \geq p$. Then P is κ-presaturated.*

Proof. Suppose $A \subset V$ and that $\mu := |A|^{V[G]} < \kappa$. Let $\underset{\sim}{f}$ and $p \in G$ be such that $p \Vdash$ "$\underset{\sim}{f} : \mu \to \underset{\sim}{A}$ is onto". Define $\mathfrak{N} := \{\mathscr{N} \prec H_\theta \mid |\mathscr{N}| < \kappa, \{f, A, p, P\} \cup \mu \subset \mathscr{N}\}$, hence $\mathfrak{N}$ is a club. Therefore we can find $\mathscr{N} \in \mathfrak{N}$ such that there is $q \geq p$ which is $\mathscr{N}$-generic. Then for every $\xi < \mu$, the set $\mathscr{D}_\xi := \{r \in \mathscr{N} \mid r$ decides $f(\xi)\} \in \mathscr{N}$ is dense above q. Hence $q \Vdash$ "$\mathscr{D}_\xi \cap G \cap \mathscr{N} \neq \emptyset$". Therefore q forces that there exist $r_\xi \in G \cap \mathscr{N}$ and $x_\xi \in \mathscr{N}$ such that $r_\xi \Vdash$ "$\underset{\sim}{f}(\xi) = x_\xi$". It follows that $q \Vdash$ "$\underset{\sim}{A} \subset \mathscr{N}$", so $p \Vdash$ "there exists $A' \in V$, $\underset{\sim}{A} \subset A'$ and $|A'| < \kappa$", A' being the model $\mathscr{N}$. $\surd$

We shall prove in Proposition 5.7 that our forcing P is ω_2-presaturated. Since presaturation is a generalization of properness, the proof will be very similar to the proof of properness. Actually, it will be slightly easier, because we will not work with arbitrary models of size ω_1 but only with such models that are in a way transitive below ω_2. We isolate the collection of such models in the following definition.

Definition 5.5. *Let $\theta > \omega_2$ be a large enough regular cardinal. Define $\mathfrak{M}_2 := \{\mathscr{M} \prec H_\theta \mid |\mathscr{M}| = \omega_1, \mathscr{E} \in \mathscr{M}, [\mathscr{M}]^\omega \subset \mathscr{M}$ [3]$\}$.*

Recall that we have assumed CH so the set $\mathfrak{M}_2$ is club in $[H_\theta]^{<\omega_2}$. If $\mathscr{M} \in \mathfrak{M}_2$ then $\mathscr{M} \cap \omega_2$ is some ordinal $\delta_M \in \omega_2$, since $\omega_1 \subset \mathscr{M}$ (see [8]). Note that $\mathrm{cf}(\delta_M) = \omega_1$. Additionally, if $A \in \mathscr{M}$ and $|A| \leq \omega_1$ then $A \subset \mathscr{M}$.

To prove the ω_2-presaturation, we first isolate a lemma which is an analogue of Lemma 4.19. Our notational conventions follow those of Section 4.

Lemma 5.6. *Let $\mathscr{N} \in \mathfrak{M}_2$, and let $r \in P$ be such that $\delta_N \in \mathcal{S}_r$. Define $\mathcal{F}_{r^*_{\mathscr{N}}} := \mathcal{F}_r \cap \mathscr{N}$, $\mathcal{S}_{r^*_{\mathscr{N}}} := (\mathcal{S}_r \cap \mathscr{N}) \cup \{\sup(M \cap N) \mid M \in \mathcal{M}_r \setminus \mathscr{N}\}$, $\mathcal{O}_{r^*_{\mathscr{N}}} := \mathcal{O}_r \cap \mathscr{N}$ and $\mathcal{M}_{r^*_{\mathscr{N}}} := \{M \cap N \mid M \in \mathcal{M}_r\}$. Then $r^*_{\mathscr{N}} := (\mathcal{F}_{r^*_{\mathscr{N}}}, \mathcal{S}_{r^*_{\mathscr{N}}}, \mathcal{O}_{r^*_{\mathscr{N}}}, \mathcal{M}_{r^*_{\mathscr{N}}})$ is a condition in $P \cap \mathscr{N}$.*

Proof. First notice that $r^*_{\mathscr{N}} \in \mathscr{N}$ since $\mathscr{N}$ contains all its countable subsets. Now we check that $r^*_{\mathscr{N}}$ is a condition. Clause (1) is trivial. For clause (2), note that if

[3]Note that this implies that $\omega_1 \subset \mathscr{M}$ and that P belongs to every element of $\mathfrak{M}_2$.

$M \notin \mathscr{N}$ then by (6b) in r we have that $\sup(M \cap N) = \sup(M \cap \delta_N) \in \mathcal{D}_r \cap \mathscr{N} = \mathcal{D}_{r^*_{\mathscr{N}}}$. Clause (3) is easily checked, and especially note that $\mathscr{M}[M \cap N] = \mathscr{M}[M] \cap \mathscr{N}$ for any $M \cap N \in \mathcal{M}_{r^*_{\mathscr{N}}}$. Clause (4) follows by (4) in r.

For (5) suppose that $\alpha \in \mathcal{D}_r \cap \mathscr{N}$ and $\sigma = \sup(M \cap N)$ for some $M \in \mathcal{M}_r$. If $\alpha \geq \sup(M)$ then the conclusion follows since $\sup(M) \in \mathcal{S}_r$ and (5) holds in r. Otherwise $\alpha < \sup(M)$. Since $\alpha \in N$ we must have $\alpha \notin M$. In particular, $\alpha < \delta_N$, so $\alpha \subset N$. Hence $\sup(M \cap \alpha) \leq \sup(M \cap N) = \sigma < \alpha$ and (6c) applies in r to conclude that $C_\alpha \cap \sup(M \cap \alpha)$ is finite, and in particular, $C_\alpha \cap \sigma$ is finite.

For clause (6a), if $\alpha \in \mathcal{D}_r \cap (M \cap N)$ for some $M \in \mathcal{M}_r$, then $C_\alpha \in \mathscr{M}[M]$ by (6a) in r. If C_α is countable then $C_\alpha \in \mathscr{N}$ by the closure of $\mathscr{N}$ under countable subsets. Otherwise, $\alpha \in \mathcal{D}_r$ and $C_\alpha = E_\alpha \setminus \beta$ for some $\beta \in \mathcal{D}_r$. Since $\alpha \in N$, also $\beta \in N$, and hence $C_\alpha \in \mathscr{N}$ since $\mathscr{E} \in \mathscr{N}$.

For (6b), suppose $\alpha \in \mathcal{D}_r \cap N$, $M \in \mathcal{M}_r$ and $\alpha \notin M \cap N$ while $\alpha < \sup(M \cap N)$. Then $\alpha \in N$, so $\alpha \notin M$ and $\alpha < \sup(M)$. By (6b) in r, $\min(M \setminus \alpha) \in \mathcal{S}_r$ and $\sup(M \cap \alpha) \in \mathcal{D}_r$. We have that $\min(M \setminus \alpha) \leq \min((M \cap N) \setminus \alpha) < \delta_N$, so $\min((M \cap N) \setminus \alpha) = \min(M \setminus \alpha) \in \mathcal{S}_r \cap N$. Similarly, $\sup((M \cap N) \cap \alpha) \leq \sup(M \cap \alpha) \leq \alpha < \delta_N$, so $\sup((M \cap N) \cap \alpha) = \sup(M \cap \alpha) \in \mathcal{D}_r \cap N$. Suppose on the other hand that $\alpha \in M \cap N$ but $\sup((M \cap N) \cap \alpha) < \alpha$, hence $\alpha \in M$ and $\sup(M \cap \alpha) < \alpha$ and we argue similarly.

For (6c) suppose that for some $\alpha \in \mathcal{D}_r \cap \mathscr{N}$ and some $M \cap N \in \mathcal{M}_{r^*_{\mathscr{N}}}$, $\alpha \notin M \cap N$, we have $\sup((M \cap N) \cap \alpha) < \alpha < \sup(M \cap N)$, and there is no $\beta \in \mathcal{D}_{r^*_{\mathscr{N}}} \setminus (\alpha + 1)$ such that $\alpha \in \mathrm{Lim}(C_\beta)$. Then $\alpha \in \mathcal{D}_r$, $\alpha \notin M$ and $\sup(M \cap \alpha) < \alpha < \sup(M)$. If there is no $\beta \in \mathcal{D}_r \setminus (\alpha + 1)$ such that $\alpha \in \mathrm{Lim}(C_\beta)$ then it follows from (6c) for r that $C_\alpha \cap \sup((M \cap N) \cap \alpha) = C_\alpha \cap \sup(M \cap \alpha)$ is finite. So suppose there is $\beta \in \mathcal{D}_r \setminus (\alpha + 1)$ such that $\alpha \in \mathrm{Lim}(C_\beta)$. In particular $\beta \geq \delta_N > \min(M \setminus \alpha)$ and $\alpha < \delta_N$. But $\min(M \setminus \alpha) \in \mathcal{S}_r$ by (6b) in r and so by (5) in r we have that $C_\beta \cap \min(M \setminus \alpha)$ is a finite set, a contradiction.

Clause (6d) is proved similarly.

Clause (7) is clear and the clause (8) follows because it is true in r and $N \cap \omega_2$ is an ordinal. For Clause (9) notice that in fact for every relevant M we have that $M \cap N = M \cap \delta_N$ and so (9) follows from (9) in r. $\surd$

Proposition 5.7. *P is ω_2-presaturated.*

Proof. Suppose that $\mathscr{N} \in \mathfrak{M}_2$ and $p \in P \cap \mathscr{N}$. We extend p to q by putting δ_N into both $\mathcal{D}_p$ and $\mathcal{S}_p$. For the corresponding club C_{δ_N} we take $E_{\delta_N} \setminus \max(\mathcal{D}_p)$. It is easy to check that $q \in P$ and that $q \geq p$. We will prove that q is $\mathscr{N}$-generic.

Suppose that r is an arbitrary extension of q, so in particular $\delta_N \in \mathcal{S}_r$. Hence $r^*_{\mathscr{N}}$ as given by Lemma 5.6 is well-defined. For a fixed dense set $\mathscr{D} \subset P$, $\mathscr{D} \in \mathscr{N}$, extend $r^*_{\mathscr{N}}$ to $s \in \mathscr{D}$. Then $s \in \mathscr{N}$. As with properness, we will prove clause by clause of Definition 4.6 that $t := (\mathcal{F}_r \cup \mathcal{F}_s, \mathcal{S}_r \cup \mathcal{S}_s, \mathcal{O}_r \cup \mathcal{O}_s, \mathcal{M}_r \cup \mathcal{M}_s)$ is a condition.

Clause (1): notice that $\mathcal{D}_s \cap \mathcal{D}_r \subset \mathcal{D}_{r^*_{\mathscr{N}}}$, so that $\mathcal{F}_r \cup \mathcal{F}_s$ is indeed a function. The rest of the clause follows easily.

Clauses (2) and (3) need no comments.

Clause (4): suppose that $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $\beta \in \mathcal{D}_s \setminus \mathcal{D}_r$, so $\beta < \delta_N$ and $\alpha \geq \delta_N$. Then $C_\alpha \cap \mathscr{N}$ is a finite set because $\delta_N \in \mathcal{S}_r$. Also, $C_\beta \subset \mathscr{N}$. Hence $\mathrm{Lim}(C_\alpha) \cap \mathrm{Lim}(C_\beta) = \emptyset$.

Clause (5): if $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $\sigma \in \mathcal{S}_s \setminus \mathcal{S}_r$ then $C_\alpha \cap \sigma \subset C_\alpha \cap \delta_N$, which is a finite set as in (4). If $\alpha \in \mathcal{D}_s \setminus \mathcal{D}_r$ and $\sigma \in \mathcal{S}_r \setminus \mathcal{S}_s$ then $\alpha < \delta_N \leq \sigma$ so clause (5) does not apply.

Clause (6): First suppose that $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $M \in \mathcal{M}_s \setminus \mathcal{M}_r$. Then $\alpha > \sup(M)$ since $\alpha \geq \delta_N$ and $M \subset \delta_N$, so no parts of (6) can apply.

Suppose then that $\alpha \in \mathcal{D}_s \setminus \mathcal{D}_r$ and $M \in \mathcal{M}_r \setminus \mathcal{M}_s$. Then $M \cap N \in \mathcal{M}_s$. For (6a) if $\alpha \in M$, then $\alpha \in M \cap N$, so $C_\alpha \in M \cap N \subset N$, by (6a) for s. For (6b), if $\alpha \notin M$ and $\alpha < \sup(M)$ then suppose first $\alpha < \sup(M \cap N)$, in which case $\min(M \setminus \alpha) = \min((M \cap N) \setminus \alpha) \in \mathcal{S}_s$ and $\sup(M \cap \alpha) = \sup((M \cap N) \cap \alpha) \in \mathcal{D}_s$. If $\alpha \geq \sup(M \cap N)$ then $\sup(M \cap \alpha) = \sup(M \cap N) \in \mathcal{S}_s \subset \mathcal{D}_s$. Also, $\min(M \setminus \alpha) = \min(M \setminus \delta_N) \in \mathcal{S}_r$ by (6b) in r. Suppose now that $\alpha \in M$ and $\sup(M \cap \alpha) < \alpha$, hence $\alpha \in M \cap N$ and $\sup((M \cap N) \cap \alpha) < \alpha$. Also, $\sup(M \cap \alpha) = \sup((M \cap N) \cap \alpha)$ and $\min(M \setminus \alpha) = \alpha = \min((M \cap N) \setminus \alpha)$. The former is in $\mathcal{D}_s$ and the latter in $\mathcal{S}_s$ by (6b) for s.

For (6c) suppose that $\alpha \notin M$ is such that $\sup(M \cap \alpha) < \alpha < \sup(M)$ and there is no $\beta \in \mathcal{D}_t \setminus (\alpha + 1)$ such that $\alpha \in \mathrm{Lim}(C_\beta)$. Then we have $\sup(M \cap \alpha) = \sup((M \cap N) \cap \alpha)$, so if $\alpha < \sup(M \cap N)$ then $C_\alpha \cap \sup(M \cap \alpha)$ is a finite set by (6c) in s. If $\sup(M \cap N) < \alpha$ then $\sup(M \cap \alpha) = \sup(M \cap N) \in \mathcal{S}_s$ and the conclusion follows by (5) in s.

For (6d), if the assumptions of (6d) apply, note that $\sup((M \cap N) \cap \alpha) = \alpha$, so the conclusion follows by (6d) in s.

Clause (7): clearly $\mathcal{O}_t$ is a finite set of half open nonempty intervals. If $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $(\beta', \beta] \in \mathcal{O}_s \setminus \mathcal{O}_r$ then $(\beta', \beta] \subset \mathscr{N}$, hence $\alpha \notin (\beta', \beta]$. Suppose now that $\alpha \in \mathcal{D}_s \setminus \mathcal{D}_r$ and $(\beta', \beta] \in \mathcal{O}_r \setminus \mathcal{O}_s$. Since $\delta_N \in \mathcal{D}_r$, we have $(\beta', \beta] \cap \mathscr{N} = \emptyset$, hence $\alpha \notin (\beta', \beta]$.

Clause (8): if $M \in \mathcal{M}_s \setminus \mathcal{M}_r$ and $(\beta', \beta] \in \mathcal{O}_r \setminus \mathcal{O}_s$ then $(\beta', \beta] \cap \mathscr{M} = \emptyset$ because $\delta_N \in \mathcal{D}_r$. Consider an $M \in \mathcal{M}_r \setminus \mathcal{M}_s$ and $(\beta', \beta] \in \mathcal{O}_s \setminus \mathcal{O}_r$. Then $(\beta', \beta]$ and $M \cap N$ satisfy (8) in s. If $(\beta', \beta] \in \mathscr{M} \cap \mathscr{N}$ then $(\beta', \beta] \in \mathscr{M}$. If $(\beta', \beta] \cap (\mathscr{M} \cap \mathscr{N}) = \emptyset$ then $(\beta', \beta] \cap \mathscr{M} = ((\beta', \beta] \cap \mathscr{N})) \cap \mathscr{M} = \emptyset$.

Clause (9): consider two models $M \in \mathcal{M}_r \setminus \mathcal{M}_s$ and $M' \in \mathcal{M}_s \setminus \mathcal{M}_r$. Then $M \cap N$ and M' are compatible in s. Notice that $M \cap M' = (M \cap N) \cap M'$ and let $\delta := \sup(M \cap M') = \sup((M \cap N) \cap M')$. If $\delta \in M$ then $(M \cap N) \cap M' \in \mathscr{M} \cap \mathscr{N}$ and so $M \cap M' = M \cap (M' \cap N) \in \mathscr{M}$. Now suppose that $\delta \notin M$ so $M \cap \delta = (M \cap N) \cap \delta = (M \cap N) \cap M' = M \cap M'$. If $\delta \in M'$ then $M \cap M' = (M \cap N) \cap M' \in \mathscr{M}'$. If $\delta \notin M'$ then $M \cap M' = (M \cap N) \cap M' = M' \cap \delta$. This establishes the compatibility.

For the fences, the M-fence for M' is contained in δ_N and so is the same set as the $M \cap N$-fence for M', which is finite and contained in $\mathcal{S}_s$. The M'-fence for M is the same as the M'-fence for $M \cap N$, and so finite and contained in $\mathcal{S}_s$. $\surd$

Corollary 5.8. *Forcing with P preserves cardinals.*

Proof. P has the ω_3-c.c. because, assuming $2^{\omega_1} = \omega_2$, $|P| = \omega_2$. Hence it preserves cardinals $\geq \omega_3$. It it preserves ω_1 because it is proper and preserves ω_2 because it is ω_2-presaturated. $\surd$

Definition 5.9. *Let $G \subset P$ be a generic set. Define $\mathcal{F} := \bigcup_{p \in G} \mathcal{F}_p$, and $\mathcal{C} := \mathrm{dom}(\mathcal{F})$.*

Proposition 5.10. *$\mathcal{C}$ is unbounded in ω_2.*

Proof. Define $\mathscr{D}_\alpha := \{p \in P \mid \max(\mathcal{D}_p) > \alpha\}$ for $\alpha < \omega_2$. Consider an arbitrary $p \in P$ and assume that $p \notin \mathscr{D}_\alpha$. Now let $\alpha' := \sup(\mathcal{D}_p \cup \bigcup \mathcal{O}_p \cup \bigcup \mathcal{M}_p) < \omega_2$ and let

$q := (\mathcal{F}_p \cup \{(\alpha'+\omega, (\alpha', \alpha'+\omega))\}, \mathcal{S}_p, \mathcal{O}_p, \mathcal{M}_p)$. Clearly, $q \in P$, $q \geq p$ and $q \in \mathscr{D}_\alpha$, hence $\mathscr{D}_\alpha$ is dense in P for every $\alpha < \omega_2$. It follows that $\mathcal{C}$ is unbounded in ω_2. $\surd$

To prove that $\mathcal{C}$ is closed, we need the following lemma, which shows the role of the part $\mathcal{O}_p$ of the conditions in P.

Lemma 5.11. *Suppose that $\alpha < \omega_2$ is a nonzero limit ordinal. Then the set $\mathscr{D}^*_\alpha := \{p \in P \mid \alpha \in \mathcal{D}_p \cup \bigcup \mathcal{O}_p\}$ is open dense in P.*

Proof. It is clear that the set is open, let us show that it is dense. Given $p \in P$ and suppose that $p \notin \mathscr{D}^*_\alpha$. We shall consider several cases.

Case 1. There is no $M \in \mathcal{M}_p$ such that $\alpha = \sup(M \cap \alpha)$.

Subcase (a). $\alpha \notin \bigcup \mathcal{M}_p$.[4]

Let $\beta' := \sup((\mathcal{D}_p \cup \bigcup \mathcal{M}_p) \cap \alpha)$, hence $\beta' < \alpha$, as α is a limit. In particular, $(\beta', \alpha] \cap M = \emptyset$ for every $M \in \mathcal{M}_p$. Let $q := (\mathcal{F}_p, \mathcal{S}_p, \mathcal{O}_p \cup \{(\beta', \alpha]\}, \mathcal{M}_p)$.

It is easy to check that q is a condition and that $q \geq p$, as the only part of the definition of the condition requiring comment is part (8), which we have specifically addressed by the choice of β'. Clearly $q \in \mathscr{D}^*_\alpha$.

Subcase (b). There is $M \in \mathcal{M}_p$ with $\alpha \in M$.

In particular, $\mathrm{cf}(\alpha) = \omega_1$ by Lemma 3.4. Suppose that $M, M' \in \mathcal{M}_p$ are such that $\alpha \in M \setminus M'$. We shall prove that $\sup(M' \cap \alpha) < \sup(M \cap \alpha)$.

If $\alpha > \sup(M \cap M')$ then $\sup(M' \cap \alpha) \leq \sup(M \cap \alpha)$, otherwise α is in the M-fence for M', hence $\alpha \in \mathcal{S}_p$ by (9) in p, a contradiction with $\alpha \notin \mathcal{D}_p$. In fact $\sup(M' \cap \alpha) = \sup(M \cap \alpha)$ can also not happen, because in this case $\sup(M' \cap \alpha) = \sup(M \cap \alpha) = \sup(M \cap M')$ and α is again in the M-fence for M'. The situation $\alpha = \sup(M \cap M')$ cannot happen because then $\mathrm{cf}(\alpha) = \omega$, a contradiction. So assume now that $\alpha < \sup(M \cap M')$. Since $\alpha \in M \setminus M'$, we see that by compatibility of M and M' in p, $M \cap M' \in M$ and $M \cap M' = M' \cap \sup(M \cap M')$. But then $\sup(M' \cap \alpha) = \sup((M' \cap M) \cap \alpha)$ and the latter is in M by elementarity. Hence $\sup(M' \cap \alpha) < \sup(M \cap \alpha)$.

Let $M^* \in \mathcal{M}_p$ be such that $\beta^* := \sup(M^* \cap \alpha) = \min\{\sup(M \cap \alpha) \mid M \in \mathcal{M}_p,\ \alpha \in M\}$. Then $\beta^* < \alpha$ and $\mathrm{cf}(\beta^*) = \omega$. There is no $\gamma \in \mathcal{D}_p$ such that $\beta^* \leq \gamma \leq \alpha$, since otherwise $\alpha = \min(M^* \setminus \gamma) \in \mathcal{S}_p$ by clause (6b) for γ and M^* in p. Let $M \in \mathcal{M}_p$ be such that $\beta^* < \sup(M \cap \alpha)$. Then there exists some $\alpha' \in (\beta^*, \sup(M \cap \alpha))$ such that $\alpha' \in M \setminus M^*$. Just as above we prove that $\beta^* = \sup(M^* \cap \alpha) \in M$. Hence, if $M \in \mathcal{M}_p$ is such that $\alpha \in M$ then either $\beta^* \in M$ or at least $\beta^* = \sup(M \cap \beta^*)$. Therefore there exists some $\beta' \geq \sup[\bigcup\{M' \cap \alpha \mid M' \in \mathcal{M}_p \mid \alpha \notin M'\} \cup (\mathcal{D}_p \cap \alpha)]$ such that $\beta' < \beta^*$. Then $(\beta', \alpha] \in M$ for every $M \in \mathcal{M}_p$ such that $\alpha \in M$, while $(\beta', \alpha] \cap M' = \emptyset$ for every $M' \in \mathcal{M}_p$ such that $\alpha \notin M'$.

Define $q := (\mathcal{F}_p, \mathcal{S}_p, \mathcal{O}_p \cup \{(\beta', \alpha]\}, \mathcal{M}_p)$. It is easily seen that q is a condition. Clauses (7) and (8) are taken care of by the choice of β', and the other clauses are irrelevant for $(\beta', \alpha]$. Clearly $q \geq p$ and $q \in \mathscr{D}^*_\alpha$.

Case 2. There is $M \in \mathcal{M}_p$ with $\alpha = \sup(M \cap \alpha)$, and $\alpha \in M'$ for every $M' \in \mathcal{M}_p$ such that $\sup(M' \cap \alpha) = \alpha$.

Let $\beta^* := \sup[\bigcup\{M'' \cap \alpha \mid \sup(M'' \cap \alpha) < \alpha,\ M'' \in \mathcal{M}_p\} \cup (\mathcal{D}_p \cap \alpha)]$. Hence $\beta^* < \alpha$. There is $\beta' \in [\beta^*, \alpha)$ such that $(\beta', \alpha] \in M'$ for every $M' \in \mathcal{M}_p$ with $\alpha = \sup(M' \cap \alpha)$. Now let $q := (\mathcal{F}_p, \mathcal{S}_p, \mathcal{O}_p \cup \{(\beta', \alpha]\}, \mathcal{M}_p)$. Like in Case 1, it is easy to check that $q \in \mathscr{D}^*_\alpha$ is a condition and that $q \geq p$. We have chosen β' so that both (7) and (8) hold.

[4] Also if $\mathcal{M}_p = \emptyset$.

Case 3. There is $M \in \mathcal{M}_p$ with $\alpha = \sup(M \cap \alpha)$ and $\alpha \notin M$.

We partition $\mathcal{M}_p$ into three disjoint sets: $\mathcal{M}_1 := \{M \in \mathcal{M}_p \mid \sup(M \cap \alpha) < \alpha\}$, $\mathcal{M}_2 := \{M \in \mathcal{M}_p \mid \sup(M \cap \alpha) = \alpha,\ \alpha \in M\}$ and $\mathcal{M}_3 := \{M \in \mathcal{M}_p \mid \sup(M \cap \alpha) = \alpha,\ \alpha \notin M\}$. Case 3 means that $\mathcal{M}_3 \neq \emptyset$ while $\mathcal{M}_1$ and $\mathcal{M}_2$ might be empty.

Fix some $M \in \mathcal{M}_3$. Then $\alpha < \sup(M)$, otherwise $\alpha = \sup(M) \in \mathcal{S}_p$, a contradiction with $\alpha \notin \mathcal{D}_p$. We shall first investigate how elements from $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ compare to M.

First pick some $M' \in \mathcal{M}_3$, $M' \neq M$. If $\sup(M \cap M') < \alpha$ then by compatibility of M and M' we cannot have that both $\alpha = \sup(M \cap \alpha)$ and $\alpha = \sup(M' \cap \alpha)$ (see Lemma 4.5 with $\alpha = \gamma$), so we conclude that $\sup(M \cap M') \geq \alpha$. If $\sup(M \cap M') = \alpha$ then $\min(M \setminus \alpha)$ is in the M-fence for M', hence $\min(M \setminus \alpha) \in \mathcal{S}_p$ by (9) in p, and therefore $\alpha \in \mathcal{D}_p$ by (6b), a contradiction. Hence $\sup(M \cap M') > \alpha$.

It follows from Lemma 3.5 that $M \cap M' \notin M$ and $M \cap M' \notin M'$. Hence, by compatibility of M and M', $M \cap \sup(M \cap M') = M \cap M' = M' \cap \sup(M \cap M')$. But then $\min(M \setminus \alpha) = \min(M' \setminus \alpha)$.

Now pick some $M' \in \mathcal{M}_2$. If $\sup(M \cap M') < \alpha$ then $\min(M \setminus \alpha)$ is in the M-fence for M', hence $\min(M \setminus \alpha) \in \mathcal{S}_p$ and $\alpha \in \mathcal{D}_p$, a contradiction. If $\sup(M \cap M') = \alpha$ then α is in the M-fence for M', hence $\alpha \in \mathcal{S}_p$, again a contradiction. Hence $\sup(M \cap M') > \alpha$.

Since $\alpha \in M' \setminus M$, we know that $M \cap M' \neq M' \cap \sup(M \cap M')$, hence $M \cap M' \in M'$ and so $M \cap M' = M \cap \sup(M \cap M')$. Therefore $\min(M \setminus \alpha) \in M'$ and consequently $\min(M \setminus \alpha) \geq \min(M' \setminus \alpha)$.

Finally pick some $M' \in \mathcal{M}_1$ and assume that $\alpha < \sup(M')$. As we shall see, if $\alpha > \sup(M')$ then M' is irrelevant for Case 3. We will prove that $\min(M \setminus \alpha) < \min(M' \setminus \alpha)$.

It is entirely possible that $\alpha > \sup(M \cap M')$. But $\min(M' \setminus \alpha)$ is in the M'-fence for M, hence $\min(M' \setminus \alpha) \in \mathcal{S}_p$. If $\min(M \setminus \alpha) \geq \min(M' \setminus \alpha)$ then $\alpha \in \mathcal{D}_p$ by (6b) applied to $\min(M' \setminus \alpha)$ and M. Therefore $\min(M \setminus \alpha) < \min(M' \setminus \alpha)$.

It is obvious that $\alpha \neq \sup(M \cap M')$, since $\sup(M' \cap \alpha) < \alpha$. So assume now that $\alpha < \sup(M \cap M')$. Since $(\sup(M' \cap \alpha), \alpha) \neq \emptyset$, there exists some $\alpha' < \alpha$ such that $\alpha' \in M \setminus M'$. But then $M \cap M' = M' \cap \sup(M \cap M')$, hence $\min(M' \setminus \alpha) \in M$. Therefore $\min(M \setminus \alpha) \leq \min(M' \setminus \alpha)$.

Subcase (a). $\min(M \setminus \alpha) = \min(M' \setminus \alpha)$ for every $M' \in \mathcal{M}_1$.

In particular, $\alpha < \sup(M \cap M')$. If $\mathcal{M}_1 \neq \emptyset$ then let $M^* \in \mathcal{M}_1$ be such that $\beta^* := \sup(M^* \cap \alpha) = \min\{\sup(M' \cap \alpha) \mid M' \in \mathcal{M}_1\} < \alpha$. If $\mathcal{M}_1 = \emptyset$ then let $\beta^* := \alpha$ and $M^* := M$. In any case, $\beta^* \leq \alpha$ and $\mathrm{cf}(\beta) = \omega$. There is no $\gamma \in \mathcal{D}_p$ such that $\beta^* \leq \gamma \leq \min(M \setminus \alpha) =: \gamma'$, since otherwise $\gamma' \in \mathcal{S}_p$ by clause (6b) for γ and M in p. But then $\alpha = \sup(M \cap \gamma') \in \mathcal{D}_p$ by (6b) for γ' and M.

Let us prove that $\beta^* \in M''$ for every $M'' \in \mathcal{M}_p$ such that $\beta^* < \sup(M'' \cap \alpha)$. Notice that this is automatically true if $\mathcal{M}_1 = \emptyset$ (i.e. $\beta^* = \alpha$). So assume that $\mathcal{M}_1 \neq \emptyset$. If $M'' \in \mathcal{M}_2$ then $\alpha \in M'' \setminus M^*$ and $\alpha < \sup(M'' \cap M^*)$. But then $M'' \cap M^* \neq M'' \cap \sup(M'' \cap M^*)$, hence $M'' \cap M^* \in M''$ and $M'' \cap M^* = M^* \cap \sup(M'' \cap M^*)$, and therefore $\beta^* = \sup((M'' \cap M^*) \cap \alpha) \in M''$ by elementarity. If $M'' \in \mathcal{M}_3$ then we argue in the same way, but instead of α we consider some $\alpha' \in (\beta^*, \alpha) \cap M'' \neq \emptyset$. If $M'' \in \mathcal{M}_1 \setminus \{M^*\}$ and $\beta^* < \alpha$ then we repeat the argument with some $\alpha' \in (\beta^*, \sup(M'' \cap \alpha)) \cap M''$. The interval $(\beta^*, \sup(M'' \cap \alpha))$ is nonempty due to the way we defined β^*.

Since $\mathrm{cf}(\beta^*) = \omega$ and $\beta^* \in M''$ for every $M'' \in \mathcal{M}_p$ such that $\beta^* < \sup(M'' \cap \alpha)$, we know that $\beta^* = \sup(M'' \cap \beta^*)$ for every $M'' \in \mathcal{M}_p$. Hence there exists some $\beta' \in (\bigcap \mathcal{M}_p) \cap \beta^*$ such that $(\beta', \beta^*) \cap \mathcal{D}_p = \emptyset$. Then $(\beta', \gamma'] \in M''$ for every $M'' \in \mathcal{M}_p$, while $(\beta', \gamma'] \cap \mathcal{D}_p = \emptyset$.

Define $q := (\mathcal{F}_p, \mathcal{S}_p, \mathcal{O}_p \cup \{(\beta', \alpha]\}, \mathcal{M}_p)$. It is easily seen that q is a condition. Clauses (7) and (8) are satisfied by the choice of β', while the other clauses do not matter for $(\beta', \alpha]$. Clearly $q \geq p$ and $q \in \mathscr{D}^*_\alpha$.

Subcase (b). $\min(M \setminus \alpha) < \min(M' \setminus \alpha)$ for every $M' \in \mathcal{M}_1$.

We can assume that $\mathcal{M}_1 \neq \emptyset$ otherwise Subcase (a) applies. Let $M^* \in \mathcal{M}_1$ be such that $\beta^* := \sup(M^* \cap \alpha) = \max\{\sup(M' \cap \alpha) \mid M' \in \mathcal{M}_1\} < \alpha$. As with Subcase (a), there is no $\gamma \in \mathcal{D}_p$ such that $\alpha \leq \gamma \leq \min(M \setminus \alpha)$. There exists some $\beta' \in [\bigcap(\mathcal{M}_2 \cup \mathcal{M}_3)] \setminus \beta^*$ such that $(\beta', \alpha) \cap \mathcal{D}_p = \emptyset$. Then $(\beta', \min(M \setminus \alpha)] \in M''$ for every $M'' \in \mathcal{M}_2 \cup \mathcal{M}_3$, while $(\beta', \min(M \setminus \alpha)] \cap M'' = \emptyset$ for every $M'' \in \mathcal{M}_1$. Also $(\beta', \min(M \setminus \alpha)] \cap \mathcal{D}_p = \emptyset$.

Define $q := (\mathcal{F}_p, \mathcal{S}_p, \mathcal{O}_p \cup \{(\beta', \alpha]\}, \mathcal{M}_p)$. We have made sure that clauses (7) and (8) are satisfied by the choice of β'. The other clauses do not matter. Clearly $q \geq p$ and $q \in \mathscr{D}^*_\alpha$.

Subcase (c). There is some $M' \in \mathcal{M}_1$ such that $\min(M \setminus \alpha) = \min(M' \setminus \alpha)$, and there is some $M'' \in \mathcal{M}_1$ such that $\min(M \setminus \alpha) < \min(M'' \setminus \alpha)$.

Let $M', M'' \in \mathcal{M}_1$ be such that $\min(M \setminus \alpha) = \min(M' \setminus \alpha)$ and $\min(M \setminus \alpha) < \min(M'' \setminus \alpha)$. We shall prove that $\sup(M'' \cap \alpha) < \sup(M' \cap \alpha)$. Suppose first that $\alpha > \sup(M' \cap M'')$. If $\sup(M'' \cap \alpha) \geq \sup(M' \cap \alpha)$ then $\min(M' \setminus \alpha)$ is in the M'-fence for M'', hence $\min(M \setminus \alpha) = \min(M' \setminus \alpha) \in \mathcal{S}_p$. But then $\alpha \in \mathcal{D}_p$ by (6b) for M and $\min(M \setminus \alpha)$, a contradiction. Suppose now that $\alpha < \sup(M' \cap M'')$. We know that $\min(M \setminus \alpha) \in M' \setminus M''$. Then, by compatibility of M' and M'', we have $M' \cap M'' \in M'$ and $M' \cap M'' = M'' \cap \sup(M' \cap M'')$, hence by Lemma 3.5, $\sup(M'' \cap \alpha) = \sup((M' \cap M'') \cap \alpha) < \sup(M' \cap \alpha)$.

Let $\beta^* := \sup(M^* \cap \alpha) = \min\{\sup(M' \cap \alpha) \mid M' \in \mathcal{M}_1, \min(M \setminus \alpha) = \min(M' \setminus \alpha)\}$ and $\beta^{**} := \sup(M^* \cap \alpha) = \max\{\sup(M'' \cap \alpha) \mid M'' \in \mathcal{M}_1, \min(M \setminus \alpha) < \min(M'' \setminus \alpha)\}$. Then $\beta^{**} < \beta^* < \alpha$ and, just as in Subcase (a), $\beta^* \in \bigcap(\mathcal{M}_2 \cup \mathcal{M}_3)$ as well as $\beta^* \in M'$ for every $M' \in \mathcal{M}_1$ such that $\min(M \setminus \alpha) = \min(M' \setminus \alpha)$ and $\beta^* < \sup(M' \cap \alpha)$. Subcase (a) also shows that there is no $\gamma \in \mathcal{D}_p$ such that $\beta^* \leq \gamma \leq \min(M \setminus \alpha) =: \gamma'$.

There exists $\beta' \in [\bigcap(\mathcal{M}_2 \cup \mathcal{M}_3 \cup \{M' \in \mathcal{M}_1 \mid \min(M \setminus \alpha) = \min(M' \setminus \alpha)\})] \cap [\beta^{**}, \beta^*)$ such that $(\beta', \beta^*) \cap \mathcal{D}_p = \emptyset$. Then $(\beta', \gamma'] \in M'$ for every $M' \in \mathcal{M}_2 \cup M_3$, and $(\beta', \gamma'] \in M'$ for every $M' \in \mathcal{M}_1$ such that $\min(M \setminus \alpha) = \min(M' \setminus \alpha)$, while $(\beta', \gamma'] \cap M'' = \emptyset$ for every $M'' \in \mathcal{M}_1$ such that $\min(M \setminus \alpha) < \min(M'' \setminus \alpha)$. At the same time, $(\beta', \gamma'] \cap \mathcal{D}_p = \emptyset$.

Define $q := (\mathcal{F}_p, \mathcal{S}_p, \mathcal{O}_p \cup \{(\beta', \alpha]\}, \mathcal{M}_p)$. The choice of β' once again made sure that clauses (7) and (8) are satisfied. Clearly $q \geq p$ and $q \in \mathscr{D}^*_\alpha$. $\surd$

Proposition 5.12. *$\mathcal{C}$ is closed in ω_2.*

Proof. Suppose for contradiction that $p \in G$ is such that $p \Vdash$ "$\alpha \in \mathrm{Lim}(\mathcal{C})$ but $\alpha \notin \mathcal{C}$" for some $\alpha < \omega_2$. Then $\alpha \notin \mathcal{D}_p$. Let q be the extension given by previous lemma. But then $q \Vdash$ "$\alpha \notin \mathrm{Lim}(\mathcal{C})$", which contradicts the fact that $p \Vdash$ "$\alpha \in \mathrm{Lim}(\mathcal{C})$". $\surd$

Sequence $\mathcal{F}$ might not be a $\square_{\omega_1}$ sequence since we have no guarantees that its domain $\mathcal{C}$ is $\mathrm{Lim}(\omega_2) \cap \omega_2$. But the next proposition shows that we can now extend our sequence to the whole $\mathrm{Lim}(\omega_2) \cap \omega_2$.

Proposition 5.13. $V[G] \models \square_{\omega_1}$.

Proof. The idea is to throw away every ordinal which is not in $\mathcal{C}$, effectively making $\mathcal{C}$ equal to ω_2. In fact, keeping only limit points of $\mathcal{C}$ will suffice. Thus, let $\mathcal{E} := \mathrm{Lim}(\mathcal{C}) \cap \omega_2$. $\mathcal{E}$ is still a club of ω_2. For every $\alpha \in \mathrm{Lim}(\mathcal{E})$ of cofinality ω_1 define $D_\alpha := C_\alpha \cap \mathcal{E}$. Since $\mathcal{E} \cap \alpha$ is a club in α for every $\alpha \in \mathrm{Lim}(\mathcal{E}) \cap$ of cofinality ω_1, D_α is a club in α. Suppose now that $\alpha \in \mathrm{Lim}(\mathcal{E})$ has cofinality ω. If there is $\beta > \alpha$ of uncountable cofinality such that α is a limit point of D_β, let $D_\alpha := D_\beta \cap \alpha$. This choice does not depend on β, as clause (4) of Definition 4.6 is valid for D_β's just as it was for clubs in $\mathcal{F}$. Otherwise, if there is no such β, let D_α be an ω-sequence cofinal in α and consisting of elements of $\mathcal{E}$.

Now suppose that $\beta \in \mathrm{Lim}(D_\alpha)$ for some $\beta < \alpha$. Then β is a limit point of both $\mathcal{E}$ and C_α, and $D_\beta = C_\beta \cap \mathcal{E} = C_\alpha \cap \beta \cap \mathcal{E} = D_\alpha \cap \beta$. Also, if $\mathrm{cf}(\alpha) = \omega$ then $|D_\alpha| = \omega$. Hence, $\langle D_\alpha \mid \alpha \in \mathrm{Lim}(\mathcal{E}) \cap \omega_2 \rangle$ is a nontrivial coherent sequence of clubs.

Let $\{\gamma_i \mid i < \omega_2\}$ be an increasing enumeration of $\mathcal{E}$. For $i \in \mathrm{Lim}(\omega_2)$ define $E_i := \{j < i \mid \gamma_j \in D_{\gamma_i}\} = \gamma^{-1}[D_{\gamma_i}]$. It is a club in i because γ is a continuous function. Let us prove that $\langle E_i \mid i \in \mathrm{Lim}(\omega_2) \rangle \in V[G]$ is a square sequence. If $i < j$ and $i \in \mathrm{Lim}(E_j)$ then $\gamma_i \in \mathrm{Lim}(D_{\gamma_j})$. Hence, $D_{\gamma_i} = D_{\gamma_j} \cap \gamma_i$. Therefore, $E_i = \gamma^{-1}[D_{\gamma_i}] = \gamma^{-1}[D_{\gamma_j} \cap \gamma_i] = \gamma^{-1}[D_{\gamma_j}] \cap i = E_j \cap i$. Also, if $\mathrm{cf}(i) = \omega$ then $\mathrm{cf}(\gamma_i) = \omega$, hence $|E_i| = |D_{\gamma_i}| = \omega$. $\surd$

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Biotehniška fakulteta, Oddelek za gozdarstvo, Večna pot 83, SI - 1000 Ljubljana, Slovenia
E-mail address: gregor.dolinar@bf.uni-lj.si

School of Mathematics, University of East Anglia, Norwich, NR4 7TJ, UK
E-mail address: m.dzamonja@uea.ac.uk
URL: http://www.mth.uea.ac.uk/~h020/