PARAGRADED STRUCTURES INSPIRED BY MATHEMATICAL LOGIC

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ABSTRACT. We use methods from mathematical logic to give new examples of paragraded structures, showing that at certain cardinals all first order structures are paragraded. We introduce the notion of bi-embeddability to measure when two paragraded structures are basically the same. We prove that the bi-embeddability of the paragraduating system gives rise to the bi-embeddability of the limiting structures. Under certain circumstances the converse is also true, as we show here. Finally, we show that one paragraded structure can have many graded substructures, to the extent that the number of the same is not always decidable by the axioms of set theory.

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1. **Introduction**

In abstract algebra, a *graded structure*, may it be a ring a module or an algebra, is defined using a *gradation*, which is a sequence of substructures indexed using a monoid, most commonly $\mathbb{N}$ or $\mathbb{Z}$. For example, a graded ring is a ring R which is a direct sum of abelian groups R_i $(i \in \mathbb{Z})$ satisfying that $R_i \cdot R_j \subseteq R_{i+j}$. Examples of graded structures are common in mathematics, and include polynomial rings and tensor algebras. These structures are classic and were introduced first by Bourbaki [1], in the case of groups and rings, but with the requirement that the graduated ring is based on an abelian graduated group. Although this requirement is fulfilled in the case of interest in [1], the requirement that the group is abelian is not necessary. This was shown in the definition given by Krasner in [5], which can also be found in Krasner [6] and in Krasner-Vuković [9]. A generalisation of graded scructures is provided by the notion of *paragraduation*, which was introduced by Krasner and Vuković in a series of papers, starting with [7], further work in [8], [10] and others. The advantage of this generalisation is that the category of paragraded structures is closed under direct and Cartesian products preserving the homogeneous parts, which is not the case with the

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classical graded structures. In further work Vuković and her collaborators, including Ilić-Georgijević, (for example [3]) found many other nice algebraic properties of the category of paragraded structures. It is natural then to ask what kinds of new examples this category provides compared with the classical case of the graded structures, apart from the ones coming from the products of graded structures. In this paper we use methods of mathematical logic to give some new examples of paragraded structures and show the surprising result that at certain cardinals all first order structures are paragraded. We also ask the natural question of when two paragraded structures are basically the same, introducing the notion of bi-embeddability to measure this. Finally, to stress the difference between the graded and the paragraded context, we show that many graded substructures can exists as substructures of a given paragraded one.

We also investigate a question of the similarity of the limiting structures of two paragraded systems, proving that the bi-embeddability of the system gives rise to the bi-embeddability of the limiting structures. We consider the question of when the converse is true and provide several answers. This is done in §3. In §4 we show that any first order structure whose size is an uncountable regular cardinal is paragraded, using elementary chains, hence showing that the main interest of paragraded structures rests in the context of the countable. In §5 we investigate the question of the graded substructures of a given paragraded structure and give an example of one paragraded structure which has $2^{\aleph_0}$ graded substructures, hence a number not even determined by the usual axioms of set theory.

All the basic definitions needed in the paper are recalled in §2. Although much of what we say can be applied to any category of paragraded algebraic structures, for clarity of presentation we limit ourselves to groups.

2. Background : the definition of a paragraded structure and elementary chains

Many readers of this volume, to a significant extent devoted to the memory of Prof. Krasner, will certainly be already familar with the idea of a paragraded structure. For the convenience of the remaining readers, we choose to repeat the definition here. For simplicity, we only give the definition in the case of groups. Our notation for groups is multiplicative. For x, y which belong to a group G, by $z(x, y)$ we denote the commutator $yxy^{-1}x^{-1}$ (also often denoted by $[x, y]$ in the literature).

Definition 2.1. *Let* $(\Delta, \leq)$ *be a partially ordered set such that for any* $\Delta' \subseteq \Delta$ *there is an infimum, denoted* $\inf \Delta'$*, and such that any non* *co-final chain $\mathcal{C} \subseteq \Delta$ has an upper bound in Δ (note [1]). We let the minimal element of Δ be denoted by 0_Δ. A* paragraduation *of a group G by the order Δ is a system of subgroups $\{G_\delta : \delta \in \Delta\}$ such that:*

(1) *$G_{0_\Delta} = \{e\}$, $\delta \leq \delta' \implies G_\delta \subseteq G_\delta$,*
(2) *each G_δ is a normal subgroup of G,*
(3) *if $\Delta' \subseteq \Delta$ then $G_{\inf \Delta'} = \bigcap_{\delta \in \Delta'} G_\delta$,*
(4) *denoting $H = \bigcup_{\delta \in \Delta} G_\delta$, which is called the set of homogeneous elements, we have that for every $A \subset G$ (proper subset) satisfying that $x, y \in A \implies xy \in H$, there exist $\delta \in \Delta$ such that $A \subseteq G_\delta$,*
(5) *H generates G by the relations $xy = z$ and $xy = z(x,y)yx$ where $z(x,y)$ is the commutator of x and y.*

We say that G is the limiting structure *of the paragraduation $\langle G_\delta : \delta \in \Delta \rangle$.*

A point that we shall make in Theorem 4.1 is that the familiar notion of an elementary chain from model theory gives a paragraduation, hence we pause to give the required definitions. For the readers not familiar with elementary submodels, the idea is that a submodel $\mathfrak{A}$ of a model $\mathfrak{B}$ is elementary if it solves all the equations with the parameters in $\mathfrak{A}$ which are solvable in $\mathfrak{B}$.

Definition 2.2. *(1) Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are models of the same language $\mathcal{L}$ and $\mathfrak{A} \subseteq \mathfrak{B}$. Then $\mathfrak{A}$ is an* elementary submodel of $\mathfrak{B}$, *written $\mathfrak{A} \prec \mathfrak{B}$, if for all formulas $\varphi(\bar{x})$ of $\mathcal{L}$ (possibly with parameters) and all elements $\bar{a}$ of $\mathfrak{A}$ such that $\varphi(\bar{a})$ is defined, we have $\mathfrak{A} \models \varphi(\bar{a})$ iff $\mathfrak{B} \models \varphi(\bar{a})$.*
(2) An elementary chain *is a sequence $\langle \mathfrak{A}_\alpha : \alpha < \alpha^* \rangle$ of models indexed by some ordinal α^* and such that for each α we have $\mathfrak{A}_\alpha \prec \mathfrak{A}_{\alpha+1}$.*

Finally, we recall the notion of a club subset of an ordinal.

Definition 2.3. *Suppose that $C \subseteq \alpha$ for some ordinal α. The set C is said to be* closed *if it contains all its limit points in the order topology of α, which means that for any $\gamma < \alpha$, if $\sup(C \cap \gamma) = \gamma$, then $\gamma \in C$. The set C is said to be* unbounded *if for every $\gamma < \alpha$ there is γ' with $\gamma < \gamma' < \alpha$ and $\gamma' \in C$. Finally, the set C is said to be* club *if it is both closed and unbounded.*

3. Bi-embeddability

Since our aim is to seearch for novel examples of paragraded structures, it is of interest to know when two paragraded structures are

[1]we wish to be able to study the case when Δ itself does not have the maximal element, hence we only require the non-cofinal chains to be bounded, not all the chains

essentially the same, even if they may be given by different paragraduations. Finding the right notion capturing this is the purpose of this section.

We start by a given graded structure $\mathfrak{A}$ (it could be a group, a ring, an algebra ...) with a gradation $\mathfrak{A}_i\,(i \in \mathbb{I})$ and we explore the question of how many essentially different paragraded structures we can construct as substructures of $\mathfrak{A}$. For purposes of readability, we shall first consider the simplest graduation, that one by $\mathbb{N}$ and the simplest graded structure, that of a graded group. The ideas presented here can easily be extended to more complex gradations and more involved algebraic structures.

Example 3.1. *Suppose that $\langle G_i : i \in \mathbb{N}\rangle$ is a graduation with the limiting group G. Let $\Delta \subseteq \mathbb{N}$ be any subset which is closed under $+$ and $\cdot$. Then, clearly, the same gradation used to construct G induces a graded structure whose homogeneous part is $\bigcup_{i\in\Delta} G_i$ and which is generated inside of G by this homogeneous part. To satisfy the definition of a graded structure we had to restrict our restriction to subsets closed under $+$ and $\cdot$. The definition of a paragraded structure (see §2) lets us exit this paradigm and concentrate on the order structure of $\mathbb{N}$, so just the underlying set ω and the natural order $\leq$ on it. For example, the chain $\langle G_{2i}\,(i \in \mathbb{N})\rangle$ gives a graduation of G, but the chain $G_0, G_{2i+1}\,(i \in \mathbb{N})$ does not. However the latter structure does give rise to a paragraduation.*

The process from Example 3.1, though, feels far from the desired generalisation since the homogeneous elements of the two paragraduations $\bigcup_{i\in\mathbb{N}} G_{2i}$ and $G_0 \cup \bigcup_{i\in\mathbb{N}} G_{2i+1}$ will generate the same group, namely G. As paragraded structures, the two chains allow homomorphic embeddings[2] into each other, so are basically the same. This example leads us to refine what we consider as essentially different paragraded structures, as we now define.

Definition 3.2. *(1) Two structures $\mathfrak{A}$ and $\mathfrak{B}$ of the same type (so in particular, groups, rings. etc.) are said to be* bi-emeddable *if there is an isomorphic embedding from $\mathfrak{A}$ to $\mathfrak{B}$ and an isomorphic embedding from $\mathfrak{B}$ to $\mathfrak{A}$.*
(2) Suppose that $\mathfrak{A}_i\,(i \in \mathbb{I})$ and $\mathfrak{B}_j\,(j \in \mathbb{J})$ are paragraduations of two structures of the same type (a group, a ring, an algebra ...). We say that the two paragraduations are bi-emeddable *if for all $i \in \mathbb{I}$ there is $j \in \mathbb{J}$ and a homomorphism $\varphi_{i,j}$ from $\mathfrak{A}_i$ to $\mathfrak{B}_j$, and vice versa, or all $j \in \mathbb{J}$ there is $i \in \mathbb{I}$ and a homomorphism $\psi_{j,i}$ from $\mathfrak{B}_j$ to $\mathfrak{A}_i$, and moreover the homomorphisms are such that $i \leq_{\mathbb{I}} i'$, $\implies$ $\varphi_{i,j} \subseteq \varphi'_{i',j}$ for any fixed j and $j \leq_{\mathbb{J}} j' \implies \psi_{j,i} \subseteq \psi_{j',i}$ for any fixed i.*

[2]following the tradition in model theory, what is called homomorphism in algebra, will from this point be called isomorphic embedding and is applicable to other structures but to those coming from algebra.

(3) Two paragraded structures of the same type are said to be essentially different *if they are not bi-emebeddable.*

We see in the above simple example of odd versus even paragraduation of a paragraded group, that the two paragraduations are bi-emeddable. Here is an observation which shows that the limiting structures of two bi-emeddable paragraded structures are also bi-embeddable.

Theorem 3.3. *Suppose that $\mathfrak{A}_i\,(i \in \mathbb{I})$ and $\mathfrak{B}_j\,(j \in \mathbb{J})$ are two bi-embeddable paragraded groups, rings or algebras, respectively. Then the limiting groups (respectively rings, algebras) generated by the homogeneous elements of the two paragraduations are also bi-embeddable.*

Proof. Let us again concentrate on the case of paragraded groups, say $G_\mathbb{I}$ and $G_\mathbb{J}$.

In the forward direction, for $i \in \mathbb{I}$ and $j \in \mathbb{J}$ let $\varphi_{i,j}$ and $\psi_{j,i}$ be the embeddings witnessing that Definition 3.2(2) is satisfied, which exist by the assumption of the bi-embeddability of the paragraduations. Let $H_\mathbb{I}$ and $H_\mathbb{J}$ denote the sets of homogeneous elements of the two paragraduations and $G_\mathbb{I}$ and $G_\mathbb{J}$ the paragraded groups generated by $H_\mathbb{I}$ and $H_\mathbb{J}$ respectively. To produce an isomorphic embedding φ from $G_\mathbb{I}$ to $G_\mathbb{J}$ it then suffices to obtain the restriction of this embedding on $H_\mathbb{I}$, and then to extend it using the equations used to generate the paragraded groups, and vice versa. An isomorphic embeddding from $H_\mathbb{I}$ to $H_\mathbb{J}$ is given by letting $\varphi(x) = y$ if there exists $i \in \mathbb{I}$ and $j \in \mathbb{J}$ such that $\varphi_{i,j}(x) = y$. Let us check that this is indeed a well-defined isomorphic embedding.

If $x \in H_\mathbb{I}$, then there exist $i \in \mathbb{I}$ such that $x \in H_i$, and therefore by the choice of the bi-embedability witnesses, there exists $j \in \mathbb{J}$ such that $\varphi_{i,j}(x)$ is defined. Moreover the value of $\varphi_{i,j}(x)$ does not depend on j, and hence the function φ is well-defined on the entire $H_\mathbb{I}$. We can similarly check that this function is an isomorphic embedding, since every equation of the type $x \cdot y = z$ in $H_\mathbb{I}$ is present already in H_i for some i, and therefore $\varphi(x) \cdot \varphi(y) = \varphi(z)$ since the analogue is true for $\varphi_{i,j}$, for any j.

We can similarly define an isomorphic embedding ψ from $H_\mathbb{J}$ to $H_\mathbb{I}$. $\bigstar_{3.3}$ $\square$

We can then ask if the limiting structures of two bi-embeddable paragraduations are in fact isomorphic. In some categories, such as that of pure sets with injection (by the Schroeder-Bernstein theorem) and that of countable torsion groups (see [4] for a proof) the notions of bi-embeddability and the isomorphism coincide, but in general bi-embedability is weaker than isomorphism, for example in the category of groups. So our example 3.1 is to some extent misleading, as it gave us isomorphic limiting structures, an upgrade of the situation promised in Theorem 3.3.

Another question is if the converse of Theorem 3.3 is true: do two bi-emebddable paragraded groups have bi-embeddable paragraduations? Note that given such two groups say $G_{\mathbb{I}}$ and $G_{\mathbb{J}}$ with **given** paragraduations $G_i\,(i \in \mathbb{I})$ and $G_j\,(j \in \mathbb{J})$, even if we suppose that $G_{\mathbb{I}}$ and $G_{\mathbb{J}}$ are actually isomorphic, say with an isomorphism φ, there is no reason to think that φ will carry the homogeneous elements of $G_i\,(i \in \mathbb{I})$ to the homogeneous elements of $G_j\,(j \in \mathbb{J})$. Therefore, we shall not be able to use the isomorphism to give us bi-embeddability between the paragraduations. Of course, an easy partial converse to Theorem 3.3 is provided by the following basic observation:

Lemma 3.4. *Suppose that $G_{\mathbb{I}}$ is a paragraded group with a given paragraduation $G_i\,(i \in \mathbb{I})$ and that $G_{\mathbb{I}}$ is isomorphic to a group G'. Then G' is paragraded and has a paragraduation $G'_i\,(i \in \mathbb{I})$ with each G'_i isomorphic with the corresponding G_i.*

Proof. Let φ be an isomorphism from $G_{\mathbb{I}}$ to G'. Define for $i \in \mathbb{I}$ the group G'_i as the image under φ of G_i; it is easily seen that this gives us a paragraduation as required. $\bigstar_{3.4}$ $\square$

A more interesting version of Lemma 3.4 shows that a similar conclusion can be made with the assumption of embeddability.

Lemma 3.5. *Suppose that $G_{\mathbb{I}}$ is a paragraded group with a given paragraduation $G_i\,(i \in \mathbb{I})$ and that $G_{\mathbb{I}}$ is isomorphically embeddable in a group G'. Then G' has a paragraded subgroup with a paragraduation $G'_i\,(i \in \mathbb{I})$ such that $G_i\,(i \in \mathbb{I})$ is element-wise embeddable in $G'_i\,(i \in \mathbb{I})$, in the sense that each G_i isomorphically embeds into the corresponding G'_i.*

Proof. Let φ be an isomorphic embedding from $G_{\mathbb{I}}$ to G' . Define for $i \in \mathbb{I}$ the group G'_i as the image under φ of G_i. Note that $G'_0 = \{e\}$, since φ is a homomorphism. It follows that for each G'_i is a normal subgroup of G', as each G_i is closed under commutators. It is then easily seen that $G'_i\,(i \in \mathbb{I})$ is a paragraduation of some subgroup G'' of G' and that it is as required, as φ embeds G_i into G'_i. $\bigstar_{3.5}$ $\square$

The corresponding versions of Lemma 3.4 and 3.5 for bi-embeddability seem to be more interesting and to depend on the underlying order $\mathbb{I}$. We shall show in §4 that the analogue holds in the natural case when $\mathbb{I}$ is an uncountable ordinal and the given paragraduation is actually a filtration.

In conclusion of this section, we see that the results obtained justify the name "essentially different" from Definition 3.2(3) because two essentially different paragraduation are not going to have the same limiting structure, at least in a many cases, while the essentially same paragraduations lead to essentially same limiting structures.

4. **Elementary chains**

It turns out that there is an example of paragraduations that is ubiquitous in model theory, as we now show.

Theorem 4.1. *Suppose that λ is a regular uncountable cardinal and $\langle G_\alpha : \alpha < \lambda\rangle$ and $\langle G'_\alpha : \alpha < \lambda\rangle$ are two filtrations of the same group G of size λ such that $G_0 = G'_0 = \{e_G\}$. Then:*

(1) each of the filtrations gives a paragraduation of G and
(2) there is a club C of λ such that for every α in C we have $G_\alpha = G'_\alpha$.

In particular, the two paragraduations are bi-embeddable.

Proof. (1) The ordinal λ certainly satisfies the properties required from the partial order Δ in the Definition 2.1, since it is a linear order where every non-empty subset has a minimal element and every non-cofinal subset is bounded from the above. Since every G_α and G'_α is an elementary substructure of G, it certainly is a normal subgroup (or closed under the relevant properties, as the case may be for the structure in question). The rest of the properties are easy to check.

(2) This follows a well known method (see [2]) from the theory of elementary models and can be obtained by closing under functions described by taking $f(\alpha)$ for $\alpha < \lambda$ to be the minimal β such that $G_\alpha \subseteq G'_\beta$ and $g(\alpha)$ to be the minimal β such that $G'_\alpha \subseteq G_\beta$. Note that both f and g are well defined, by the regularity of λ (which simply means, by definition, that λ has no cofinal subsets of size $< \lambda$). In other words, let C_0 be the set of ordinals $\alpha < \lambda$ such that for every $\gamma < \alpha$ we have $f(\gamma), g(\gamma) < \alpha$. To show that this set is closed, suppose that $\alpha < \lambda$ is a limit of ordinals in C_0 and that $\gamma < \lambda$. Then there is β with $\gamma < \beta < \alpha$ and $\beta \in C_0$. Therefore $f(\gamma), g(\gamma) < \beta < \alpha$. To check that the set C_0 is unbounded in λ, start with any $\alpha < \lambda$. Let $\alpha_0 = \sup\{f(\beta), g(\beta) : \beta < \alpha\} \cup \{\alpha\}$. By the regularity of λ, we have that $\alpha_0 < \lambda$. We continue this definition by induction on $n < \omega$, defining $\alpha_{n+1} = \sup\{f(\beta), g(\beta) : \beta < \alpha_n\}$, hence each $\alpha_n < \lambda$ by the same argument. At the end let us take $\alpha^* = \sup_{n<\omega} \alpha_n$ and then observe that α^* is closed under f and g by construction and that $\alpha^* < \lambda$ since λ is regular uncountable. Hence $\alpha^* \geq \alpha$ and $\alpha^* \in C_0$.

Now let C be the set of limit points in C_0, so those $\alpha \in C_0$ satisfying that $\alpha = \sup(C_0 \cap \alpha)$. It is well known (see [11] or check by hand) that C is still a club of λ. Now it is easy to verify that C satisfies that for every α in C we have $G_\alpha = G'_\alpha$. Namely, if $\alpha \in C$ and $x \in G_\alpha$, then there is $\beta < \alpha$ such that $x \in G_\beta$, as α is a limit ordinal, so $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. Therefore $f(\beta) < \alpha$ and hence $x \in G'_\beta \subseteq G'_\alpha$. Therefore $G_\alpha \subseteq G'_\alpha$ and the other direction is symmetric, so $G_\alpha = G'_\alpha$ $\bigstar_{4.1}$ □

It is interesting to note that any first order structure whose size is an infinite regular cardinal λ allows for a filtration (see again [2]) and hence by Theorem 4.1, it can be seen as a paragraded structure. Theorem 4.1 really shows to a logician what the interest is in paragraded structures: they are mostly interesting for *countable* structures or structures whose cardinality has countable cofinality. At such cardinals we do not have the general analog of the combinatorics of club sets and filtrations and hence paragraded structures provide us with a version of filtrations, at least for algebraic structures such as groups, rings and modules!

We note another application of filtrations which gives a theorem that was announced at the end of §3.

Theorem 4.2. *Suppose that λ is an uncountable regular cardinal, G is a paragraded group with a filtration $\langle G_\alpha : \alpha < \lambda \rangle$ and that H is a group bi-embeddable with G. Then H has a filtration that is bi-embeddable with $\langle G_\alpha : \alpha < \lambda \rangle$.*

Proof. Let $\varphi : G \to H$ and $\psi : H \to G$ be isomorphic embeddings. Let us first note that this implies that $|H| = \lambda$. Indeed the image $\varphi'' G$ of G is a subset of H of size λ, hence $H \geq \lambda$, but also $|H| = |\psi'' H| \subseteq G$ and hence $|H| \leq |G| \leq \lambda$. Let $\langle H_\alpha : \alpha < \lambda \rangle$ be any filtration of H. Now for each $\alpha < \lambda$ we can find $\beta_\alpha < \lambda$ such that $\varphi'' G_\alpha \subseteq H_{\beta_\alpha}$, and we hence define for $\beta \geq \beta_\alpha$ the embedding $\varphi_{\alpha,\beta} = \varphi \restriction G_\alpha$. We can similarly define the embeddings $\psi_{\alpha,\gamma}$ for large enough γ, going from H_α to G_γ. It is clear that these embeddings provide bi-embeddability between the paragraduations. $\bigstar_{4.2}$ $\square$

5. Paragraduation versus graduation

We finish the paper by an example which shows that one paragraduation can lead to a large number of graded substructures and which also shows a context in which it is much more natural to speak of paragraduation than of graduation. This example also shows that the number of graded substructures of a given paragraded structure cannot always be decided by the axioms of set theory, even in the case of countable structures.

An example of a non-linear partial order which satisfies the requirement of Definition 2.1 is the infinite binary tree ${}^{<\omega}2$ of finite sequences of 0s and 1s, ordered by $s \leq t$ if s is an initial segment of t, denoted by $s \trianglelefteq t$. We shall use this example to obtain an example of a paragraded structure which is not graded. By recursion on the length of $t \in {}^{<\omega}2$, we define the group H_t.

Let $H_\emptyset$ be the trivial group $\{e\}$. Given H_s for all $s \triangleleft t$, let H_t be the abelian group freely finitely generated by $\bigcup_{s \triangleleft t} H_s$ except for the equations $s \cdot t = t$ for $s \triangleleft t$. It is clear that $\langle H_t : t \in {}^{<\omega}2 \rangle$ is a paragraded system of groups which gives rise to a limiting paragraded group, say H.

The paragraded group H does not give rise to a natural graded structure, yet it has many graded subgroups. In particular, every infinite branch ρ of ${}^{<\omega}2$ gives rise to a graded subgroup H_ρ of H given by the gradation $\langle H_t : t \triangleleft \rho \rangle$. We recall that the number of infinite branches of ${}^{<\omega}2$ is $2^{\aleph_0}$, whose value in terms of the $\aleph$-hierarchy is not decidable in the ordinary set theory ZFC, by the well known work of Cohen (see [11]). As a corollary we obtain the following theorem:

Theorem 5.1. *The number of graded substructures of a given paragraded structure in general is not decidable in ZFC.*

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