FORCING AXIOMS, FINITE CONDITIONS AND SOME MORE

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Abstract. We survey some classical and some recent results in the theory of forcing axioms, aiming to present recent breakthroughs and interest the reader in further developing the theory. The article is written for an audience of logicians and mathematicians not necessarily familiar with set theory.

0. Introduction

We shall work within the axioms of the Zermelo-Fraenkel set theory with Choice (ZFC). These axioms were introduced basically starting from 1908 and improving to a final version in the 1920s as an attempt to axiomatize the foundations of mathematics. There have been other such attempts at about the same time and later, but it is fair to say that for the purposes of much of modern mathematics the axioms of ZFC represent the accepted foundation (see [13] for a detailed discussion of foundational issues in set theory). Gödel's Incompleteness theorems [16] prove that for any consistent theory T which implies the Peano Axioms and whose axioms are presentable as a recursively enumerable set of sentences, so for any reasonable theory one would say, there is a sentence φ in the language of T such that T does not prove or disprove φ. In some sense the discussion of which axioms to use is made less interesting by these theorems, which can be interpreted as saying that a perfect choice of axioms does not exists. We therefore do like the most, we concentrate on the axioms that correctly model most of mathematics, and for the rest, we try to understand the limits and how we can improve them. For us ZFC is a basis for a foundation which in some circumstances can be extended to a larger set of axioms which provide an insight into various parts of mathematics. In here we concentrate on the forcing axioms (and their negations).

Key words and phrases. forcing, proper, semiproper, iteration, support.

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1. The discovery of forcing

The proof of Gödels' Incompleteness theorems is not constructive and in particular it does not construct an independent sentence φ, it only proves its existence. It is therefore quite amazing that for the theory of ZFC such a sentence φ turned out to be the following simple statement formulated by Cantor as early as 1878 [8] (as an implicite conjecture only at that point):

Continuum Hypothesis (CH): For every infinite subset A of the reals $\mathbb{R}$, either there is a bijection between A and $\mathbb{R}$ or there is a bijection between A and $\mathbb{N}$.

This statement tormented Cantor, who could not prove it or disprove it. With a good reason, since it was finally proved by Cohen in [9] that if ZFC is consistent then so is ZFC with the negation of CH. Since Gödel [17] had proved that if ZFC is consistent then so is ZFC along with CH, it follows that CH is independent of ZFC. To obtain his proof Cohen introduced the technique of forcing. It is a technique to extend a universe $\mathbf{V}$ of set theory to another one, $\mathbf{V}[G]$, so that $\mathbf{V}[G]$:

- has the same ordinals
- (most often) has the same cardinals and
- satisfies a desired formula ϕ.

One way to think of this technique is to imagine that we are actually working within some large ambient model of ZFC and seeing only a small submodel which we call $\mathbf{V}$. This submodel may even be assumed to be countable. Being so small, $\mathbf{V}$ has a rather particular opinion of the reality, for example it esteems that every infinite cardinal $\aleph_\alpha$ is some ordinal $\beta(\aleph_\alpha)$ among the ordinals β that actually belong to $\mathbf{V}$ (we denote this by $\aleph_\alpha^{\mathbf{V}}$). For Cohen's proof we may also assume that $\mathbf{V}$ satisfies CH- since if it does not we have already violated CH. What we aim to do is to extend $\mathbf{V}$ to a larger model which will contain $\aleph_2^{\mathbf{V}}$ many reals from our ambient universe, while $\mathbf{V}[G]$ and $\mathbf{V}$ will actually agree on their opinion of what is $\aleph_1$ and $\aleph_2$ (they will have the same cardinals). Then in $\mathbf{V}[G]$ we can choose any set A of only $\aleph_1^{\mathbf{V}}$ reals to demonstrate that A is not bijective with either $\mathbb{N}$ or $\mathbb{R}$, hence CH fails. This construction rests upon a combinatorial method which adds these new reals while preserving the cardinals. We may imagine this as a sort of inductive construction, but one in which the desired object is not added using a linearly ordered set of approximations but rather a partially ordered set. For example, thinking of a real as a function from ω to 2 (as there is a bijection between $\mathbb{R}$ and $\mathcal{P}(\omega)$), we may add a real by considering the partial order of finite partial functions from ω to 2 in their increasing order, some coherent subset of which will be glued together to give us a total function from ω to 2. The coherent subset is our G, the generic filter. The fact that such a subset can be chosen is one of the main ingredients of the method. The actual proof of the negation of CH requires us to work with functions from $\omega_2 \times \omega$ to 2, but the idea is the same.

Partial orders considered in the theory of forcing have the property of having the smallest element and are often called *forcing notions*. Elements of a forcing notion are usually called *conditions*. As we are looking for coherent subsets of a forcing notion, an important point is to consider for given two conditions if they are coherent, which means that they have a common extension. We say that conditions having such an extension are *compatible*, otherwise they are *incompatible*. A set of conditions is called *an antichain* if it consists of pairwise incompatible conditions.[1] The moral opposite of an antichain is a *filter*, which is a set in which every two conditions are compatible, moreover with a common extension in the filter itself. We also assume that filters are closed under weakenings of the conditions within the filter. The generic then is a very special kind of filter.

We digress to say that many authors consider forcing notions as partial orders which have the largest element and in which smaller elements give more information than the larger ones. The intuition may be that at the beginning we have a misty view of what our generic object is going to be, and that with every stronger condition we clear the myst and restrict the vision to a smaller relevant part, leading in the end to a single object which is the generic. The former approach was used by Cohen who discovered forcing, the latter was used by Solovay who quickly took over from Cohen to become a leading figure of set theory for many years. The two approaches are obviously equivalent, here we shall use the former one.

Cohen's discovery led to a large number of independence results, leading to many mathematical and philosophical developments. We shall concentrate on the mathematical ones. In particular we shall discuss the notion of a forcing axiom and the related concept of iterations of forcing. We shall often discuss the situation of relative consistency of a statement φ, that is to say the situation that if ZFC is consistent then so is the conjunction of ZFC and our statement φ. We shorten this description by saying "φ is consistent", and leave it to the reader to remember that this in fact only relates to relative consistency.

[1]Note that the notion of an antichain here differs from that one in the theory of order, where the conditions in an antichain are simply required to be pairwise incomparable.

2. Iterated Forcing and Martin's Axiom

We have described in §1 that CH is independent of ZFC, but it turns out that so are various other statements coming from a large number of fields of mathematics. For example, the statement that every ccc[2] Boolean algebra of size less than the continuum supports a measure and that CH fails, is consistent with ZFC. Call this statement $\mathcal{BA}$. To prove this we need more than just a single step forcing extension described in the introduction and used for the failure of CH. The reader may imagine trying to prove this statement by going through some list of "small" ccc Boolean algebras and generically adding a measure to each of them. So we need to iterate the method described in the introduction. Only, this is not completely trivial as it can be shown that if we proceed naively, taking one generic after another and unions at the limit, already after ω many steps we shall no longer have a model of ZFC. Another issue which is more subtle is that even if we manage to preserve ZFC, it is easy to destroy the cardinals, in the sense that our final model will have learned that what $\mathbf{V}$ in its restricted opinion had considered to be cardinals, in fact are just bare small ordinals. For example, it might have learned that the ω_1 of $\mathbf{V}$ is countable, so the final theorem will not be about ω_1. A forcing which does not put us in this situation is said to *preserve cardinals*. An example of such a forcing is the so called *ccc* forcing, where the word ccc is used to denote the fact that every antichain is countable.

The way out of these difficulties is the iterated forcing. We shall not describe it in detail, but we may imagine it as a huge forcing notion consisting of sequences of elements where each coordinate corresponds to a name for an element of a forcing notion, not in $\mathbf{V}$ but in some extension of it intermediate between $\mathbf{V}$ and $\mathbf{V}[G]$. Every sequence in this object has a set of nontrivial coordinates, which is to say places in where it is not equal to the trivial smallest element of the corresponding forcing notion. This set is called the *support* of a condition. A major advance in the theory was the following theorem:

Theorem 2.1. *(Martin and Solovay) An iteration with finite supports of ccc forcing is ccc.*

The history of this theorem is that Solovay and Tennebaum in [31] proved the consistency of the Souslin Hypothesis (i.e. there are no Souslin lines, meaning that the reals are characterised by being a complete order dense set with no first or last element in which every family

[2]this means that every family of pairwise disjoint elements in the algebra is countable

of pairwise disjoint sets is countable) using the technique of iterating a certain forcing. Their original result was an iteration of the specific forcing destroying Souslin lines. However, Martin realised that their technique could be extended to prove Theorem 2.1 and introduced the Martin's Axiom (see below), which appears in Section 6 of [31] and the paper [22] by Martin and Solovay. The point is that iterated forcing is quite a complex technique and there is no reason to expect that a mathematician not working in set theory but interested in the possible independence of some concrete statement in mathematics should be learning the technique of iterated forcing, or of course that a set theorist will be able to work in any given part of mathematics to answer the question- although many examples of both the former and the latter are known in the literature. However, this is exactly where the forcing axioms come in, and the first one was discovered thanks to Theorem 2.1. It is *Martin's Axiom*:

Martin's Axiom (MA) : For every ccc forcing notion $\mathbb{P}$ and every family $\mathcal{F}$ of $< \mathfrak{c}$ many dense sets in $\mathbb{P}$, there is a filter in $\mathbb{P}$ which intersects all elements of $\mathcal{F}$.

A *dense set* in a forcing notion is a subset such that every condition in the forcing notion has an extension in the dense set. Intersecting dense sets is what corresponds to genericity, because in fact the definition of a *generic* filter (over $\mathbf{V}$) is that it intersects every dense set which is in $\mathbf{V}$. Martin's Axiom actually follows from CH, as can be proved by induction, but the point is that it is also consistent with the negation of CH, in fact with CH being as large as we wish. This has had far reaching consequences. The point is that formulating this axiom separates the two parts of the technique of forcing: the logical one and the combinatorial one, to the extent that the "end user" of this axiom does not need to know anything about logic, it suffices to concentrate on the combinatorics involved. This in fact is often not too difficult and a careful reader of this article could easily now go away and prove for himself the consistency of the statement $\mathcal{BA}$ above, simply using MA + $\neg$ CH. This is why MA has had a large success in the mathematical community and a large number of independence results were obtained using it. Many of these developments are documented in Fremlin's book [19], and new developments come up regularly.

3. Beyond ccc

Many nice forcing notions are not ccc. An example is *Sacks forcing* which adds a real of a minimal Turing degree [27]. In fact, some very natural statements in mathematics are known to be independent of set theory but it is also known that these independence results cannot be shown using MA. An example is the following: say that a subset A of the reals is $\aleph_1$-dense if for every $a < b$ in A there are $\aleph_1$-many reals in the intersection $A \cap (a, b)$. Baumgartner proved in [4] that it is consistent that every two such sets are order isomorphic and CH fails. Yet, Abraham and Shelah [3] proved that this statement does not follow from MA+$\neg$ CH. Baumgartner's proof uses PFA, the proper forcing axiom. Properness is a more general notion than that of ccc and is expressed in a less combinatorial way. It was invented by Shelah in the 1980s (see [30] for the majority of references relating to proper forcing in this section), in a response to a growing need of the set theorists to have an iterable notion of forcing which preserves cardinals (or at least $\aleph_1$) and is not necessarily ccc.

The history of this development is that Laver showed in [21] the consistency of the Borel conjecture, which postulated that all sets of reals which have strong measure zero are countable. Laver showed that this is the case in a model obtained by adding $\aleph_2$ many Laver reals to a model of GCH, using countable supports in the iteration. It is exactly this notion of a countable support that is used in the proper forcing axiom. Namely,

Theorem 3.1. *(Shelah) An iteration with countable supports of proper forcing is proper.*

Shelah also showed that Theorem 3.1 is not true when countable supports are replaced by the finite ones. It should be noted that Laver's forcing also inspired the notion of Property A, introduced by Baumgartner [5], which is a notion implied by ccc and implying properness, and which is iterable using countable support. Proper forcing is a more general notion and hence more useful. Shelah's iteration theorem is a step in Baumgartner's proof [5] that from the assumption of the consistency of the existence of one supercompact cardinal, one can prove the consistency of a forcing axiom he formulated, the proper forcing axiom PFA. PFA is the statement

Proper Forcing Axiom (PFA) : For every proper forcing notion $\mathbb{P}$ and every family $\mathcal{F}$ of $\aleph_1$ many dense sets in $\mathbb{P}$, there is a filter in $\mathbb{P}$ which intersects all elements of $\mathcal{F}$.

A careful reader may wonder why in the formulation of Martin's Axiom we have the possibility to use $< \mathfrak{c}$ many dense sets and in the PFA we can only use $\aleph_1$ many. In fact, the two boil down to same, since Veličković and Todorčević proved in [34] and [7] that PFA implies $\mathfrak{c} = \aleph_2$. Proper forcing has many applications, in the set theory of the reals, combinatorial properties of ω_1 and topology, algebra and analysis.

Many of these developments can be found in Shelah's monumental book mentioned above [30] and Baumgartner's article [6]. Let us give a definition of properness for those readers who would like to see the details. It is a somewhat complicated definition, involving the use of elementary models - which exactly was behind the revolution that it has created, as it was a totally new way of looking at forcing.

Let χ be a regular cardinal, which means that $\mathcal{H}(\chi)$[3] models all the (finitely) many instances of the axioms of set theory that will be used in our argument and that it contains all objects we need[4].

Definition 3.2. *Let $\mathbb{P}$ be a forcing notion and suppose that N is a countable elementary submodel of $\mathcal{H}(\chi)$ such that $\mathbb{P} \in N$. We say that $q \in \mathbb{P}$ is an* $(N, \mathbb{P})$-generic *condition if for every dense subset $\mathcal{D}$ of $\mathbb{P}$ with $\mathcal{D} \in N$, we have that $\mathcal{D} \cap N$ is dense above q.*

$\mathbb{P}$ is said to be proper *if for every N and p as above with $p \in N$, there is $q \geq p$ which is $(N, \mathbb{P})$-generic.*

The interested reader should plan to spend a few happy hours reading either [30] or [5], where excellent introductions are given. Good questions to test the understanding of the topic is to prove that every ccc forcing is proper and that proper forcing preserves ω_1.

4. Away from properness

Proper forcing preserves ω_1 and the stationary subsets of it, but it is not the only forcing with these preservation properties. For example, the Prikry forcing changing the cofinality of a measurable cardinal to ω preserves stationary subsets of ω_1 and is not proper. Shelah (see [30]) defined a larger class of forcing, the semiproper forcing, which does preserve ω_1 and includes the class of proper forcing and the Prikry forcing. Then one can formulate the semiproper forcing axiom SPFA in a similar way as PFA and prove it consistent from a supercompact cardinal using a proof simillar to that of Baumgartner for PFA. With one major difference: the iteration has to be done with the so called

[3]this is the set of all sets whose transitive closure under $\in$ has size $< \chi$

[4]The reader should recall that there are infinitely many axioms of set theory, as the Axiom of Replacement and the Axiom of Comprehension are actually infinite axioms schemes, giving one axiom for each formula of set theory. Any given argument will only use finitely many axioms and for any such finite portion ZFC* of ZFC there is a χ such that $\mathcal{H}(\chi)$ models ZFC*. The way to think of the argument to come is similar to the choice of ϵ in the continuity arguments in analysis: we know the argument works independently of the choice of ϵ, or χ in our case, and we know that the ϵ could have been chosen so that it works for the situation in question- hence the proof follows.

revised countable support. This is quite an involved concept, going way beyond the scope of this article. Foreman, Magidor and Shelah studied in [12] a similar forcing axiom, where the entries are all forcings that preserve stationary subsets of ω_1, so it is called the Martin Maximum. In fact, it turns out that SPFA and MM are the same axiom, as was proved by Shelah in [29].

The word Maximum in some sense also indicates the outlook of the subject after the invention of MM. Clearly, stronger axioms than MM could not be invented, or at least not in an obvious way, and hence research concentrated for some years on studying the weakenings of these axioms, especially weakenings of PFA. Some of the popular choices are OCA, the open colouring axiom (it has two versions, introduced respectively by Abraham, Rubin and Shelah in [1] and Todorčević in [33]), BPFA, which is the bounded PFA, introduced by Goldstern and Shelah in [18], MPR introduced by Moore in [24] and, of particular interest, PID, the P-ideal dichotomy which has the surprising feature to be consistent with the continuum hypothesis, as shown by Abraham and Todorčević in [2]. The interest of these axioms is exactly in their relative weakness, as they allow us to get the relative consistency of statements that are seemingly contradictory, such as CH and certain consequences of PID.

Another direction suggesting itself here are the forcing axioms on cardinals above ω_1. Generalised Martin Axioms were developed by Baumgartner in [5] and Shelah in [28] and their basic form is that they apply to κ^+ for some cardinal κ satisfying $\kappa = \kappa^{<\kappa}$ and to forcing satisfying some strong version of the κ^+-chain condiion (the ordinary version won't do), being $(< \kappa)$-closed[5] or some similar condition, and some condition like well-metness: every two compatible conditions have the least upper bound. The version of the κ^+-chain condition appearing in Shelah's work is called stationary κ^+-cc. The consistency of these axioms is proved by a proof similar to that of the consistency of Martin's Axiom, but using supports of size $(< \kappa)$. The situation with the generalisation of proper focing is much more complicated, and in spite of a series of papers by Rosłanowski and Shelah (see e.g. [26]) where partial solutions are found, it is fair to say that the right geneeralisation does not exist for the moment. Exciting new work by Neeman seems to be able to obtain exactly that, as we discuss in the next section.

[5]this means that every increasing sequence of length $< \kappa$ in the forcing has a common upper bound

Before leaving this section let us also discuss forcing axioms at another kind of cardinals, the successor of a singular cardinal. Work by Džamonja and Shelah in [11] gives, modulo a supercompact cardinal, the consistency of a forcing axiom at a supercompact cardinal that will be made to have cofinality ω by a certain Prikry extension. The axiom has a different form than the ones that we have seen so far, as it applies in the universe before we do the Prikry extension, rather than the final universe. It states that 2^κ is large (can be made as large as we like it), there is a normal measure $\mathcal{D}$ on κ which is obtained as an increasing union of κ^{++} filters (which allows for a very nice prediction of $\Pr(\mathcal{D})$ names for objects on κ^+) and that the Generalised Martin's Axiom holds for κ^+-stationary chain condition $(< \kappa)$-directed closed (every directed system of $< \kappa$ many conditions has a common upper bound) well-met forcing notions. Using this axiom Džamonja and Shelah obtained the consistency (modulo a supercompact cardinal) of the existence of a family of κ^{++} graphs on κ^+ for κ singular strong limit of cofinality ω, together with 2^κ being as large as we wish. Work in progress by Cummings, Džamonja, Magidor, Morgan and Shelah promises to extend this to cofinalities larger than ω as well as to $\aleph_\omega$.

5. Iterating properness with finite supports- it is possible after all

As mentioned above, it is well known that one cannot iterate proper forcing axiom with finite supports and guarantee that the properness is preserved. Yet a recent result of Neeman in [25] shows that in some sense we can do exactly this, as he gave an alternative proof of the consistency of PFA using conditions which have finite support. Assuming some bookekepping function f coming from the Laver diamond and giving the list of all proper focings (this is the standard part of any known proof of the consistency of PFA), Neeman introduced the forcing which consists of pairs $(\mathcal{M}_p, w_p)$ where $\mathcal{M}_p$ is a finite $\in -increasing$ sequence of elementary submodels of $\mathcal{H}(\chi)$, each either countable or of the form $\mathcal{H}(\alpha)$ for some $\alpha < \chi$, and the sequence is closed under intersections. The working part w_p is also finite and it has a support of finitely many α satisfying that at any nontrivial stage α the coordinate $w_p(\alpha)$ is forced by $\mathbb{P}_\alpha$ to be $(M_{G_\alpha}, \underset{\sim}{Q}_\alpha)$-generic for all $M \in \mathcal{M}_p$ such that $\mathbb{P}_\alpha \in M$.

Neeman's work is very revolutionary but it also builds up on some earlier developments, such as the papers by Friedman [14] and Mitchell [23] in which these authors independently obtained forcing notions to force a club to ω_2 using finite conditions and certain systems of models.

This idea was also used in [10] to add a square to ω_2 , while the idea of using elementary submodels as side conditions goes back to Todorčević ([32]), with the difference that there only countable models were used. Koszmider [20] used models organised along a morass to obtain the consistency of the existence of a chain of length ω_2 in $\mathcal{P}(\omega_1)/\text{fin}$. A major ingredient in Neeman's work is also the notion of strong properness, which was used by Mitchell in [23]. Applications of Neeman's method in giving elegant proofs of several known difficult consistency results can be found in the paper [35] by Veličković and Venturi. Very new developements inspired by Neeman's work include a proof by Gitik and Magidor [15] that SPFA can be obtained by a sort of "revised finite support", and Neeman's yet unwritten work which shows how a generelisation of his method can be used to obtain a workable larger cardinal analogue of PFA.

6. Challenges

Some of the main directions of present research are indicated in the above: generalising PFA to larger cardinals and forcing axioms at the successor of a singular cardinal. There are many open questions that arise. Another direction on which we have not commented on too much and which forms a major line of research is the search for a forcing axiom on ω_1 which would be consistent with CH and give some concrete combinatorial statements, such as "measuring". Note that almost all the forcing we discussed in the context of ω_1 adds reals (which is why it is so interesting that PID is consistent with CH). There is a large theory behind this and the interested reader may start with [30], especially Chapter V.

Challenges in this field also come from applications to other fields, and without having more space to spend on the numerous applicatons of the axioms we have so far, we can just mention that applications have been found in fields as varied as the theory of Boolean algebras, topology, algebra, Banach space theory, measure theory and C*-algebras.

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