Some strong logics within combinatorial set theory and the logic of chains

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Abstract

In this note we report on a project in progress, where we study compactness of infinitary logics, including the logic of chains. The motivation of this project is to find logical reasons for the set-theoretic phenomenon of compactness at singular cardinals. [1] [2]

Dedication. This paper is dedicated to Prof. Mirjana Vuković, on the occasion of her 70th birthday, with warmest wishes for a happy and long continuation of a lifework of achievements in mathematics and selfless contributions to the mathematical community.

1 Introduction

In the world of infinite cardinals, combinatorial properties of singular cardinals are somewhat special. This is especially visible by the fact that they often exhibit a compactness behaviour. The celebrated Shelah's inequality [15]

$$[(\forall n<\omega)2^{\aleph_n}<\aleph_\omega] \implies 2^{\aleph_\omega}<\aleph_{\omega_4}.$$

is an example of such a behaviour, because we can interpret it as saying that if powers of cardinals smaller than the singular cardinal $\aleph_\omega$ are bounded,

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[2]MSC 2010 Classification: 03C75, 03C55, 03C80, 03C85, 03E75

then so is the power of $\aleph_\omega$. There are many compactness theorems about singular cardinals, some of which we shall mention below. The cardinal $\aleph_0$ is also very special, often because of the compactness. An important example is the compactness of the first order logic. Therefore it is natural to ask if there is a compact logic associated to singular cardinals, a question that we explore. This paper reports on results in progress obtained as part of a larger project and represents an extended version of a talk given by the first author at the conference "Modern Algebra and Analysis" organised by ANUBiH in Sarajevo in September 2018. Full exposition of the results mentioned here and other results will be written as a full length paper in the future.

2 Infinitary Logic

A logic that might serve the purpose of compactness at a singular cardinal was discovered by Carol Karp. She introduced the chain logic in her Ph.D. thesis in 1959 with Henkin [7], and continued working on it, on her own, and with her students, throughout her career. Her motivation was to generalise recursion theory through the use of infinitary logics. The part which is most relevant to us concerns the work of Ellen Cunningham from her Ph.D. thesis (1974, two years after Karp's death) [1].

The beginning of this work is to consider logics of the form $L_{\kappa,\lambda}$. Here κ and λ are infinite cardinals and we are allowed to make conjunctions of length $< \kappa$ and iterations of $< \lambda$-quantifiers, with other logical rules transported from first order logic, which in this notation becomes $L_{\omega,\omega}$. An interesting question is to find pairs κ, λ which give the nice properties that we have for $L_{\omega,\omega}$, may it be completeness, compactness and so on. This was an important research topic in the 1960s and 1970s, much about which can be found in the books by Jerry Keisler [10] on $L_{\omega_1,\omega}$ and Max Dikcmann [2] on $L_{\kappa,\lambda}$. It was found that if we want to recover the properties of first order logic for κ, λ regular, most often we need to work with $\kappa = \lambda$ some large cardinal. Let us, for example, review the case of strongly compact and the case of weakly compact cardinals.

We say that a set of sentences is *κ-satisfiable* if every subset of size $< \kappa$ has a model. Tarski [17] defined a *strongly compact* cardinal to be an uncountable κ such that every κ-satisfiable set of $L_{\kappa,\kappa}$-sentences is satisfiable. As we know, strong compactness is a large cardinal notion, equivalently defined in various other ways. Tarski [17] also defined a *weakly compact* cardinal to be an uncountable κ such that every κ-satisfiable set of $L_{\kappa,\kappa}$-sentences involving at most κ non-logical symbols, is satisfiable. This is another large cardinal notion, of course. An important exception to the large cardinal rule is the case of $L_{\omega_1,\omega}$ which shares some important properties of first order logic, notably completeness (see [10]).

2.1 Completeness and compactness

We recall the relation between the completeness and the compactness properties of a logic. It is easy to obtain the compactness of the first order logic as a consequence of its completeness. Namely, suppose that Σ is a set of first order sentences that it is not satisfiable. By completeness, Σ proves a contradiction. But the proof must be finite, so it only involves a finite subset $\Sigma_0 \subseteq \Sigma$. Hence Σ_0 is not satisfiable and so Σ is not finitely satisfiable. Let us note that this argument works because the notion of satisfaction and the notion of deduction are so well matched. However, there are logics which are complete but not compact, and this is the case of $L_{\omega_1,\omega}$. Karp proved that this logic is complete in [8], yet let us observe by a simple example that this logic is not compact. Namely, let $c_0, c_1, \ldots, c_\omega$ be constant symbols and let Σ be the following set of $L_{\omega_1,\omega}$ sentences:

$$\{(\forall x) \bigvee_{n<\omega} x = c_n, c_\omega \neq c_0, c_\omega \neq c_1, \ldots, c_\omega \neq c_n, \ldots\}.$$

Then Σ is finitely satisfiable but not satisfiable.

This difference between the relative behaviour of completeness and the compactness in the cases of these two different logics comes from the fact that when changing logic we have to use different rules of inference than those of the first order logic. For example, we, naturally, need to use the axiom $\bigwedge \Phi \implies \varphi$ for any countable set Φ of formulas with $\varphi \in \Phi$. Yet, we still keep the same notion of the finiteness in a proof, which is now less well-matched with the rules of inference. In this way we obtain that for infinitary logics compactness is harder than completeness.

Karp's Ph.D. student Judy Green [6] considered logics $L_{\kappa,\omega}$ searching for results analogous to those for $L_{\omega_1,\omega}$, in particular completeness. She used different but similar techniques in two cases: κ successor of a regular cardinal or κ singular or successor of a singular. Green defined proof systems for these logics, with proofs of length $< \kappa$ in a way $L_{\kappa,\omega}$ becomes complete and shares many other nice properties of the first order logic.

2.2 Chains

The next new idea that Karp brought to this subject is to consider not just the logic but also the structure of the underlying model. In this way she was able to approach the logic $L_{\kappa,\kappa}$, where κ is a singular cardinal of countable cofinality. See Karp's lecture [8]. She defined the notion of a chain model of size κ as an ordinary model of size κ along with a decomposition of it into an increasing union of submodels of length $\mathrm{cf}(\kappa)$. The most interesting case is that:

- $\mathrm{cf}(\kappa) = \omega$
- the chain consists of sets of strictly increasing cardinalities.

A typical chain model A with decomposition $\langle A_n : n < \omega \rangle$ is denoted by $(A_n)_n$. It is mostly interesting when κ is a strong limit and $2^{|A_n|} < |A_{n+1}|$. In order to define the logic of chain models we need to change the definition of $\models$, defining the new notion $\models^c$, given as follows. For formulas $\varphi(\bar{x})$ of $L_{\kappa,\kappa}$ (so $\bar{x}$ is a sequence of length $< \kappa$):

$$\text{“}(A_n)_n \models^c \exists \bar{x} \varphi(\bar{x})\text{” iff there is } n \text{ such that “} A_n \models \exists \bar{x} \varphi(\bar{x})\text{”.}$$

There is a natural way to define a logic out of this, which we denote by $L^c_{\kappa,\kappa}$. Karp and Cunnigham [1] proved that $L^c_{\kappa,\kappa}$ satisfies completeness, and has other nice properties, such as the Downward (to κ) Lowenheim-Skolem theorem. [3] The spirit here is that $L^c_{\kappa,\kappa}$ behaves very much like $L_{\omega_1,\omega}$.

In our joint work [4], we analysed the family of chain models coded as the elements of the topological space κ^ω, κ strong limit, $\mathrm{cf}(\kappa) = \omega$ (as well as other cofinalities). The *orbit* of a chain model coded by $f \in \kappa^\omega$ is the set of all g which code models chain-isomorphic to the model. The main theorem of [4] is:

Theorem 2.1 *The orbit of a chain model A is always a Σ^1_1set. The orbit is Δ^1_1 if and only if there is a tree T of height and size κ with no branches of length κ such that for any chain model B, player I has a winning strategy in $EFD^{c,<\kappa}_T(A, B)$ if and only if $A \approx^c B$.*

This theorem has since had several applications, notably as an input to the work of Vincenzo Dimonte, Luca Moto Ross and Xianghui Shi [3] which further develops descriptive set theory of such κ^ω. One may say that Theorem 2.1 completed the classical analysis of the chain logic.

3 The present project

Completeness of the first order logic has many applications, yet the above completeness theorems seem purely abstract, and so is the case of Theorem

[3]To understand these results properly, one has to make a distinction between weak chain models and proper chain models, which is a bit out of the scope of this paper.

2.1. Our present project is to obtain combinatorial theorems about singular cardinals such as $\beth_\omega$ as consequence of the known properties of strong logics, in particular the chain logic and its fragments. Fixing a singular strong limit cardinal κ of cofinality ω, we may try to obtain the following known theorems as a test of the method.

Theorem 3.1 *(Erdös-Tarski [5]) If a Boolean algebra has an antichain of any size $< \kappa$, then it has an antichain of size κ.*

Shelah's Singular Cardinal Compactness theorem, or just some consequences of it, such as:

Theorem 3.2 *(Shelah [14][4]) If every subset of size $< \kappa$ of a graph G has the coloring number $\leq \lambda < \kappa$, then so does G.*

We would also like to address some open conjectures and questions, such as:

Conjecture 3.3 *If a Banach space has a (semi)-biorthogonal system of every length $< \kappa$, then it has one of length κ.*

or

Question 3.4 *If a complete $L_{\omega_1,\omega}$-sentence has a model of size $\aleph_n$ for every n, does it then have a model of size $\aleph_\omega$?*

A well known question coming from Shelah's work is:

Question 3.5 *If every subset of of size $< \kappa$ of a graph G has the chromatic number $\leq \lambda < \kappa$, then does so G?*

4 Which logics are compact

Our first candidate for a logic compact at a singular cardinal is chain logic. However, the following results we were able to prove, although not completely conclusive, indicate that this logic is not compact. Namely, we have been able to compare the chain logic with other logics which are known not to have singular compactness, notably the logic $L_{\kappa,\omega}$. For this we used the idea of a Chu transform, defined as follows:

[4] A much simplified version of the proof of the Singular Compactness Theorem by Shelah himself is to appear in Sarajevo Journal of Mathematics.

Definition 4.1 *A* Chu space *over a set K is a triple (A, r, X) where A is a set of points, X is a set of states and the function $r : A \times X \to K$ is a K-valued binary relation between the elements of A and the elements of X. When $K = \{0, 1\}$ we just speak of Chu spaces and r becomes an ordinary relation.*

A Chu transform *between Chu spaces (A, r, X) and $(A', r,' X')$ over the same set K is a pair of functions (f, g) where $f : A \to A'$, $g : X' \to X$ and which satisfies the* adjointness condition *$r'(f(a), x')) = r(a, g(x'))$.*

This is relevant for us because of the following results.

We shall consider Chu spaces $(L, \models, S)$ where L is a set of sentences closed under conjunctions, S a set or a class of structures of the same signature as the sentences in L and $\models$ a relation between the elements of S and the elements of L, whose interpretation is a satisfaction relation which satisfies Tarski's definition of truth for the quantifier-free formulas.

Definition 4.2 *We say that $(L, \models, S) \leq (L', \models', S')$ if there is a Chu transform (f, g) between $(L, \models, S)$ and $(L', \models', S')$ where f preserves the logical operations and such that the range of g is* dense *in the following sense*

- *for every $\phi \in L$ for which there is $s \in S$ with $s \models \phi$, there is $s \in \mathrm{ran}(g)$ with $s \models \phi$.*

As an example, any g which is onto will clearly satisfy the density condition.

Theorem 4.3 *Suppose that $(L, \models, S) \leq (L', \models', S')$ and $(L', \models', S')$ is compact. Then so is $(L, \models, S)$.*

Proof Let (f, g) be the Chu transform which witnesses $(L, \models, S) \leq (L', \models' , S')$. Suppose that $\Sigma \subseteq L$ is finitely satisfiable and let $\Sigma' = \{f(\varphi) : \varphi \in \Sigma\}$. We now claim that Σ' is finitely satisfiable. Namely any finite $\Gamma' \subseteq \Sigma'$ is of the form $\{f(\varphi) : \varphi \in \Gamma\}$ for some finite $\Gamma \subseteq \Sigma$. Therefore there is $M \in S$ with $M \models \varphi$ for all $\varphi \in \Gamma$. Since g is not necessarily onto, we cannot use it to obtain from M an element of S'.

However, we have that $\bigwedge \Gamma$ is a sentence of L, by the closure under conjunctions. Since $\models$ satisfies Tarski's definition of truth for the quantifier-free formulas, we have that the fact that $M \models \varphi$ for all $\varphi \in \Gamma$ implies that $M \models \bigwedge \Gamma$. By the density requirement on g, there is $M' \in S'$ such that $g(M') \models \bigwedge \Gamma$ and hence $M' \models' f(\bigwedge \Gamma)$. By the preservation of the logical operations by f, we have that $f(\bigwedge \Gamma) = \bigwedge_{\varphi \in \Gamma} f(\varphi)$ so that $M' \models' f(\varphi)$ for all $\varphi \in \Gamma$ and $M' \models' \Gamma'$. So Γ' is finitely satisfiable in S', which by the assumption implies that there is $N' \in S'$ with $N' \models \Sigma'$. Therefore $g(N') \models \Sigma$. $\bigstar_{4.3}$

The proof of Theorem 4.3 with easy changes goes through for the higher degrees of compactness, let us specify.

Theorem 4.4 *Suppose that $(L, \models, S) \leq (L', \models', S')$ as witnessed by a pair (f, g) and that the following conditions are satisfied:*

1. *L, L' are closed under conjunctions of $< \lambda$ sentences,*
2. *$\models$ satisfies Tarski's definition of truth for the quantifier-free formulas, including the conjunctions and disjunctions of size $< \lambda$,*
3. *f preserves the conjunctions and disjunctions of size $< \lambda$.*

Then, for any θ, if $(L', \models', S')$ is (λ, θ)-compact, so is $(L, \models, S)$.

Chain logic comes in several different versions, which we shall not define right now, but one of them is the logic of *weak chain models*, denoted by $L^{c,w}_{\kappa,\kappa}$. Using Chu transforms, we were able to prove

Theorem 4.5 $(L_{\kappa,\omega}, \models, \mathcal{M}) \leq L^{c,w}_{\kappa,\kappa}$.

and then conclude thanks to Theorem 4.3 that

Corollary 4.6 *The logic $L^{c,w}_{\kappa,\kappa}$ is not κ-compact.*

We are still studying the question of the transformation of this proof which would allow us to conclude:

Conjecture 4.7 *Chain logic is not κ-compact.*

Some other logics are known to be κ-compact, notably two logics considered by Keisler in [9]: the ordered logic and the logic with an extra quantifier 'exists at least κ'. We are considering other candidates, such as certain fragments of the chain logic and Shelah's logic L^1_κ [16].

Once we have a supply of compact logics, we still need to see how we can get any combinatorial theorems as a consequence. A question that we are considering at the moment is if Theorem 5.4 is a consequence of Keisler's ordered logic.

5 Model Existence Theorems

Proofs of Completeness from $L_{\omega_1,\omega}$ are based on a version of Henkin's argument involving the so called *Consistency Properties.* They prove Model Existence Theorem (MET). As Keisler states in his book [10], the Model Existence Theorem based on Consistency Properties is frequently used in $L_{\omega_1,\omega}$ where Compactness is used in $L_{\omega,\omega}$. Consistency Properties were invented by Michael Makkai [12], also using ideas from earlier work by R. Smullyan.

A *consistency property* is a judiciously chosen set of sentences of a logic. The precise definition depends on the logic, but the point is to be able to prove the following type of theorem:

Theorem 5.1 *(Makkai [12]) A sentence of $L_{\omega_1,\omega}$ has a model iff it belongs to a consistency property.*

We call such theorems MET. As an example of an application, in Keisler's book [10] there is a proof based on MET of the following theorem, known as the undefinability of Well Order):

Theorem 5.2 *(Morley [13], Lopez-Escobar [11]) Let T be a countable set of sentences of $L_{\omega_1,\omega}$ and let $U, <$ be a unary and binary relation symbol of $L_{\omega,\omega}$. Suppose that for all $\alpha < \omega_1$, T has a model $\mathfrak{A}_\alpha = (A_\alpha, U_\alpha, <, ...)$ such that $<$ linearly orders U and $(\alpha, <) \subseteq (U_\alpha, <)$. Then T has a model $\mathfrak{B} = (B, U, <, ...)$ such that $<$ linearly orders B and $(U, <)$ contains a copy of $\mathbb{Q}$.*

Consistency properties were found by Green for logics of the form $L_{\lambda,\omega}$ and by Cunnigham for $L^c_{\kappa,\kappa}$, both working with or under the influence of Karp, as explained above. The definition of a consistency property depends on the logic and is somewhat lengthy, so we are not going into the details of such a definition here. The point is that it is a non-trivial matter to develop the right kind of consistency property and for a logic to have it, and the proofs are very long.

In our work in progress we are interested in the second order or restricted second order versions of $L^c_{\kappa,\kappa}$ since the application in questions, as seen above; are sometimes expressed in that way. In this context, set variables are bounded by an element of the chain. We are in the process of verifying the following theorem, which at this stage we still address as a conjecture:

Conjecture 5.3 *$L^{2,c}_{\kappa,\kappa}$ has a consistency property, so that a sentence of $L^{2,c}_{\kappa,\kappa}$ has a model iff it belongs to a consistency property.*

Recall that the full logic $L_{\kappa,\kappa}$ does not have the consistency property or MET but $L^c_{\kappa,\kappa}$ does. This is because it is easier for a sentence to have a chain model than a full model, as the following example shows.

Example 5.4 *Consider the sentence "$<$ is a well order". We can construct a chain model of this sentence which is not a real model by taking increasing disjoint blocks of size $\beth_1, \beth_2$ etc. and putting them below each other in the order $<$. The chain model so obtained satisfies that $<$ is a well order because no bounded piece of it contains an infinite $<$-decreasing sequence, yet the actual model contains such a sequence.*

Let us finish by proving that even the chain models are not going to help us to obtain a compact, or even countably compact second order logic.

Theorem 5.5 *Second order logic is not countably compact even in chain models.*

Proof Let θ be a second order sentence which says that $<$ is a well-order on some predicate P. In chain models of θ we have no guarantee that $<$ is really a well-order because a descending sequence may cross over all the sets A_i.

Let ϕ be the second order sentence $\exists X \forall y (P(y) \rightarrow X(y))$. The chain models of ϕ are the chain models in which P is contained in one level of the chain. In models of ϕ we have full second order quantification over subsets of P.

In models of $\theta \wedge \phi$ we know by the above that $<$ is really a well-order because any potential descending chain is a subset of P and hence a subset of some A_i. We can now form a finitely consistent theory $\{\theta, \phi\}\} \cup \bigcup_{n<\omega}\{P(c_n)\} \cup \{c_0 > c_1 > c_2 > \ldots\}$, which has no chain models. $\bigstar_{5.5}$

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