TYPES AND REPRESENTATIONS OF p-ADIC SYMPLECTIC GROUPS

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Shaun Ainsley Ross Stevens

Department of Mathematics
King's College London
University of London

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ABSTRACT

Let F be a non-archimedean local field and let $G = \mathbb{G}(F)$ be the F-points of a reductive group defined over F. Bushnell and Kutzko have described a strategy to classify the representations of G via the theory of *types*, which associates to each inertial class in the Bernstein spectrum a pair (K, ρ) consisting of a compact open subgroup K of G and an irreducible representation ρ of K.

We impose the restriction that the residual characteristic of F not be 2.

In this thesis we begin the construction of types associated to certain discrete series (in particular, to supercuspidal) representations of $G = \mathrm{Sp}_{2N}(F)$ by transferring Bushnell and Kutzko's construction for $\mathrm{GL}_{2N}(F)$ to our situation. Certain objects in the construction, in particular the simple characters, transfer simply by restriction.

In a certain case, we complete the construction of the type (K, ρ) and hence construct new supercuspidal representations in the wildly ramified case. In this case, we are also able to describe a (tentative) transfer map from certain supercuspidal representations of $\mathrm{GL}_{2N}(F)$ to supercuspidal representations of $\mathrm{Sp}_{2N}(F)$, which associates to each representation π of $\mathrm{GL}_{2N}(F)$ a set $\Pi(\pi)$ of representations of $\mathrm{Sp}_{2N}(F)$.

CONTENTS

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INTRODUCTION

The aim of this thesis is to begin the systematic examination of the admissible dual of the symplectic group $\mathrm{Sp}_{2N}(F)$ over a non-archimedean local field, of residual characteristic not 2, via the theory of *types* (see below).

Let F be a non-archimedean local field and let $\mathbb{G}$ be a reductive group defined over F. Let $G = \mathbb{G}(F)$ be the F-points of $\mathbb{G}$ and let $\mathfrak{R}(G)$ be the category of smooth representations of G.

For $i = 1, 2$, let M_i be a Levi subgroup of G and let σ_i be an irreducible supercuspidal representation of M_i. We say that the pairs (M_i, σ_i) are *inertially equivalent* if there exist $g \in G$ and χ an unramified quasicharacter of M_2 such that $M_1 = gM_2g^{-1}$ and $\sigma_2 \otimes \chi \simeq \sigma_1^g$. We write $\mathfrak{s} = [M_1, \sigma_1]_G$ for the inertial equivalence class of (M_1, σ_1) with respect to this equivalence relation and write $\mathcal{B}$ for the set of such equivalence classes (called the Bernstein spectrum).

For each inertial equivalence class $\mathfrak{s} = [M, \sigma]_G$, we can define a subcategory $\mathfrak{R}^{\mathfrak{s}}(G)$ of $\mathfrak{R}(G)$, which is the full subcategory of representations π of G such that any irreducible subquotient of π is equivalent to an irreducible subquotient of the parabolically induced representation $\mathrm{Ind}_{L,P}^{G}\tau$, for some $(L, \tau) \in \mathfrak{s}$, P a parabolic subgroup of G with Levi factor L. Then we have the Bernstein decomposition $\mathfrak{R}(G) = \prod_{\mathfrak{s}\in\mathcal{B}} \mathfrak{R}^{\mathfrak{s}}(G)$ (**[B]** or see **[BK2]**).

In **[BK2]**, Bushnell and Kutzko describe a strategy for classifying the representations of the group G via *types*: these are pairs (K, ρ) consisting of a compact open subgroup K and an irreducible representation ρ of K. The pair (K, ρ) is then called an $\mathfrak{s}$-type, for $\mathfrak{s} = [M, \sigma]_G$, if, for π any irreducible representation of G, π contains ρ (i.e. $\mathrm{Hom}_K(\rho, \pi) \neq 0$) if, and only if, $\pi \in |\mathfrak{R}^{\mathfrak{s}}(G)|$. If this is the case, then we have an equivalence of categories

$$M_\rho : \mathfrak{R}^{\mathfrak{s}}(G) \xrightarrow{\simeq} \mathcal{H}(G, \rho)\text{-Mod},$$

where $\mathcal{H}(G, \rho)$ is the spherical Hecke algebra and $\mathcal{H}(G, \rho)$-Mod is the category of left modules over $\mathcal{H}(G, \rho)$.

So the strategy is first to construct types for each inertial equivalence class $\mathfrak{s}$ and then to describe $\mathcal{H}(G, \rho)$-Mod. This has been done for $G = GL_N(F)$ (see **[BK]**, **[BK1]**) and, partially, for G a division algebra over F (see **[Br1]**) and for $G = SL_N(F)$ (see **[BK3]**, **[BK4]**).

If $\mathfrak{s} = [M, \sigma]_G$ and M is a proper Levi subgroup of G then **[BK2]** §8 describes a method for approaching the construction of an $\mathfrak{s}$-type via *covers*, if you already have an $[M, \sigma]_M$-type (i.e. a type for the decomposition of $\mathfrak{R}(M)$). This has been used to construct types for certain classical groups (see **[Bl]**, **[Au]**) and, in this way, types have been constructed for every $[M, \sigma]_G$, $M \lneq G = \mathrm{Sp}_4(F)$ if the residual characteristic is not 2 ([**BB**]). Types have also been constructed for principal series representations of split groups ([**Ro**]).

For $\mathfrak{s} = [G, \sigma]_G$ (i.e. for supercuspidal representations of G), types have been constructed in the tame case ([**Ad**], [**Kim**], [**M2**], [**M3**]).

In this thesis we look at the case $G = \mathrm{Sp}_{2N}(F)$, for F of residual characteristic not 2, and take some steps toward the construction of types in the arbitrarily ramified case. The construction procedure follows closely [**BK**] and we often call upon results from there. It would also be possible to extend the results here to all unitary groups (indeed, many of the results are given in this generality), a work which is postponed to a later date.

At this point, we should remark that, in the recent paper [**Ka**], Kariyama generalizes the methods of Carayol (in [**Ca**]) to the symplectic group, for arbitrarily ramified tori. We shall discuss the similarities and differences with this work at the end of this introduction.

Let F be a non-archimedean local field of residual characteristic not 2 and V a $2N$-dimensional F-vector space, equipped with a nondegenerate alternating form h. This form then induces an adjoint involution $^-$ on $A = \mathrm{End}_F(V)$. The symplectic group G can then be seen as the set of $g \in A$ such that $g\overline{g} = 1$ and the symplectic Lie algebra $\mathfrak{g} = A_-$ as the set of $x \in A$ such that $x + \overline{x} = 0$.

A *skew stratum* in A is a quadruple $[\mathfrak{A}, n, r, \beta]$ which is a stratum in the sense of [**BK**] (1.5) (i.e. $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in A, $n > r$ are integers and $\beta \in \mathfrak{P}^{-n}$, where $\mathfrak{P}$ is the Jacobson radical of $\mathfrak{A}$) subject to the additional conditions that $\mathfrak{A}$ is stable under the involution and $\beta \in A_-$.

In the case of a stratum of the form $[\mathfrak{A}, n, n-1, \beta]$ for $\mathrm{GL}_N(F)$, the stratum corresponds to a character ψ_β of the group $U^n(\mathfrak{A})$ which is trivial on $U^{n+1}(\mathfrak{A})$, where $U^r(\mathfrak{A})$, $r \in \mathbb{Z}$ is the standard filtration (by powers of the Jacobson radical) of the parahoric subgroup $\mathfrak{A}^\times$ of $\mathrm{GL}_N(F)$. If a representation π of $\mathrm{GL}_N(F)$ contains the representation ψ_β then we say that π contains the stratum.

For the symplectic group G the situation is similar, with the parahoric subgroup $P = \mathfrak{A} \cap G$ and the standard filtration $P^r(\mathfrak{A}) = U^r(\mathfrak{A}) \cap G$. In fact, the character ψ_β of $P^n(\mathfrak{A})$ is just the restriction of the character ψ_β of $U^n(\mathfrak{A})$.

The construction procedure for simple types begins with a skew *simple* stratum $[\mathfrak{A}, n, 0, \beta]$ (see chapter 2 or [**BK**] (1.5.5) for the definition). These skew simple strata have the property that their intertwining is, in some sense, as small as possible. We then construct some special characters (called *simple characters*, see chapter 6 for definitions) of certain compact open subgroups $H_-^{m+1}(\beta, \mathfrak{A}) \subset P^{m+1}(\mathfrak{A})$, for $0 \leq m \leq n-1$. In fact, these turn out to be nothing other than the restrictions of simple characters for $\mathrm{GL}_{2N}(F)$ ([**BK**] (3.2)). These simple characters also have a "good" intertwining formula.

Given a simple character θ of $H^1_-(\beta,\mathfrak{A})$, there exists a unique irreducible (Heisenberg) representation η of another compact open subgroup $J^1_-(\beta,\mathfrak{A}) \supset H^1_-(\beta,\mathfrak{A})$ which contains θ. This representation also has a "good" intertwining formula.

The final step then consists in extending the representation η to a representation κ of a compact open subgroup $J_-(\beta,\mathfrak{A}) \supset J^1_-(\beta,\mathfrak{A})$ which also has "small" intertwining (for $\mathrm{GL}_{2N}(F)$, such a κ is called a *β-extension* of η). We complete this step in the case where the field extension $F[\beta]/F$ is maximal in A (i.e. it is of degree $2N$).

The simple characters and the groups $H^k_-(\beta,\mathfrak{A})$, $J^k_-(\beta,\mathfrak{A})$ are defined by an inductive process. For $[\mathfrak{A},n,0,\beta]$ a (skew) simple stratum, there exists an integer r such that the stratum $[\mathfrak{A},n,r-1,\beta]$ is simple but $[\mathfrak{A},n,r,\beta]$ is not - it is only *pure* (see chapter 2 or **[BK]** (1.5.5)).

The main result in **[BK]** which allows the induction procedure to be set up says that there is an element γ such that the stratum $[\mathfrak{A},n,r,\gamma]$ is simple and equivalent to $[\mathfrak{A},n,r,\beta]$ ([**BK**] (2.4.1)). Then the objects for the stratum $[\mathfrak{A},n,0,\beta]$ are defined in terms of those for $[\mathfrak{A},n,0,\gamma]$.

In our case, we must prove the analogous result: for $[\mathfrak{A},n,r,\beta]$ a skew pure stratum there exists an element $\gamma \in A_-$ such that the stratum $[\mathfrak{A},n,r,\gamma]$ is skew simple and equivalent to $[\mathfrak{A},n,r,\beta]$ (see (5.4.7)).

This construction procedure will not produce all types for $\mathrm{Sp}_{2N}(F)$ and we now discuss the reasons for this.

The idea for $\mathrm{GL}_N(F)$ is that, given a positive level representation π of $\mathrm{GL}_N(F)$, we can find a stratum which is contained in π and is *fundamental* (nondegenerate in [**MP**]). i.e. the coset $\beta+\mathfrak{P}^{1-n}$ contains no nilpotent elements.

However, for the symplectic group (the situation is similar for other reductive groups) the results of Moy and Prasad ([**MP**]) require us to use more than just the standard filtrations of parahoric subgroups in order to obtain a similar result - in general the filtrations come from *self-dual lattice functions* (see [**Br3**]). Recent work **[PY]** has shown that for the symplectic group the only filtrations which are really needed are the standard ones and Morris's "C-chain" filtrations (see also [**M3**]).

Here we consider only the standard filtrations; however, the results of chapter 1 were written by Morris for C-chain filtrations (indeed they are valid for any self-dual lattice function filtration) and the results of chapter 2 are, in a sense, generalizable to this case.

In searching for supercuspidal representations we next require the notion of a *split* stratum: if a stratum is split then we want any representation containing it to be non-supercuspidal (so any supercuspidal will contain a non-split stratum). In the case of $\mathrm{GL}_N(F)$ this is fairly simply expressed in terms of a characteristic polynomial of the stratum (see **[BK]** (2.3.3) or chapter 5 for the definition of characteristic polynomial) - a stratum is split if and only if its characteristic polynomial has at least two distinct irreducible factors. This definition is the correct one because every maximal torus T of $\mathrm{GL}_N(F)$ contained in no proper Levi subgroup is of the form $T = E^\times$ for E a field extension of F of degree N.

Here we use the same definition of a split stratum but this will not be the correct one in general. As Morris remarked **[M3]** (1.3) (see also **[Kim]** (1.1)) the maximal tori in $G = \mathrm{Sp}_{2N}(F)$ contained in no proper Levi take the form

$$N_1(E_1) \times \cdots \times N_1(E_r),$$

where E_i/F is a field extension of degree $2n_i$ contained in A, $i = 1, \ldots, r$, $\sum_{i=1}^r n_i = N$, $\overline{E}_i = E_i$ but $\bar{}$ is non-trivial on E_i, $i = 1, \ldots, r$, and $N_1(E_i) = \{e \in E_i : e\bar{e} = 1\}$ are the norm 1 elements of E_i.

So we are restricting considerably the supercuspidals we hope to construct (even in $\mathrm{Sp}_4(F)$ we will miss some). However, we can hope methods similar to those of "semisimple types" **[BK1]** may allow us to obtain the remaining supercuspidals in the future.

We are able to show (as a corollary to the results of chapter 5) that if an irreducible representation π contains a nonsplit fundamental skew stratum then it contains a skew simple stratum.

Note that there is a general definition of split stratum for any reductive group due to Lemaire **[Le]**. This definition is in terms of the building of G and a lattice-theoretic translation has yet to be done.

There are several differences between this work and the paper **[Ka]**. In that paper, arbitrary maximal anisotropic tori are considered. This has the advantage that those strata which appear split (by our definition) are not excluded. However, considering only maximal tori will certainly not be enough - it is not so even for $\mathrm{GL}_N(F)$. Further, only elements which are a sum of *minimal* elements (see **[BK]** (1.4.14)) are considered, which again is insufficient even for $\mathrm{GL}_N(F)$. As in this thesis, only the standard filtrations of parahoric subgroups are considered.

We now give a brief summary of the contents of each chapter.

In chapter 1 we present some preliminary results, most of which are due to Morris (**[M1]**,**[M2]**,**[M3]**). In chapter 2 we check that we have the exact sequences analogous to those in **[BK]** (1.4) and we prove the intertwining theorem for a skew simple stratum. In chapter 3 we introduce the notion of a *residual subspace* of a lattice chain $\mathfrak{L}$ (due to Bushnell) and prove some results about block decompositions for $\mathfrak{A}$. In chapter 4 we introduce the analogue of the (W, E)-decomposition

[BK] (1.2.6) for the symplectic Lie algebra and use it to begin the refinement process. In chapter 5 we prove some results on Jordan decompositions and prove that a nonsplit fundamental skew stratum is "contained in" a skew simple stratum (cf. **[BK]** (2.3.4)) and that a skew pure stratum is equivalent to a skew simple one (cf. **[BK]** (2.4.1)). In chapter 6 we define the groups $H_-^{m+1}(\beta, \mathfrak{A})$ and $J_-^{m+1}(\beta, \mathfrak{A})$, define simple characters and calculate their intertwining. Finally, in chapter 7, we consider the case where the field extension associated to the skew simple stratum is a maximal extension of F in A; in this case we complete the construction of the type and examine a tentative transfer map from certain supercuspidals of $\mathrm{GL}_{2N}(F)$ to supercuspidals of G.

1
PRELIMINARIES

In this chapter we recall some results concerning self-dual lattice chains and parahoric subgroups of symplectic and unitary groups. Most of the results in this section can be found in Morris's papers [**M1**], [**M2**], [**M3**].

(1.1) Notation

Let F be a non-archimedean field equipped with a Galois involution $x \mapsto \overline{x}$ with fixed field F_0. We allow the possibility that $F_0 = F$. We will use the following notation throughout:

$\mathfrak{o}_F$ the discrete valuation ring in F;
$\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$;
k_F $= \mathfrak{o}_F/\mathfrak{p}_F$, the residue class field of F;
p the residual characteristic of F;
q $= p^f = \#k_F$.

(1.1.1) Assumption *We assume throughout that $p \neq 2$.*

We have similar notation $\mathfrak{o}_0, \mathfrak{p}_0, k_0$ for the same objects in F_0. We also use

ψ_0 some fixed continuous character of the additive group of F_0, with conductor $\mathfrak{p}_0$;
ψ_F $= \psi_0 \circ \mathrm{tr}_{F/F_0}$, where tr_{F/F_0} denotes trace;
V an F-vector space of finite dimension N;
A $= \mathrm{End}_F(V)$.

Note that F/F_0 is at worst tamely ramified, since the residual characteristic of F is not 2, so the character ψ_F of $(F,+)$ has conductor $\mathfrak{p}_F$.

If F/F_0 is unramified, we put $\pi_F = \pi_0$, a uniformizer of F_0; if F/F_0 is ramified, we choose π_F to be a uniformizer of F such that $\pi_F + \overline{\pi_F} = 0$ and put $\pi_0 = \pi_F^2$. In either case we have $\pi_F = \pm\overline{\pi_F}$.

Let $h : V \times V \to F$ be a nondegenerate ϵ-hermitian form on V, with $\epsilon = \pm 1$ (see [**Sch**] (7.1.2), for example). If $F = F_0$ we exclude the case $\epsilon = 1$ (i.e. we rule out orthogonal groups). For definiteness, assume that h is F-linear in the first variable. So

$$h(\lambda v, w) = \lambda h(v,w) = \lambda \epsilon \overline{h(w,v)}, \qquad \text{for } v, w \in V,\ \lambda \in F.$$

Then h induces an adjoint involution on A, also denoted $\bar{\ }$, given, for $a \in A$, by

$$h(av, w) = h(v, \overline{a}w), \qquad \text{for all } v, w \in V.$$

Note that for F embedded diagonally in A, the two involutions coincide.

We set $G = \{g \in \mathrm{Aut}_F(V) : h(gv, gw) = h(v, w) \text{ for all } v, w \in V\}$
$\qquad = \{g \in \mathrm{Aut}_F(V) : g\overline{g} = 1\}$.
This is the group of F_0-points of a unitary (or symplectic) group defined over F_0.

For X an additive subgroup in A invariant under the involution, define

$$\begin{aligned} X_- &= \{x \in X : \overline{x} = -x\}, && \text{the skew elements;} \\ X_+ &= \{x \in X : \overline{x} = x\} \quad , && \text{the symmetric elements.} \end{aligned}$$

These are both additive groups and if X is an $\mathfrak{o}_F$-lattice then

$$\begin{aligned} X = &\quad X_+ \;\oplus\; X_- \\ x = &\tfrac{1}{2}(x + \overline{x}) + \tfrac{1}{2}(x - \overline{x}) \end{aligned}$$

since $2 \in \mathfrak{o}_F^\times$. Set $\mathrm{tr}_0 = \mathrm{tr}_{F/F_0} \circ \mathrm{tr}_{A/F} : A \to F_0$ where tr denotes trace. Then this is an orthogonal direct sum decomposition with respect to tr_0 since, for $x \in X_-, y \in X_+$, we have

$$\begin{aligned} \mathrm{tr}_0(xy) = \mathrm{tr}_{F/F_0}(\mathrm{tr}_{A/F}(xy)) &= \mathrm{tr}_{F/F_0}(\overline{\mathrm{tr}_{A/F}(\overline{xy})}) \\ &= \mathrm{tr}_{F/F_0}(\mathrm{tr}_{A/F}(\overline{y}\,\overline{x})) = -\mathrm{tr}_0(yx) = -\mathrm{tr}_0(xy), \end{aligned}$$

and hence $\mathrm{tr}_0(xy) = 0$.
Note that A_- is just the Lie algebra of G.

For L an $\mathfrak{o}_F$-lattice in V, define the dual lattice of L to be

$$L^\# = \{x \in V : h(x, L) \subset \mathfrak{p}_F\}.$$

Then $L^{\#\#} = L$ and $L^\#$ can be identified with $\mathrm{Hom}(L, \mathfrak{p}_F)$ by the non-degeneracy of h. For $\mathfrak{L} = \{L_k : k \in \mathbb{Z}\}$ an $\mathfrak{o}_F$-lattice chain, define the dual chain

$$\mathfrak{L}^\# = \{L_k^\# : k \in \mathbb{Z}\}.$$

We say that $\mathfrak{L}$ is self-dual if $\mathfrak{L}^\# = \mathfrak{L}$.

(1.1.2) Lemma *Let $\mathfrak{L}$ be a self-dual lattice chain in V. Then there exists a unique $d \in \mathbb{Z}$ such that $L_k^\# = L_{d-k}$ for $k \in \mathbb{Z}$.*

Proof: $L_k^\# = L_{k'}$ for some $k' = k'(k) \in \mathbb{Z}$ and $L_k^{\#\#} = L_k$ so $k \mapsto k'$ is an order-reversing bijection $\mathbb{Z} \to \mathbb{Z}$. So it is of the form $k \mapsto d - k$. ∎

In fact, by changing the indexing of the lattices in $\mathfrak{L}$ (so that $L_{[\frac{d+1}{2}]}$ becomes L_0), we may assume that $d = 0$ or -1.

(1.2) Hereditary orders

Let $\mathfrak{L}$ be a lattice chain in V and let $\mathfrak{A} = \mathrm{End}^0_{\mathfrak{o}_F}(\mathfrak{L})$ be the associated hereditary $\mathfrak{o}_F$-order in A (see **[BK]** (1.1)). Let $\mathfrak{P}$ be the Jacobson radical of $\mathfrak{A}$. Then $\mathfrak{P}$ is invertible as a fractional ideal of $\mathfrak{A}$ and the $\mathfrak{P}^n$, for $n \in \mathbb{Z}$, give a filtration of A. There is also a valutation map $\nu_{\mathfrak{A}}$ associated to the hereditary order $\mathfrak{A}$, given by

$$\nu_{\mathfrak{A}}(x) = \max\{n \in \mathbb{Z} : x \in \mathfrak{P}^n\}, \qquad x \in A,$$

with the understanding that $\nu_{\mathfrak{A}}(0) = \infty$.

(1.2.1) Lemma *Let $\mathfrak{L}$ be a self-dual lattice chain in V and let $\mathfrak{A}$ be the associated hereditary $\mathfrak{o}_F$-order, with Jacobson radical $\mathfrak{P}$. Then we have $\overline{\mathfrak{P}^n} = \mathfrak{P}^n$, for $n \in \mathbb{Z}$.*

Proof: Choose $x \in \mathfrak{P}^n$; so $xL_k \subset L_{k+n}$ for all $k \in \mathbb{Z}$. Let d be the unique integer such that $L_k^{\#} = L_{d-k}$ for $k \in \mathbb{Z}$ given by (1.1.2). Fix $k \in \mathbb{Z}$ and let $v \in L_k$; then

$$h(\overline{x}v, L_{k+n}^{\#}) = h(v, xL_{d-k-n}) \subset h(v, L_{d-k}) \subset \mathfrak{p}_F$$

so $\overline{x}v \in L_{k+n}^{\#\#} = L_{k+n}$. This is true for all $k \in \mathbb{Z}$ so $\overline{x} \in \mathfrak{P}^n$. ∎

In particular, in the situation of (1.2.1), we have $\overline{\mathfrak{A}} = \mathfrak{A}$ and, for $n \in \mathbb{Z}$, we may define the additive groups $\mathfrak{P}^n_-$ and $\mathfrak{P}^n_+$ as in (1.1) above; so $\mathfrak{P} = \mathfrak{P}^n_- \oplus \mathfrak{P}^n_+$.

(1.2.2) Lemma ([M3] (2.8)) *Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A such that $\overline{\mathfrak{A}} = \mathfrak{A}$ and let $\mathfrak{b}$ be a left ideal in $\mathfrak{A}$. Suppose that $x \in \mathfrak{b}$ satisfies $\overline{x} = \eta x$ for $\eta = \pm 1$. Then $x = y + \eta\overline{y}$ for some $y \in \mathfrak{b}$.*

Proof: $2 \in \mathfrak{o}_F^\times$ so $\frac{1}{2}.1_A \in \mathfrak{A}$. Then set $y = (\frac{1}{2}.1_A)x \in \mathfrak{b}$. ∎

Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A such that $\overline{\mathfrak{A}} = \mathfrak{A}$ and let $\mathfrak{b} \subset \mathfrak{P}$ be a left ideal in $\mathfrak{A}$ such that $\mathfrak{b} = \overline{\mathfrak{b}}$. We now define the Cayley transform C on $\{x \in A_- : \det(1 - \frac{x}{2}) \neq 0\}$ by

$$C(x) = (1 + \tfrac{x}{2})(1 - \tfrac{x}{2})^{-1}.$$

It is easy to check that $\mathrm{im}\, C \subset G$. Moreover, if $x \in \mathfrak{b}_-$ then $C(x)$ exists since $\mathfrak{b}^n \subset \mathfrak{P}^n \subset \mathfrak{p}_F^{j(n)}\mathfrak{A}$, for some $j(n) \in \mathbb{Z}$, and $j(n) \to \infty$ as $n \to \infty$ so $(1 - \frac{x}{2})^{-1}$ can be defined by the usual power series expansion; then also $C(x) \in (1 + \mathfrak{b}) \cap G$. Note that this is not quite the same Cayley transformation as that used by Morris in **[M1]**.

(1.2.3) Lemma (cf. [M1] (2.13)(c)) *With notation as above, we have a bijection*

$$\begin{aligned} \mathfrak{b}_- &\to (1 + \mathfrak{b}) \cap G \\ x &\mapsto C(x) \end{aligned}$$

Proof: The map is certainly injective since, if $u \in (1 + \mathfrak{b}) \cap G$ then, for $u = C(x)$, we must have $x = -2(1 - u)(1 + u)^{-1}$.

Now let $u \in (1+\mathfrak{b}) \cap G$; then $u = 1+y$, for some $y \in \mathfrak{b}$, so $1+u = 2+y = 2(1+\frac{y}{2})$. Then $2 \in \mathfrak{o}_F^\times$ implies $\frac{y}{2} \in \mathfrak{b}$ so $1+u$ is indeed invertible and we have

$$\begin{aligned}
-2(1-u)(1+u)^{-1} - 2\overline{(1-u)(1+u)^{-1}} &= 0, \\
(1+\overline{u})(1-u) + (1-\overline{u})(1+u) &= 0, \\
2 &= 2\overline{u}u, \\
u\overline{u} &= 1.
\end{aligned}$$

So x is indeed skew and $x = y(1 - \frac{y}{2} + (\frac{y}{2})^2 - \dots) \in \mathfrak{b}$; so $x \in \mathfrak{b}_-$. ■

We now give a multiplicative version of (1.2.2)

(1.2.4) Lemma *Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A such that $\overline{\mathfrak{A}} = \mathfrak{A}$ and let $\mathfrak{b} \subset \mathfrak{P}$ be a left ideal in $\mathfrak{A}$. Let $u \in 1+\mathfrak{b} \subset U^1(\mathfrak{A})$.*
(i) If $\mathfrak{b} = \overline{\mathfrak{b}}$ and $u\overline{u} = 1$ then there exists $v \in 1+\mathfrak{b}$ such that $u = v\overline{v}^{-1}$.
(ii) If $u = \overline{u}$ then there exists $v \in 1+\mathfrak{b}$ such that $u = v\overline{v}$.

Proof: (i) This follows from (1.2.3) since $u \in (1+\mathfrak{b}) \cap G$ so $u = C(x)$ for some $x \in \mathfrak{b}_-$. Then we put $v = 1 + \frac{x}{2}$.

(ii) For each $n \in \mathbb{N}$, we will find inductively $v_n \in 1+\mathfrak{b}$ such that $v_n - v_{n-1} \in \mathfrak{b}^{(n-1)}$ and $v_n\overline{v}_n \equiv u \mod \mathfrak{b}^n$. Then, as $\mathfrak{A}$ is compact and $\mathfrak{b}^n \subset \mathfrak{P}^n$, there exists $v \in \mathfrak{A}$ such that $v_n \to v$; then $v \in 1+\mathfrak{b}$ and $v\overline{v} = u$ as required.

We can take $v_1 = 1$, so assume we have found v_n as required, i.e. $v_n^{-1}u\overline{v_n}^{-1} \equiv 1 \mod \mathfrak{b}^n$. Write $v_n^{-1}u\overline{v_n}^{-1} = 1+x$, $x \in \mathfrak{b}^n$; then $\overline{v_n^{-1}u\overline{v_n}^{-1}} = v_n^{-1}u\overline{v_n}^{-1}$ implies $\overline{x} = x$. By (1.2.2), there exists $y \in \mathfrak{b}^n$ such that $y + \overline{y} = x$. We obtain

$$(1-y)v_n^{-1}u\overline{v_n}^{-1}(1-\overline{y}) = 1 + x - y - \overline{y} + O(\mathfrak{b}^{2n}) \in 1 + \mathfrak{b}^{(n+1)}$$

So we set $v_{n+1} = v_n(1-y)^{-1}$. ■

In particular, we will apply the preceding two lemmas to $\mathfrak{b} = \mathfrak{P}^n$.

Before proceeding we give some further preliminary results which will be useful later.

(1.2.5) Lemma ([M4] (6.1)) *Let $X \supset Y$ be $\mathfrak{o}_F$-lattices in A which are stable under the involution, so that X/Y inherits an involution and $(X/Y)_-$ is defined. Then*
(i) the natural map
$$\begin{aligned} I_- &\to (I/J)_- \\ x &\mapsto x + J \end{aligned}$$
is surjective;
(ii) the map
$$\begin{aligned} I_-/J_- &\to (I/J)_- \\ x + J_- &\mapsto x + J \end{aligned}$$
is an isomorphism.

Proof: (i) Let $y + J \in (I/J)_-$, so that $y + \overline{y} = a \in J$. Since $a = \overline{a}$, (1.2.2) implies that $a = b + \overline{b}$ for some $b \in J$. Then set $y' = y - b$. Then part (ii) is clear. ■

There is an analogous lemma in the case of $(X/Y)_+$. Further

$$\begin{aligned} X/Y &= \quad (X/Y)_- \oplus (X/Y)_+ \\ x + Y &= \tfrac{1}{2}(x - \overline{x}) + Y \;+\; \tfrac{1}{2}(x + \overline{x}) + Y \end{aligned}$$

(1.2.6) Lemma *Let X, Y be additive subgroups of A invariant under the involution. Then*

$$(X+Y)_- = X_- + Y_-$$

Proof: We clearly have the containment $\supset$. Let $x+y \in (X+Y)_-$; so $x+\overline{x} = -(y+\overline{y})$. We have $x+\overline{x} \in (X \cap Y)_+$ and, by (1.2.2), there exists $z \in (X \cap Y)$ such that $x+\overline{x} = z+\overline{z}$. Then set $x' = x-z$, $y' = y+z$. ■

We now give a multiplicative version of the previous lemma.

(1.2.7) Lemma *Let K, K' be subgroups of GL_{2N} such that $K \cap K' = 1+\mathfrak{b}$, where $\mathfrak{b}$ is a left ideal in some hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$, both invariant under the involution and with $\mathfrak{b} \subset \mathfrak{P} = rad\,\mathfrak{A}$. Then $(K.K') \cap G = K \cap G \;.\; K' \cap G$.*

Proof: Let $kk' \in (K_1K_2) \cap G$. Then $k'\overline{k'} = k^{-1}\overline{k^{-1}}$ lies in $K_1 \cap K_2$ and, by (1.2.4)(ii), there exists $u \in K_1 \cap K_2$ such that $k'\overline{k'} = u\overline{u}$. Then we set $k_1 = ku$, $k_1' = u^{-1}k'$. ■

(1.3) Parahoric subgroups and filtrations

Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A; then we have a parahoric subgroup in $\mathrm{GL}_{2N}(F)$ given by $U(\mathfrak{A}) = \mathfrak{A}^\times$, the groups of units in $\mathfrak{A}$. This has standard filtration given by

$$U^n(\mathfrak{A}) = 1 + \mathfrak{P}^n \qquad \text{for } n \geq 1.$$

Set $P = P(\mathfrak{A}) = \mathfrak{A} \cap G$, a parahoric subgroup of G; this comes equipped with its standard filtration $P^n = P^n(\mathfrak{A}) = \{x \in P : x \equiv 1 \bmod \mathfrak{P}^n\} = U^n(\mathfrak{A}) \cap G$, for $n \geq 1$. This gives a filtration of $P(\mathfrak{A})$ by normal open subgroups and $[P^n, P^m] \subseteq P^{m+n}$ for $m, n \in \mathbb{Z}$. Then we can apply (1.2.3) to give

(1.3.1) Proposition ([M1] (2.13)(c)) *For each $n \geq 1$, the Cayley map provides a bijection $\mathfrak{P}^n_- \to P^n$; $x \mapsto C(x)$.*

(1.3.2) Corollary ([M3] (2.1.4)(b)) *If $2n \geq m \geq n \geq 1$, the map $\mathfrak{P}^n_- \to P^n$ in (1.3.1) induces an isomorphism of abelian groups*

$$\mathfrak{P}^n_-/\mathfrak{P}^m_- \simeq P^n/P^m$$
$$x \mapsto 1+x$$

Proof: We only need to check that the map is indeed a group homomorphism but this follows since $2n \geq m$. ■

As $\mathfrak{P}$ is stable under the involution, $\mathfrak{A}/\mathfrak{P}$ is a k_F-algebra which inherits an involution. Set

$$\begin{aligned}\overline{P}(\mathfrak{A}) &= \{x \in \mathfrak{A}/\mathfrak{P} : x\overline{x} = 1\} \\ &= \{x + \mathfrak{P} : x\overline{x} - 1 \in \mathfrak{P}\}\end{aligned}$$

(1.3.3) Proposition ([M3] (2.11)) *The natural map $P \to \overline{P}(\mathfrak{A})$ is surjective.*

Proof: Suppose $x \in \mathfrak{A}$ and $x + \mathfrak{P} \in \overline{P}(\mathfrak{A})$. Let $x_1 = x$. For each n, we will find inductively $x_n \in \mathfrak{A}$ such that $x_n - x_{n-1} \in \mathfrak{P}^{n-1}$ and $x_n\overline{x_n} \equiv 1 \bmod \mathfrak{P}^n$. Then, as $\mathfrak{A}$ is compact, there exists $t \in \mathfrak{A}$ such that $x_n \to t$; then $t - x \in \mathfrak{P}$ and $t\overline{t} = 1$, i.e. $t \in G \cap \mathfrak{A} = P$ and t maps to $x + \mathfrak{P}$.

We have x_1 already so assume we have found x_n as required, for some $n \geq 1$. Then $\overline{x_n\overline{x_n} - 1} = x_n\overline{x_n} - 1$ so, applying (1.2.2), there exists $a \in \mathfrak{P}^n$ such that

$$x_n\overline{x_n} = 1 + a + \overline{a}.$$

Then $(1+a)(1+\overline{a}) = x_n\overline{x_n} + a\overline{a}$ and $a\overline{a} \in \mathfrak{P}^{2n} \subset \mathfrak{P}^{n+1}$ so

$$1 \equiv ((1+a)^{-1}x_n)\overline{((1+a)^{-1}x_n)} \qquad \bmod \mathfrak{P}^{n+1}.$$

So set $x_{n+1} = (1+a)^{-1}x_n$. ■

(1.3.4) Corollary ([M3] (2.14)(a)) *The natural map in (1.3.3) induces an isomorphism $P/P^1 \simeq \overline{P}(\mathfrak{A})$*

Proof: Consider the natural map $\varphi : P \hookrightarrow \mathfrak{A} \to \mathfrak{A}/\mathfrak{P}$. By (1.3.3), $\operatorname{im}\varphi = \overline{P}(\mathfrak{A})$. Further

$$\begin{aligned}\ker\varphi &= \{x \in P : x + \mathfrak{P} = 1 + \mathfrak{P}\}\\ &= \{x \in P : x \in 1 + \mathfrak{P}\} = P^1.\end{aligned}$$ ■

Then we see that $\overline{P}(\mathfrak{A})$ is the reductive quotient of the parahoric subgroup P.

(1.4) Characters

Recall that $\mathrm{tr}_0 = \mathrm{tr}_{F/F_0} \circ \mathrm{tr}_{A/F}$. If X is an $\mathfrak{o}_F$-lattice in A (hence an $\mathfrak{o}_0$-lattice in A) define

$$X^* = \{a \in A : \mathrm{tr}_0(aX) \subset \mathfrak{p}_0\},$$

which is also an $\mathfrak{o}_F$-lattice. Since $2 \in \mathfrak{o}_F^\times$ and F/F_0 is of degree at most 2, F is at worst tamely ramified over F_0 so

$$X^* = \{a \in A : \mathrm{tr}_{A/F}(aX) \subset \mathfrak{p}_F\}$$

which is the same as the usual definition of X^* (as in **[BK]** (1.1.4)). In particular

$$(\mathfrak{P}^n)^* = \mathfrak{P}^{1-n}\ , \qquad \text{for all } n \in \mathbb{Z}$$

(see **[Bu]** p.190).

If X is also stable under the involution, we can define

$$(X_-)^* = \{a \in A_- : \mathrm{tr}_0(aX_-) \subset \mathfrak{p}_0\}.$$

Then, as the direct sum $X = X_- \oplus X_+$ is orthogonal with respect to tr_0, we have

$$(X_-)^* = (X^*)_-$$

and, in particular,

$$(\mathfrak{P}_-^n)^* = \mathfrak{P}_-^{1-n}.$$

Recall that ψ_0 is a character of the additive group of F_0 with conductor $\mathfrak{p}_0$. As in **[W]** (II.5), the map

$$\begin{aligned} A_- &\to (A_-)\hat{\ } \\ x &\mapsto (y \mapsto \psi_0(\mathrm{tr}_0(xy))) \end{aligned}$$

is an isomorphism of abelian groups, where $\hat{\ }$ denotes the Pontrjagin dual.

Given an $\mathfrak{o}_0$-lattice L in A_-, set $L_{\hat{}} = \{\chi \in (A_-)\hat{\ } : \chi(L) \equiv 1\}$. Then the identification $A_- \to (A_-)\hat{\ }$ enables us to identify $L_{\hat{}}$ with L^*. Moreover, if $L_1 \supset L_2$ then $(L_1/L_2)\hat{\ } \simeq L_{2\hat{}}/L_{1\hat{}} \simeq L_2^*/L_1^*$. We have obtained:

(1.4.1) Lemma ([M2] (4.19)) *If $2n \geq m \geq n \geq 1$ there is a P-equivariant isomorphism of abelian groups*

$$\begin{array}{rcl} \mathfrak{P}_-^{1-m}/\mathfrak{P}_-^{1-n} & \tilde{\to} & (P^n/P^m)\hat{\ } \\ b + \mathfrak{P}_-^{1-n} & \mapsto & \psi_b \end{array}$$

where $\psi_b(p) = \psi_0(\mathrm{tr}_0(b(p-1)))$ for $p \in P^n$.

Then ψ_b is the restriction to P^n of the character ψ_b of $U^n(\mathfrak{A})$ defined in **[BK]** (1.1.6), since $\psi_0 \circ \mathrm{tr}_0 = \psi_F \circ \mathrm{tr}_{A/F}$ and ψ_F has conductor $\mathfrak{p}_F$.

2

EXACTNESS AND INTERTWINING

In this chapter we show that the exact sequences of **[BK]** (1.4) restrict well to the symplectic Lie algebra. This will follow from the fact that if E/F is a subfield of A stable under the involution then there is a *tame corestriction* on A relative to E/F which commutes with the involution. (See below or **[BK]** (1.3) for definitions.).

These exact sequences, together with the formula for the intertwining in GL_{2N} of *simple strata* **[BK]** (1.5.8), allow us to calculate the intertwining in G of *skew* simple strata, showing that it is, in some sense, as small as possible. Note that a similar intertwining theorem has been obtained in the case E/F tamely ramified by Morris (**[M3]** (3.13)).

(2.1) Exact Sequences

Let E/F be a subfield of A such that $E = \overline{E}$ and let E_0 be the fixed field of the involution. Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A which is invariant under the involution $^-$ and suppose $E^\times$ normalizes $\mathfrak{A}$. Then we define

$$\begin{aligned} B &= \mathrm{End}_E(V), \text{ the } A\text{-centralizer of } E; \\ \mathfrak{B} &= \mathfrak{A} \cap B, \qquad \mathfrak{Q} = \mathfrak{P} \cap B. \end{aligned}$$

Then, by **[BK]** (1.2.4), $\mathfrak{B}$ is a hereditary $\mathfrak{o}_E$-order with Jacobson radical $\mathfrak{Q}$ and $\mathfrak{Q}^n = \mathfrak{P}^n \cap B$. Note also that $E = \overline{E}$ implies that $B = \overline{B}$.

Recall, from **[BK]** (1.3.3), that a *tame corestriction* on A (relative to E/F) is a (B, B)-bimodule homomorphism $s : A \to B$ such that $s(\mathfrak{A}) = \mathfrak{A} \cap B$ for any hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in A which is normalized by $E^\times$. By **[BK]** (1.3.2), a tame corestriction is uniquely determined up to multiplication by a unit $u \in \mathfrak{o}_E^\times$. For E/F tamely ramified, we can take s to be orthogonal projection, relative to the pairing $(x, y) \mapsto \mathrm{tr}_{A/F}(xy)$ (see **[BK]** (1.3.8)(*ii*)).

We now prove that there is a tame corestriction which commutes with the involution on A.

(2.1.1) Lemma *There exists a tame corestriction on A relative to E/F such that $s(\overline{x}) = \overline{s(x)}$ for all $x \in A$.*

Proof: Recall that we have the character ψ_0 of the additive group of F_0, with conductor $\mathfrak{p}_0$; then $\psi_F = \psi_0 \circ \mathrm{tr}_{F/F_0}$ is a character of the additive group of F, with conductor $\mathfrak{p}_F$ since F/F_0 is at worst tamely ramified. We put $\psi_A = \psi_F \circ \mathrm{tr}_{A/F} = \psi_0 \circ \mathrm{tr}_0$, as in (1.1), where $\mathrm{tr}_0 = \mathrm{tr}_{F/F_0} \circ \mathrm{tr}_{A/F}$.

Similarly, let ψ_{E_0} be a character of the additive group of E_0, with conductor $\mathfrak{p}_{E_0}$; then $\psi_E = \psi_{E_0} \circ \mathrm{tr}_{E/E_0}$ is a character of the additive group of E, with conductor $\mathfrak{p}_E$. We put $\psi_B = \psi_E \circ \mathrm{tr}_{B/E} = \psi_{E_0} \circ \mathrm{tr}_0^E$, where $\mathrm{tr}_0^E = \mathrm{tr}_{E/E_0} \circ \mathrm{tr}_{B/E}$.

As in [**BK**] (1.3.4), there exists a unique map $s : A \to B$ such that

$$\psi_A(ab) = \psi_B(s(a)b), \qquad a \in A, b \in B$$

and this is a tame corestriction on A relative to E/F. Then, for $a \in A_-$, $b \in B_+$, we have $\psi_B(s(a)b) = \psi_A(ab) = \psi_0(\mathrm{tr}_0(ab)) = \psi_0(0) = 1$ as A_-, A_+ are orthogonal with respect to tr_0. Hence $s(a)$ is orthogonal to B_+ with respect to tr_0^E, that is, $s(a) \in B_-$. Similarly, we have that $s(A_+) \subset B_+$.
Now, for $a \in A$, $a = a_+ + a_-$ with $a_+ \in A_+$ and $a_- \in A_-$ so we have

$$\begin{aligned}\overline{s(a)} &= \overline{s(a_+)} + \overline{s(a_-)}\\ &= s(a_+) - s(a_-)\\ &= s(a_+ - a_-) = s(\overline{a})\end{aligned}$$

as required. ■

From now on, let s be a tame corestriction given by (2.1.1); it is uniquely determined up to multiplication by $u \in \mathfrak{o}_E^\times$ such that $u = \overline{u}$. Then s splits as $s_- : A_- \to B_-$ and $s_+ : A_+ \to B_+$.

Now let $\beta \in A_-$ be such that the algebra $E = F[\beta]$ is a field and consider the adjoint map

$$\begin{aligned}a_\beta : A &\to A\\ x &\mapsto \beta x - x\beta\end{aligned}$$

This is a (B, B)-bimodule homomorphism with kernel B. Then

$$\overline{a_\beta(x)} = \overline{\beta x - x\beta} = \overline{x}\overline{\beta} - \overline{\beta}\overline{x} = -\overline{x}\beta + \beta\overline{x} = a_\beta(\overline{x})$$

So a_β also splits as $a_\beta^- : A_- \to A_-$ and $a_\beta^+ : A_+ \to A_+$. In particular we have an infinite exact sequence

$$\cdots \xrightarrow{s} A \xrightarrow{a_\beta} A \xrightarrow{s} A \xrightarrow{a_\beta} \cdots$$

which splits as

$$\cdots \xrightarrow{s_-} A_- \xrightarrow{a_\beta^-} A_- \xrightarrow{s_-} A_- \xrightarrow{a_\beta^-} \cdots$$

$$\cdots \xrightarrow{s_+} A_+ \xrightarrow{a_\beta^+} A_+ \xrightarrow{s_+} A_+ \xrightarrow{a_\beta^+} \cdots$$

Let β be as above and let $k \in \mathbb{Z}$. Define

$$\mathfrak{N}_k = \mathfrak{N}_k(\beta, \mathfrak{A}) = \{x \in \mathfrak{A} : a_\beta(x) \in \mathfrak{P}^k\}.$$

Then $\overline{\mathfrak{N}_k} = \mathfrak{N}_k$ and $\mathfrak{N}_k$ is an $\mathfrak{o}_F$-lattice in A. In particular, $\mathfrak{N}_k$ also splits as ${\mathfrak{N}_k}^+ \oplus {\mathfrak{N}_k}^-$.

Let $n = -\nu_{\mathfrak{A}}(\beta)$; then $a_\beta(\mathfrak{A}) \subset \mathfrak{P}^{-n}$ whence $\mathfrak{N}_k = \mathfrak{A}$ for $k \leq -n$. Suppose $E \neq F$. Define

$$k_0 = k_0(\beta, \mathfrak{A}) = \max\{k \in \mathbb{Z} : \mathfrak{N}_k \not\subset \mathfrak{B} + \mathfrak{P}\}$$

which exists, by **[BK]** (1.4.4), and we have $k_0 \geq -n$. If $E = F$ then set $k_0(\beta, \mathfrak{A}) = -\infty$.

(2.1.2) Remark Let $e = e(E|F)$ be the ramification index and $\nu = \nu_E(\beta)$. We say the element β is *minimal* if it satisfies
(i) $\gcd(\nu, e) = 1$;
(ii) $\pi_F^{-\nu}.\beta^e + \mathfrak{p}_E$ generates the residue class field extension k_E/k_F.
Then, by **[BK]** (1.4.15), $k_0(\beta, \mathfrak{A}) = -n$ if, and only if, β is minimal. Moreover, we then have

$$\mathfrak{N}^{r-n}(\beta, \mathfrak{A}) = \mathfrak{B} + \mathfrak{P}^r \quad \text{for } r \geq 0$$

and, in particular, $\mathfrak{N}_{k_0} = \mathfrak{N}_{-n} = \mathfrak{A}$.

(2.1.3) Proposition ([BK] (1.4.7)) *Let $k, r \in \mathbb{Z}$ and suppose $k \geq k_0(\beta, \mathfrak{A})$, $r \geq 1$. Then the following sequences are exact:*

$$0 \to \mathfrak{N}_k/\mathfrak{N}_{k+r} \xrightarrow{a_\beta} \mathfrak{P}^k/\mathfrak{P}^{k+r} \xrightarrow{s} \mathfrak{Q}^k/\mathfrak{Q}^{k+r} \to 0$$

$$0 \to \mathfrak{N}_k/\mathfrak{B} \xrightarrow{a_\beta} \mathfrak{P}^k \xrightarrow{s} \mathfrak{Q}^k \to 0$$

By (1.2.5) all the terms in the exact sequence are stable under the (induced) involution and split as a sum of skew elements and symmetric elements. Further, we have seen that a_β and s both preserve the skew elements and the symmetric elements. Therefore these exact sequences split in two. In particular, we have:

(2.1.4) Proposition *Let $k, r \in \mathbb{Z}$ and suppose $k \geq k_0(\beta, \mathfrak{A})$, $r \geq 1$. Then the following sequences are exact:*

$$0 \to \mathfrak{N}_k^-/\mathfrak{N}_{k+r}^- \xrightarrow{a_\beta^-} \mathfrak{P}_-^k/\mathfrak{P}_-^{k+r} \xrightarrow{s_-} \mathfrak{Q}_-^k/\mathfrak{Q}_-^{k+r} \to 0$$

$$0 \to \mathfrak{N}_k^-/\mathfrak{B}_- \xrightarrow{a_\beta^-} \mathfrak{P}_-^k \xrightarrow{s_-} \mathfrak{Q}_-^k \to 0$$

Note also that we have

$$\begin{aligned} \overline{\mathfrak{Q}^m\mathfrak{N}_k} = \overline{\mathfrak{N}_k}\ \overline{\mathfrak{Q}^m} &= \mathfrak{N}_k\mathfrak{Q}^m \\ &= \mathfrak{Q}^m\mathfrak{N}_k \quad \text{by } \textbf{[BK]} \text{ (1.4.8)} \end{aligned}$$

so, from **[BK]** (1.4.10) and (1.2.5), we also have

(2.1.5) Corollary *For $m, k \in \mathbb{Z}$, $k \geq k_0(\beta, \mathfrak{A})$ the following sequences are exact:*

$$0 \to (\mathfrak{Q}^m\mathfrak{N}_k)_-/\mathfrak{Q}_-^m \xrightarrow{a_\beta^-} \mathfrak{P}_-^{m+k} \xrightarrow{s_-} \mathfrak{Q}_-^{m+k} \to 0$$

$$0 \to (\mathfrak{Q}^m\mathfrak{N}_k)_-/(\mathfrak{Q}^m\mathfrak{N}_{k+1})_- \xrightarrow{a_\beta^-} \mathfrak{P}_-^{m+k}/\mathfrak{P}_-^{m+k+1} \xrightarrow{s_-} \mathfrak{Q}_-^{m+k}/\mathfrak{Q}_-^{m+k+1} \to 0$$

Note that if E/F is tamely ramified then, taking s to be the projection pr:$A \to B$, all these exact sequences are split by the inclusion $B \hookrightarrow A$.

Recall from **[BK]** (1.3.10) that a lattice L is called E-exact if $s(L) = L \cap B$.

(2.1.6) Lemma *Let $\gamma \in B$, $r \in \mathbb{Z}$; then $L = \mathfrak{P}^r + \gamma\mathfrak{P}^r\gamma^{-1}$ is E-exact.*

Proof: From **[BK]** (1.3.16), the lattice $L\gamma = \mathfrak{P}^r\gamma + \gamma\mathfrak{P}^r$ is E-exact and, further, an $\mathrm{Ad}\,(E^\times)$-invariant $(\mathfrak{A}(E), \mathfrak{o}_E)$-bilattice. Then $L = (L\gamma)\gamma^{-1}$ is also an $\mathrm{Ad}\,(E^\times)$-invariant $(\mathfrak{A}(E), \mathfrak{o}_E)$-bilattice, and hence is E-exact by **[BK]** (1.3.12). ■

In particular, if $\gamma \in B \cap G$ then $(\mathfrak{P}_+^r + \gamma\mathfrak{P}_+^r\gamma^{-1}) \cap B = (\mathfrak{P}^r + \gamma\mathfrak{P}^r\gamma^{-1})_+ \cap B = (\mathfrak{Q}^r + \gamma\mathfrak{Q}^r\gamma^{-1})_+ = \mathfrak{Q}_+^r + \gamma\mathfrak{Q}_+^r\gamma^{-1}$.

Let $k = k_0(\beta, \mathfrak{A})$ and write $\mathfrak{N}$ for $\mathfrak{N}_{k_0(\beta,\mathfrak{A})}$. Finally in this section we present the multiplicative analogue of $(\mathfrak{Q}^m\mathfrak{N})_-$ for the symplectic group G. For $m \geq 1$ define

$$Q^m = (1 + \mathfrak{Q}^m\mathfrak{N}) \cap G.$$

Then, by (1.2.3), there is a bijection $(\mathfrak{Q}^m\mathfrak{N})_- \to Q^m$ given by $x \mapsto C(x)$. Note that for β minimal over F, $\mathfrak{N} = \mathfrak{A}$ so $Q^m = P^m(\mathfrak{A})$.

(2.2) Intertwining of simple strata

Recall (from **[BK]** (1.5)) that a *stratum* in A is a 4-tuple $[\mathfrak{A}, n, r, b]$ consisting of a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in A, integers $n \geq r$, and an element $b \in A$ such that $\nu_{\mathfrak{A}}(b) \geq -n$. If $[\mathfrak{A}_i, n_i, r_i, b_i]$ are strata in A, $i = 1, 2$, and $\mathfrak{P}_i = \mathrm{rad}(\mathfrak{A}_i)$, we say they are *equivalent*, denoted

$$[\mathfrak{A}_1, n_1, r_1, b_1] \cong [\mathfrak{A}_2, n_2, r_2, b_2], \quad \text{if} \quad b_1 + \mathfrak{P}_1^{-r_1} = b_2 + \mathfrak{P}_2^{-r_2}.$$

If they are equivalent then, by **[BK]** (1.5.2), $\mathfrak{A}_1 = \mathfrak{A}_2$, $r_1 = r_2$ and, if we have $\nu_{\mathfrak{A}_i}(b_i) = -n_i$ for $i = 1, 2$, also $n_1 = n_2$.

(2.2.1) Definition *A stratum $[\mathfrak{A}, n, r, b]$ in A is called* skew *if $b + \overline{b} = 0$ and $\mathfrak{A}$ is invariant under the involution on A.*

(2.2.2) Definition ([BK] (1.5.5)) *Let $[\mathfrak{A}, n, r, \beta]$ be a stratum in A. It is* pure *if*
(i) the algebra $E = F[\beta]$ is a field,
(ii) $E^\times \subset \mathfrak{K}(\mathfrak{A})$,
(iii) $\nu_{\mathfrak{A}}(\beta) = -n$.
It is called simple *if, in addition,*
(iv) $r < -k_0(\beta, \mathfrak{A})$.

We now define the formal intertwining in G of a skew stratum $[\mathfrak{A}, n, r, b]$

$$\mathcal{I}_G[\mathfrak{A}, n, r, b] = \{x \in G : x^{-1}(b + \mathfrak{P}_-^{-r})x \cap (b + \mathfrak{P}_-^{-r}) \neq \emptyset\}$$

(2.2.3) Theorem *Let $[\mathfrak{A}, n, r, \beta]$ be a simple stratum with $\beta \in A_-$ and $\mathfrak{A}$ stable under the involution $^-$. Write $k = k_0(\beta, \mathfrak{A})$ and $\mathfrak{N} = \mathfrak{N}_k(\beta, \mathfrak{A})$ and, for $m \geq 1$, define Q^m as in (2.1). Then*

$$\mathcal{I} = \mathcal{I}_G[\mathfrak{A}, n, r, \beta] = Q^{-(k+r)}(B \cap G)Q^{-(k+r)}$$

Before proceeding with the proof, we remark that $B \cap G$ is a unitary group (we will examine it more closely in chapter 4). It will therefore be necessary for us to prove many results (concerning refinement of strata etc.) for unitary groups as well as for symplectic groups.

Proof: Let $\mathcal{J} = \{x \in G : x^{-1}(b + \mathfrak{P}^{-r})x \cap (b + \mathfrak{P}^{-r}) \neq \emptyset\}$. This is the $\mathrm{GL}(2N)$-intertwining of the stratum (in the sense of **[BK]** (1.5.7)) contained in G. Clearly $\mathcal{I} \subset \mathcal{J}$. Suppose, on the other hand, $x \in \mathcal{J}$; then there exist $b_i \in \mathfrak{P}^{-r}$, $i = 1, 2$ such that

$$x(\beta + b_1)x^{-1} = \beta + b_2.$$

We now write $b_i = u_i + v_i$, $u_i \in \mathfrak{P}_+^{-r}$, $v_i \in \mathfrak{P}_-^{-r}$, for $i = 1, 2$ so

$$x(\beta + v_1)x^{-1} + xu_1x^{-1} = (\beta + v_2) + u_2.$$

Now $xA_-x^{-1} \subset A_-$ and $xA_+x^{-1} \subset A_+$ so $A = A_- \perp A_+$ implies

$$x(\beta + v_1)x^{-1} = \beta + v_2,$$

i.e. $x \in \mathcal{I}$. So we see that $\mathcal{I} = \mathcal{J} = (1 + \mathfrak{Q}^d\mathfrak{N})B^\times(1 + \mathfrak{Q}^d\mathfrak{N}) \cap G$ by **[BK]** (1.5.8), where $d = -r - k > 0$.

We continue with two lemmas:

(2.2.4) Lemma *We have*

$$(1 + \mathfrak{Q}^d\mathfrak{N})B^\times(1 + \mathfrak{Q}^d\mathfrak{N}) \cap G \subset (1 + \mathfrak{Q}^d\mathfrak{N})(B \cap G)(1 + \mathfrak{Q}^d\mathfrak{N}).$$

Proof: Let $t \in (1+\mathfrak{Q}^d\mathfrak{N})B^\times(1+\mathfrak{Q}^d\mathfrak{N}) \cap G$ with $\nu_{\mathfrak{A}}(t) = m$. So $t = (1+x_1)^{-1}b(1+y_1)$ with $b \in B^\times$, $x_1, y_1 \in \mathfrak{Q}^d\mathfrak{N} \subset \mathfrak{P}^d$. Then we have

$$\begin{aligned}
&(1 + x_1)t = b(1 + y_1),\\
&(1 + x_1)(1 + \overline{x_1}) = b(1 + y_1)(1 + \overline{y_1})\overline{b},\\
&b\overline{b} - 1 = x_1 + \overline{x_1} + x_1\overline{x_1} - b(y_1 + \overline{y_1} + y_1\overline{y_1})\overline{b}.
\end{aligned}$$

So $b\overline{b} \equiv 1 \pmod{\mathfrak{P}_+^d + b\mathfrak{P}_+^d\overline{b}}$. We will find inductively $b_n \in U^d(\mathfrak{B})bU^d(\mathfrak{B})$ such that

(2.2.5) $$b_n\overline{b_n} \equiv 1 \pmod{\mathfrak{P}_+^{nd} + b_n\mathfrak{P}_+^{nd}\overline{b_n}}$$

and $b_{n+1} \equiv b_n \pmod{\mathfrak{P}^{m+nd}}$. Granting this, we have $\nu_{\mathfrak{A}}(b_n) = \nu_{\mathfrak{A}}(t) = m$ so $b_n\overline{b_n} \equiv 1 \pmod{\mathfrak{P}_+^{nd} + \mathfrak{P}_+^{nd+2m}}$. Then the sequence $\{b_n\}$ converges to $b' \in U^d(\mathfrak{B})bU^d(\mathfrak{B})$ (since the double coset is compact) and $b'\overline{b'} = 1$. So $t \in (1 + \mathfrak{Q}^d\mathfrak{N})(B \cap G)(1 + \mathfrak{Q}^d\mathfrak{N})$ since $\mathfrak{Q}^d \subset \mathfrak{Q}^d\mathfrak{N}$.

We are reduced to proving (2.2.5). The case $n = 1$ is immediate: take $b_1 = b$. So let $n \geq 1$ and assume we have found the required element b_n. We write $b_n = (1+x)t(1+y)^{-1}$ with $x, y \in \mathfrak{Q}^d\mathfrak{N} \subset \mathfrak{P}^d$. Then, as above,

$$b_n\overline{b_n} - 1 = x + \overline{x} + x\overline{x} - b_n(y + \overline{y} + y\overline{y})\overline{b_n}$$

Now $b_n\overline{b_n} - 1 \in (\mathfrak{P}_+^{nd} + b_n\mathfrak{P}_+^{nd}\overline{b_n}) \cap B = \mathfrak{Q}_+^{nd} + b_n\mathfrak{Q}_+^{nd}\overline{b_n}$ by the remark following (2.1.6). So there exist $v', w' \in \mathfrak{Q}_+^{nd}$ such that $b_n\overline{b_n} - 1 = v' - b_n w'\overline{b_n}$. Then let $v = \frac{1}{2}v'$, $w = \frac{1}{2}w'$; $v, w \in \mathfrak{Q}_+^{nd}$ since $2 \in \mathfrak{o}_F^\times$ and

$$b_n\overline{b_n} - 1 = v + \overline{v} - b_n(w + \overline{w})\overline{b_n} = x + \overline{x} + x\overline{x} - b_n(y + \overline{y} + y\overline{y})\overline{b_n}$$

We set $b_{n+1} = (1-v)b_n(1-w)^{-1}$. Then

$$\begin{aligned}
& v + \overline{v} - b_{n+1}(w + \overline{w})\overline{b_{n+1}} \\
=& (v + \overline{v}) - (1-v)b_n(1-w)^{-1}(w+\overline{w})(1-\overline{w})^{-1}\overline{b_n}(1-\overline{v}) \\
\equiv& (1-v)[(v+\overline{v}) - b_n(1-w)^{-1}(w+\overline{w})(1-\overline{w})^{-1}\overline{b_n}](1-\overline{v}) \qquad \pmod{\mathfrak{P}_+^{2nd}} \\
\equiv& (1-v)[v + \overline{v} - b_n(w+\overline{w})\overline{b_n}](1-\overline{v}) \qquad \pmod{\mathfrak{P}_+^{2nd} + b_{n+1}\mathfrak{P}_+^{2nd}\overline{b_{n+1}}} \\
\equiv& (1-v)[x + \overline{x} + x\overline{x} - b_n(y+\overline{y}+y\overline{y})\overline{b_n}](1-\overline{v}) \qquad \pmod{\mathfrak{P}_+^{2nd} + b_{n+1}\mathfrak{P}_+^{2nd}\overline{b_{n+1}}} \\
\equiv& (1-v)(x+\overline{x}+x\overline{x})(1-\overline{v}) - (1-v)b_n(1-w)^{-1}(y+\overline{y}+y\overline{y})(1-\overline{w})^{-1}\overline{b_n}(1-\overline{v}) \\
& \qquad \pmod{\mathfrak{P}_+^{(n+1)d} + b_{n+1}\mathfrak{P}_+^{(n+1)d}\overline{b_{n+1}}} \\
\equiv& x + \overline{x} + x\overline{x} - b_{n+1}(y+\overline{y}+y\overline{y})\overline{b_{n+1}} \qquad \pmod{\mathfrak{P}_+^{(n+1)d} + b_{n+1}\mathfrak{P}_+^{(n+1)d}\overline{b_{n+1}}}
\end{aligned}$$

Now $(1-v)(1+x)t = b_{n+1}(1-w)(1+y)$ so, misappropriating the O-notation in the obvious way, we get

$$\begin{aligned}
& (1-v)(1+x)(1+\overline{x})(1-\overline{v}) = b_{n+1}(1-w)(1+y)(1+\overline{y})(1-\overline{w})\overline{b_{n+1}}, \\
\text{hence} \quad & 1 - v - \overline{v} + x + \overline{x} + x\overline{x} + O(\mathfrak{P}_+^{(n+1)d}) = \\
& \qquad b_{n+1}\overline{b_{n+1}} - b_{n+1}(-w - \overline{w} + y + \overline{y} + y\overline{y} + O(\mathfrak{P}_+^{(n+1)d}))\overline{b_{n+1}} \\
\text{and} \quad & b_{n+1}\overline{b_{n+1}} \equiv 1 \pmod{\mathfrak{P}_+^{(n+1)d} + b_{n+1}\mathfrak{P}_+^{(n+1)d}\overline{b_{n+1}}}.
\end{aligned}$$

This completes our induction and the proof of the lemma. ∎

(2.2.6) Lemma *Let $t \in G$; then*

$$(1 + \mathfrak{Q}^d\mathfrak{N})t(1 + \mathfrak{Q}^d\mathfrak{N}) \cap G = Q^d t Q^d.$$

Proof: Let $h \in (1+\mathfrak{Q}^d\mathfrak{N})t(1+\mathfrak{Q}^d\mathfrak{N}) \cap G$ and $\nu_{\mathfrak{A}}(t) = m$. We will find inductively $t_n \in Q^d h Q^d$ such that

$$\textbf{(2.2.7)} \qquad t_n \equiv t \pmod{\mathfrak{Q}^{nd}\mathfrak{N}t_n + t\mathfrak{Q}^{nd}\mathfrak{N}}.$$

Granting this, we have $\nu_{\mathfrak{A}}(t_n) = \nu_{\mathfrak{A}}(t) = m$ so $t_n \equiv t \pmod{\mathfrak{P}^{nd+m}}$. Then the sequence $\{t_n\}$ converges to t. The double coset $Q^d h Q^d$ is compact in G so we conclude that $t \in Q^d h Q^d$, or $h \in Q^d t Q^d$, as required.

So we are reduced to proving (2.2.7). The case $n = 1$ is immediate; take $t_1 = h$. So let $n \geq 1$ and assume we have found the required element t_n satisfying (2.12). Then there exist $x, y \in \mathfrak{Q}^{nd}\mathfrak{N}$ such that $t_n = t - xt_n + ty$, i.e. $(1+x)t_n = t(1+y)$, and we have

$$\begin{aligned} (1+x)(1+\overline{x}) &= t(1+y)(1+\overline{y})\overline{t}, \\ \text{hence } x - ty\overline{t} &\equiv -(\overline{x} - t\overline{y}\overline{t}) \pmod{(\mathfrak{Q}^{2nd}\mathfrak{N})_+ + t(\mathfrak{Q}^{2nd}\mathfrak{N})_+\overline{t}}, \end{aligned}$$

i.e. $x - ty\overline{t} \in (\mathfrak{Q}^{nd}\mathfrak{N} + t\mathfrak{Q}^{nd}\mathfrak{N}\overline{t})_- + (\mathfrak{Q}^{2nd}\mathfrak{N})_+ + t(\mathfrak{Q}^{2nd}\mathfrak{N})_+\overline{t}$. Now we have $(\mathfrak{Q}^{nd}\mathfrak{N} + t\mathfrak{Q}^{nd}\mathfrak{N}\overline{t})_- = (\mathfrak{Q}^{nd}\mathfrak{N})_- + t(\mathfrak{Q}^{nd}\mathfrak{N})_-\overline{t}$ so there exist $v, w \in (\mathfrak{Q}^{nd}\mathfrak{N})_-$ such that

$$\textbf{(2.2.8)} \qquad x - ty\overline{t} \equiv v - tw\overline{t} \pmod{\mathfrak{Q}^{2nd}\mathfrak{N} + t\mathfrak{Q}^{2nd}\mathfrak{N}\overline{t}}.$$

We set $t_{n+1} = C(v)t_nC(-w) \in Q^d h Q^d$. Then, from (2.2.8), we have

$$\begin{aligned} & (x-v)t \equiv t(y-w) \pmod{\mathfrak{Q}^{2nd}\mathfrak{N}t + t\mathfrak{Q}^{2nd}\mathfrak{N}} \\ \textit{i.e.}\quad & (x-v)(1+x)C(-v)t_{n+1}C(w)(1+y)^{-1} \equiv t(y-w) \\ & \qquad\qquad \pmod{\mathfrak{Q}^{2nd}\mathfrak{N}t + t\mathfrak{Q}^{2nd}\mathfrak{N}} \\ \textit{i.e.}\quad & (x-v)(1+x)C(-v)t_{n+1} \equiv t(y-w)(1+y)C(-w) \\ & \qquad\qquad \pmod{\mathfrak{Q}^{2nd}\mathfrak{N}t_{n+1} + t\mathfrak{Q}^{2nd}\mathfrak{N}} \\ \textit{i.e.}\quad & (x-v)t_{n+1} \equiv t(y-w) \pmod{\mathfrak{Q}^{2nd}\mathfrak{N}t_{n+1} + t\mathfrak{Q}^{2nd}\mathfrak{N}} \end{aligned}$$

Then $(1+x)C(-v)t_{n+1} = t(1+y)C(-w)$ implies

$$\begin{aligned} & (1 + (x-v) + \mathfrak{Q}^{2nd})t_{n+1} = t(1 + (y-w) + \mathfrak{Q}^{2nd}) \\ \text{and} \quad & t_{n+1} \equiv t \pmod{\mathfrak{Q}^{2nd}\mathfrak{N}t_{n+1} + t\mathfrak{Q}^{2nd}\mathfrak{N}}. \end{aligned}$$

Now $n \geq 1$ so $t_{n+1} \equiv t \pmod{\mathfrak{Q}^{(n+1)d}\mathfrak{N}t_{n+1} + t\mathfrak{Q}^{(n+1)d}\mathfrak{N}}$ as required. ∎

Returning to the proof of the theorem, we have

$$\begin{aligned} \mathcal{I} &= (1+\mathfrak{Q}^d\mathfrak{N})B^\times(1+\mathfrak{Q}^d\mathfrak{N}) \cap G \\ &= (1+\mathfrak{Q}^d\mathfrak{N})(B \cap G)(1+\mathfrak{Q}^d\mathfrak{N}) \cap G, && \text{by (2.2.4)}, \\ &= Q^d(B \cap G)Q^d, && \text{by (2.2.6)}. \end{aligned}$$ ∎

Note that, following the proof of (2.2.6), we find

$$\textbf{(2.2.9)} \qquad U^n(\mathfrak{A})tU^n(\mathfrak{A}) \cap G = P^n(\mathfrak{A})tP^n(\mathfrak{A}),$$

for $t \in G$ and $n \geq 1$. Further, replacing A with B, $\mathfrak{A}$ with $\mathfrak{B}$ etc. we have

$$U^n(\mathfrak{B})tU^n(\mathfrak{B}) \cap G = P^n(\mathfrak{B})tP^n(\mathfrak{B}), \tag{2.2.10}$$

for $t \in B \cap G$, $n \geq 1$ and where $P^n(\mathfrak{B}) = U^n(\mathfrak{B}) \cap G = P^n(\mathfrak{A}) \cap B$.

(2.2.11) Proposition *Let $t \in B \cap G$, $n \geq 1$. Then*

$$P^n(\mathfrak{A})tP^n(\mathfrak{A}) \cap B = P^n(\mathfrak{B})tP^n(\mathfrak{B}).$$

Proof:
$$\begin{aligned} P^n(\mathfrak{A})tP^n(\mathfrak{A}) \cap B &= U^n(\mathfrak{A})tU^n(\mathfrak{A}) \cap G \cap B, && \text{by (2.2.9)},\\ &= U^n(\mathfrak{B})tU^n(\mathfrak{B}) \cap G, && \text{by } \textbf{[BK]} \text{ (1.6.1)},\\ &= P^n(\mathfrak{B})tP^n(\mathfrak{B}), && \text{by (2.2.10)}. \end{aligned}$$
■

3
RESIDUAL SUBSPACES

In this chapter we introduce the notion of a *residual subspace* of a lattice chain in V. This is due to Bushnell, as are the results (3.1.1–9). These will be seen to correspond to subspaces of V which *split* $\mathfrak{L}$ in some sense. They will play a role similar to the $\mathbb{Z}/e\mathbb{Z}$-graded algebras in [**M3**].

We can then apply the results obtained about residual subspaces to find "nice" block decompositions for the hereditary order associated to a self-dual lattice chain in V (cf. [**BK**] **(2.5.1)**). In particular, the involution on A will permute the blocks with respect to this decomposition.

This chapter is given in the generality of unitary groups so, as in (1.1), h will be an ϵ-hermitian from on V.

(3.1) Residual subspaces

Let $\mathfrak{L}$ be an $\mathfrak{o}_F$-lattice chain in V of period e. Then we have an additive norm $\nu_{\mathfrak{L}}$ (in the sense of [**BT**] (1.1)) on V associated to $\mathfrak{L}$ given by

$$\nu_{\mathfrak{L}}(v) = \sup_{k\in\mathbb{Z}}\{k : v \in L_k\}.$$

Then $\nu_{\mathfrak{L}}$ satisfies the following properties:

$$\begin{aligned} \nu_{\mathfrak{L}}(xv) &= e\nu_F(x) + \nu_{\mathfrak{L}}(v) \quad \text{ for } x \in F^\times, v \in V; \\ \nu_{\mathfrak{L}}(v+w) &\geq \inf(\nu_{\mathfrak{L}}(v), \nu_{\mathfrak{L}}(w)) \quad \text{ for } v, w \in V; \\ \nu_{\mathfrak{L}}(v) &= \infty \iff v = 0. \end{aligned}$$

(3.1.1) Definition *An $\mathfrak{o}_F$-basis of $\mathfrak{L}$ is a basis $\{v_1, \ldots, v_N\}$ of V which splits $\nu_{\mathfrak{L}}$, in the sense that*

$$\nu_{\mathfrak{L}}(\sum_{i=1}^{N} x_i v_i) = \inf_{1\leq i\leq n} \nu_{\mathfrak{L}}(x_i v_i).$$

This is not the usual definition of an $\mathfrak{o}_F$-basis, as in [**BK**] (1.1.7), but is somewhat weaker. Indeed, for $0 \leq i \leq e-1$, let $\{v_{ij} : 1 \leq j \leq \dim_{k_F} L_i/L_{i+1}\}$ be a basis for L_i/L_{i+1}, and choose $\hat{v}_{ij} \in V$ such that $\hat{v}_{ij} + L_{i+1} = v_{ij}$. Then $\{\hat{v}_{ij}\}$ is an $\mathfrak{o}_F$-basis of $\mathfrak{L}$ in the sense of [**BK**].

(3.1.2) Lemma *With notation as above, $\{\hat{v}_{ij}\}$ is an $\mathfrak{o}_F$-basis of $\mathfrak{L}$ in the sense of (3.1.1).*

Proof: Let $v = \sum x_{ij}\hat{v}_{ij}$ and let $l = \inf \nu_{\mathfrak{L}}(x_{ij}\hat{v}_{ij})$. Then certainly $\nu_{\mathfrak{L}}(v) \geq l$. Let $\{(i_k, j_k) : 1 \leq k \leq r\}$ be the set of indices for which $\nu_{\mathfrak{L}}(x_{ij}\hat{v}_{ij}) = l$. Since $0 \leq \nu_{\mathfrak{L}}(\hat{v}_{ij}) \leq e-1$ for all i, j, we have

$$\begin{aligned} \nu_{\mathfrak{L}}(\hat{v}_{i_1 j_1}) &= \cdots = \nu_{\mathfrak{L}}(\hat{v}_{i_r j_r}); \\ \nu_F(x_{i_1 j_1}) &= \cdots = \nu_F(x_{i_r j_r}). \end{aligned}$$

So we may assume $\nu_F(x_{i_kj_k}) = 0$ and $l = \nu_{\mathfrak{L}}(\hat{v}_{i_kj_k})$ for $1 \le k \le r$. However, by definition, the $\hat{v}_{i_kj_k} + L_{l+1}$ are linearly independent over k_F so $v \notin L_{l+1}$ and $\nu_{\mathfrak{L}}(v) = l$. ■

Conversely, an $\mathfrak{o}_F$-basis $\{v_1, \ldots, v_N\}$ of $\mathfrak{L}$ in the sense of (3.1.1) is an $\mathfrak{o}_F$-basis in the sense of **[BK]** if and only if $0 \le \nu_{\mathfrak{L}}(v_i) \le e-1$, for $1 \le i \le N$.

(3.1.3) Definition *Let $\mathfrak{L}$ be an $\mathfrak{o}_F$-lattice chain in V. A* residual subspace *of $\mathfrak{L}$ is a family $\mathcal{V} = \{\mathcal{V}_k : k \in \mathbb{Z}\}$ such that*
(i) $\mathcal{V}_k$ is a k_F-subspace of L_k/L_{k+1} for each $k \in \mathbb{Z}$
(ii) for any $x \in F^\times$ with valuation $\nu = \nu_F(x)$, we have $x\mathcal{V}_k = \mathcal{V}_{k+e\nu}$, where $e = e(\mathfrak{L})$.

Since, by definition, the $\mathcal{V}_k$ are k_F-subspaces, it is enough to verify (ii) for π_F, rather than every $x \in F^\times$. It is easy to produce these residual subspaces. Let W be an F-subspace of V, and define

$$\mathfrak{L}(W)_k = \frac{W \cap L_k}{W \cap L_{k+1}} = \frac{(W \cap L_k) + L_{k+1}}{L_{k+1}}, \quad k \in \mathbb{Z}.$$

The family $\{\mathfrak{L}(W)_k : k \in \mathbb{Z}\}$ is then surely a residual subspace of $\mathfrak{L}$.

(3.1.4) Proposition *Let $\{\mathcal{V}_k : k \in \mathbb{Z}\}$ be a residual subspace of the lattice chain $\mathfrak{L}$. Then:*
(i) there exists a subspace W of V such that $\mathfrak{L}(W)_k = \mathcal{V}_k$ for all $k \in \mathbb{Z}$;
(ii) for any subspace W satisfying (i), we have

$$\dim_F(W) = \sum_{i=0}^{e-1} \dim_{k_F}(\mathcal{V}_i);$$

(iii) let W^1, W^2 be subspaces of V. Then $\mathfrak{L}(W^1)_k = \mathfrak{L}(W^2)_k$, for all $k \in \mathbb{Z}$, if and only if there exists $x \in U^1(\mathfrak{A})$ such that $W^2 = xW^1$.

Proof: For each i, $0 \le i \le e-1$, choose a basis $\{v_{ij}\}$ of $\mathcal{V}_i$ and let $\hat{v}_{ij} \in V$ be such that $\hat{v}_{ij} + L_{i+1} = v_{ij}$. Let W be the F-span of the $\hat{v}_{ij}$. Then $\{\hat{v}_{ij}\}$ is part of an $\mathfrak{o}_F$-basis of $\mathfrak{L}$ so $\mathcal{V}_i = \mathfrak{L}(W)_i$ and we have proved (i).
(ii) comes from the fact that, for any subspace W of V, $\dim_F(W) = \dim_{k_F} \frac{W \cap L_0}{W \cap L_e}$.
(iii) If $x \in U^1(\mathfrak{A})$ then $xW^1 \cap L_k = x(W^1 \cap L_k)$ and x acts as the identity on L_k/L_{k+1} so $\mathfrak{L}(W^1)_k = \mathfrak{L}(xW_1)_k$. Conversely, suppose $\mathfrak{L}(W^1)_k = \mathfrak{L}(W^2)_k$ for all $k \in \mathbb{Z}$. Choose a basis $\{w^k_{ij}\}$ for W^k such that $\{w^k_{ij}\} \pmod{L_{i+1}}$ is a basis of $\mathfrak{L}(W^k)_i$, for $0 \le i \le e-1$, $k = 1, 2$. There exists a subset $\mathcal{B}$ of V such that $\mathcal{B} \cup \{w^k_{ij}\}$ is an $\mathfrak{o}_F$-basis of $\mathfrak{L}$ for both values of k. There exists $x \in \mathrm{Aut}_F(V)$ such that $xw^1_{ij} = w^2_{ij}$ and such that $xb = b$ for $b \in \mathcal{B}$. Then x stabilises each L_i and acts as the identity on each quotient L_i/L_{i+1} so $x \in U^1(\mathfrak{A})$. ■
In particular, the dimension of a residual subspace $\mathcal{V}$ is well defined: $\dim_{k_F}\mathcal{V} = \sum_{i=0}^{e-1} \dim_{k_F}(\mathcal{V}_i)$, which is the dimension of any subspace W of V such that $\mathcal{V} = \mathfrak{L}(W)$.

(3.1.5) Proposition *Let $\mathfrak{L}$ be a lattice chain in V, with residual subspaces $\mathcal{V}^1$, $\mathcal{V}^2$, such that*

$$L_i/L_{i+1} = (\mathcal{V}^1)_i \oplus (\mathcal{V}^2)_i$$

for all $i \in \mathbb{Z}$. Let V^1, V^2 be subspaces of V such that $(\mathcal{V}^k)_i = \mathfrak{L}(V^k)_i$ for all i, k. Then $V = V^1 \oplus V^2$, and the pair $(\mathfrak{L}(V^1)_i, \mathfrak{L}(V^2)_i)$ determines the pair (V^1, V^2) up to translation by an element of $U^1(\mathfrak{A})$.
Further, for any such pair (V^1, V^2), we have

$$L_k = (L_k \cap V^1) \oplus (L_k \cap V^2), \quad k \in \mathbb{Z}$$

Proof: Choose bases $\{v_{ij}^k\}$ of $\mathcal{V}_i^k$, for $0 \le i \le e-1$, $k = 1, 2$ and let $\hat{v}_{ij}^k \in L_i$ be such that $\hat{v}_{ij}^k + L_{i+1} = v_{ij}^k$. Let V^k be the F-span of the $\hat{v}_{ij}^k$, for $k = 1, 2$. Then $\{\hat{v}_{ij}^k : k = 1, 2\}$ is an $\mathfrak{o}_F$-basis of $\mathfrak{L}$ so $\mathcal{V}^k = \mathfrak{L}(V^k)$, for $k = 1, 2$ and $V = V^1 \oplus V^2$. The uniqueness up to translation by an element of $U^1(\mathfrak{A})$ follows as in (3.1.4).
We clearly have $(L_k \cap V^1) \oplus (L_k \cap V^2) \subset L_k$ for $k \in \mathbb{Z}$. The converse holds since $\{\hat{v}_{ij}^k : k = 1, 2\}$ is an $\mathfrak{o}_F$-basis of $\mathfrak{L}$. ■

(3.1.6) Proposition *Let $\mathfrak{L} = \{L_k : k \in \mathbb{Z}\}$ be a self-dual lattice chain in V. Let $d \in \mathbb{Z}$ be the unique integer such that $L_k^\# = L_{d-k}$ for all $k \in \mathbb{Z}$, given by (1.1.2). Then for each $i \in \mathbb{Z}$, h induces a nondegenerate k_F-sesquilinear pairing*

$$\boldsymbol{h}_i : \frac{L_i}{L_{i+1}} \times \frac{L_{d-i-1}}{L_{d-i}} \to k_F \tag{3.1.7}$$

Proof: For $i \in \mathbb{Z}$, $x \in L_i$, $y \in L_{d-i-1}$, we have $\pi_F x \in L_{i+1}$ so $h(\pi_F x, y) \in \mathfrak{p}_F$. Hence $h(x, y) \in \mathfrak{o}_F$ and we can reduce modulo $\mathfrak{p}_F$ to get the pairing $\boldsymbol{h}_i$. For nondegeneracy we need that if $y \in L_{d-i-1}$ satisfies $h(L_i, y) \subset \mathfrak{p}_F$ then $y \in L_{d-i}$, which is immediate, and symmetrically. ■

Now let $\mathcal{V} = \{\mathcal{V}_i\}$ be a residual subspace of our self-dual lattice chain $\mathfrak{L}$. We say that $\mathcal{V}$ is *nondegenerate* if the pairing

$$\mathcal{V}_i \times \mathcal{V}_{d-i-1} \to k_F$$

induced by (3.1.7) is nondegenerate. Irrespective of this condition, given a residual subspace $\mathcal{V}$, we can define its orthogonal complement $\mathcal{V}^\perp = \{\mathcal{V}_i^\perp : i \in \mathbb{Z}\}$ by

$$\mathcal{V}_i^\perp = \{x \in L_i/L_{i+1} : \boldsymbol{h}_i(x, v) = 0, v \in \mathcal{V}_{d-i-1}\}.$$

Then $\mathcal{V}^\perp$ is indeed a residual subspace of $\mathfrak{L}$ and we have the identity

$$\mathcal{V}^{\perp\perp} = \mathcal{V}.$$

In the usual way, $\mathcal{V}$ is nondegenerate if and only if $\mathcal{V} \cap \mathcal{V}^\perp = \{0\}$, or, more precisely,

$$L_i/L_{i+1} = \mathcal{V}_i \oplus \mathcal{V}_i^\perp,$$

for all i.

(3.1.8) Proposition *(i) Let $\mathcal{V}$ be a nondegenerate residual subspace of the self-dual lattice chain $\mathfrak{L}$. Let W be a subspace of V such that $\mathfrak{L}(W) = \mathcal{V}$. Then $h|_{W\times W}$ is nondegenerate.*
(ii) Let W be any subspace of V such that $\mathfrak{L}(W)$ is nondegenerate. Then $\mathfrak{L}(W^\perp)$ is nondegenerate and

$$\mathfrak{L}(W^\perp) = \mathfrak{L}(W)^\perp.$$

Proof: (i) Let $w \in W$ be such that $\nu_{\mathfrak{L}}(w) = i$; so $w \in L_i$ but $w \notin L_{i+1}$. By hypothesis there exists $w' \in W \cap L_{d-i-1}$ such that $\boldsymbol{h}_i(w + L_{i+1}, w' + L_{d-i-1}) \neq 0$. Hence $h(w, w') \notin \mathfrak{p}_F$ and, in particular, $h(w, w') \neq 0$.

(ii) We certainly have $\mathfrak{L}(W^\perp)_i \subset \mathfrak{L}(W)_i^\perp$ for all $i \in \mathbb{Z}$. The result then follows by comparing dimensions. ■

(3.1.9) Corollary *Let $\mathcal{V}^1$, $\mathcal{V}^2$ be nondegenerate residual subspaces of $\mathfrak{L}$ such that $L_i/L_{i+1} = \mathcal{V}_i^1 \perp \mathcal{V}_i^2$ for $i \in \mathbb{Z}$. Then there exist subspaces V^1, V^2 of V such that $V = V^1 \perp V^2$ and $\mathfrak{L}(V^k) = \mathcal{V}^k$ for $k = 1, 2$.*

(3.1.10) Lemma *Let W be a totally isotropic subspace of V. Then $\mathfrak{L}(W^\perp) = \mathfrak{L}(W)^\perp$.*

Proof: The proof is the same as (3.1.8)(ii). ■

We call a residual subspace $\mathcal{V}$ of $\mathfrak{L}$ *totally isotropic* if $\boldsymbol{h}_i(\mathcal{V}_i, \mathcal{V}_{d-i-1}) \equiv 0$, for all $i \in \mathbb{Z}$.

(3.1.11) Proposition *Let $\mathcal{W}$ be a totally isotropic subspace of $\mathfrak{L}$. Then there exists a totally isotropic subspace W of V such that $\mathcal{W} = \mathfrak{L}(W)$.*

Proof: Put $n = \dim_{k_F}\mathcal{W}$. The idea of the proof is to split $\mathcal{W}$ into (totally isotropic) one-dimensional residual subspaces, contained in mutually orthogonal nondegenerate two-dimensional subspaces X^i of V, $1 \leq i \leq n$. Hence we reduce to the case $\dim_{k_F}\mathcal{W} = 1$ and $\dim_F V = 2$.

We choose a non-zero $w_1 \in \mathcal{W}_{r_1}$, for some $r_1 \in \mathbb{Z}$. By the nondegeneracy of $\boldsymbol{h}_{r_1}$, there exists $v_1 \in \mathcal{V}_{d-r_1-1}$ such that $\boldsymbol{h}_{r_1}(w_1, v_1) = 1$. Let $\mathcal{Y}^1$, $\mathcal{Z}^1$ be the residual subspaces given by

$$\mathcal{Y}_k^1 = \begin{cases} \widetilde{\pi_F}^s \langle w_1 \rangle_{k_F} & \text{if } k = r_1 + se; \\ 0 & \text{otherwise,} \end{cases}$$
$$\mathcal{Z}_k^1 = \begin{cases} \widetilde{\pi_F}^s \langle v_1 \rangle_{k_F} & \text{if } k = d - r_1 - 1 + se; \\ 0 & \text{otherwise,} \end{cases}$$

where $e = e(\mathfrak{L})$ is the $\mathfrak{o}_F$-period of $\mathfrak{L}$, $\langle v \rangle_{k_F} = k_F v$ is the k_F-span of v and $\widetilde{\pi_F}$ is the k_F-linear isomorphism $L_k/L_{k+1} \to L_{k+e}/L_{k+e+1}$ induced by multiplication by π_F (see (3.3) for more details). We put $\mathcal{X}^1 = \mathcal{Y}^1 + \mathcal{Z}^1$. The residual subspace $\mathcal{X}^1$ is nondegenerate so, by (3.1.8), there exists a subspace X^1 of V such that $\mathfrak{L}(X^1) = \mathcal{X}^1$ and $h_{X^1 \times X^1}$ is nondegenerate.

Replacing V by $X^{1\perp}$ and $\mathcal{W}$ by $\mathcal{W} \cap \mathcal{X}^{1\perp}$, we may repeat the above to obtain, for $1 \leq i \leq n$, residual vectors w_i, v_i and nondegenerate subspaces X^i of V such that $X^i \perp X^j$, for $i \neq j$, $\dim_F X^i$=2 and $\mathcal{W}$ is generated by $\langle w_1, ...w_n \rangle_{k_F}$ and $\widetilde{\pi_F}$.

For each i we will find $\hat{w}_i \in X^i$ such that $\hat{w}_i + L_{r_i+1} = w_i$ and such that $h(\hat{w}_i, \hat{w}_i) = 0$. Then we put $W = \langle \hat{w}_i : 1 \le i \le n \rangle_F$ and we are done.

We drop the subscript i. Let $x \in X$ be such that $x + L_{r+1} = w$. Let $s \in \mathbb{Z}$ be such that $r + se \ge d - r - 1 > r + (s-1)e$ so

$$L_{r+se+1} \subseteq L_r^{\#} \subsetneq L_{r+(s-1)e+1}.$$

Suppose first that $r + se > d - r - 1$; then $\pi_F^s x \in L_{r+se} \subseteq L_r^{\#}$ so $h(x,x) \in \mathfrak{p}_F^{1-s}$. Otherwise, $r + se = d - r - 1$; then $\pi_F^s x \in L_{d-r-1}$. Then $h(\pi_F^s x, x) + \mathfrak{p}_F = \boldsymbol{h}_{d-r-1}(\widetilde{\pi_F}^s w, w) = 0$ so $h(x,x) \in \mathfrak{p}_F^{1-s}$ in this case also.

For $n \in \mathbb{N}$ we find, by induction $x_n \in X$ such that $w = x_n + L_{r+1}$, $x_{n+1} - x_n \in L_{r+(n-1)e+1}$ and $h(x_n, x_n) \in \mathfrak{p}_F^{n-s}$. Then we put $\hat{w} = \lim_{n\to\infty} x_n$ and we are done.

We may take $x_1 = x$ so assume we have found x_n as required. If $h(x_n, x_n) \in \mathfrak{p}_F^{n+1-s}$ then we may take $x_{n+1} = x_n$ so assume $h(x_n, x_n) \in \mathfrak{p}_F^{n-s} \backslash \mathfrak{p}_F^{n+1-s}$; say $h(x_n, x_n) = \alpha_n \pi_F^{n-s}$, $\alpha_n \in \mathfrak{o}_0^{\times}$, $\pi_F^{n-s} \in F_0$. By the nondegeneracy of $\boldsymbol{h}_r : \mathcal{V}_r \times \mathcal{V}_{d-r-1} \to k_F$, there exists $y_n \in L_{d-r-1}$ such that $h(x_n, y_n) + \mathfrak{p}_F = \frac{1}{2}\alpha_n$. Then $h(y_n, y_n) \in \mathfrak{p}_F^s$ since $L_{d-r-1} \subset L_{r+(s-1)e+1} = \pi_F^{s-1} L_{d-r-1}^{\#}$ and $\pi_F^{n-s} y_n \in L_{d-r-1+(n-s)e} \subset L_{r+(s-1)e+1+(n-s)e} = L_{(n-1)e+r+1}$. Then we put $x_{n+1} = x_n - \pi_F^{n-s} y_n$ and we have

$$\begin{aligned} h(x_{n+1}, x_{n+1}) &= h(x_n, x_n) - \pi_F^{n-s} h(x_n, y_n) - \pi_F^{n-s} h(y_n, x_n) + \pi_F^{2(n-s)} h(y_n, y_n) \\ &= \alpha_n \pi_F^{n-s} - \frac{1}{2}\alpha_n \pi_F^{n-s} + O(\mathfrak{p}_F^{n+1-s}) \\ &\qquad - \frac{1}{2}\alpha_n \pi_F^{n-s} + O(\mathfrak{p}_F^{n+1-s}) + O(\mathfrak{p}_F^{2n-s}) \\ &\in \mathfrak{p}_F^{n+1-s}, \qquad \text{since } n \ge 1. \end{aligned}$$

■

(3.2) Splittings of self-dual lattice chains

We continue with the notation of the previous section; in particular h is a nondegenerate ϵ-hermitian form on V. Let $\mathfrak{L}$ be a self-dual lattice chain in V.

(3.2.1) Definition *A* splitting *of $\mathfrak{L}$ is a decomposition of V*

$$V = V_\infty \oplus V_{-r} \oplus \cdots \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus \cdots \oplus V_r$$

into F-subspaces such that the following hold:
(i) For all $k \in \mathbb{Z}$ we have $L_k = \bigoplus_{\omega\in\Omega} L_k \cap V_\omega$, where Ω is the set of $\omega \in \{\infty, -r, \ldots -1, 0, 1, \ldots, r\}$ such that $V_\omega \ne 0$;
(ii) $\mathfrak{L}^\omega := \{L_k \cap V_\omega : k \in \mathbb{Z}$ is a lattice chain of period 1, for $\omega \in \Omega$;
(iii) $V_\omega^\perp = \bigoplus_{v \ne -\omega} V_v$ for all $\omega \in \Omega$ (with the understanding that $-\infty = \infty$).

(3.2.2) Proposition *Let $\mathfrak{L}$ be a self-dual lattice chain in V. Then there exists a splitting of $\mathfrak{L}$.*

Proof: Let $d = 0$ or -1 be the integer such that $L_k^{\#} = L_{d-k}$ for $k \in \mathbb{Z}$, given by (1.1.2), and let $e = e(\mathfrak{L})$ be the $\mathfrak{o}_F$-period of the lattice chain. Define residual subspaces $\mathcal{V}^{(i)}$, for $i \in \mathbb{Z}$, by

$$\mathcal{V}_k^{(i)} = \begin{cases} L_k / L_{k+1} & i \equiv k \pmod{e} \\ 0 & \text{otherwise.} \end{cases}$$

Then h induces a duality $\mathcal{V}^{(i)} \leftrightarrow \mathcal{V}^{(d-i-1)}$ by (3.1.6). Set $\mathcal{W}^{(j)} = \mathcal{V}^{(j-1)} + \mathcal{V}^{(d-j)}$ for $j \in \mathbb{Z}$; this is a direct sum if $2j \not\equiv d+1 (\text{mod } g)$, while $\mathcal{V}^{(j-1)} = \mathcal{V}^{(d-j)}$ otherwise.

(3.2.3) Lemma *The residual subspaces $\mathcal{W}^{(j)}$ are nondegenerate.*

Proof: We have to show that the pairing

$$\mathcal{W}_k^{(j)} \times \mathcal{W}_{d-k-1}^{(j)} \to k_F$$

is nondegenerate. But if $\mathcal{W}_k^{(j)} \neq 0$ then, without loss of generality, $\mathcal{V}_k^{(d-j)} = L_k/L_{k+1} \neq 0$ so $k \equiv d-j \pmod g$. Then $d-k \equiv j \pmod g$ so $\mathcal{W}_{d-k}^{(j)} = \mathcal{V}_{d-k}^{(j)} = L_{d-k}/L_{d-k+1}$. Then by (3.1.6) the pairing is nondegenerate. ■

Set $f = [\frac{e+1-d}{2}]$; then $L_k/L_{k+1} = \mathcal{W}_k^{(d+1)} \oplus \mathcal{W}_k^{(d+2)} \oplus \cdots \oplus \mathcal{W}_k^{(d+f)}$ for all $k \in \mathbb{Z}$. The residual subspaces $\mathcal{W}^{(d+1)}, \ldots, \mathcal{W}^{(d+f)}$ are orthogonal so, by (3.1.9), there exist subspaces $W^{(d+1)}, \ldots, W^{(d+f)}$ of V such that

$$V = W^{(d+1)} \perp \cdots \perp W^{(d+f)}$$

and $\mathcal{W}^{(j)} = \mathfrak{L}(W^{(j)})$ for $d+1 \leq j \leq d+f$. So we can write $h = h_{d+1} \perp \cdots \perp h_{d+f}$ with h_j a nondegenerate ϵ-hermitian form on $W^{(j)}$.

(3.2.4) Lemma $\mathfrak{L}^j = \{L_k \cap W^{(j)} : k \in \mathbb{Z}\}$ *is a self-dual lattice chain in* $W^{(j)}$, $d+1 \leq j \leq d+f$. *Moreover,*

$$e_j = e(\mathfrak{L}_j) = \begin{cases} 1 & \text{if } 2j \equiv d+1 \pmod e \\ 2 & \text{otherwise.} \end{cases}$$

Proof: That $\mathfrak{L}^j$ is a lattice chain of period e_j in $W^{(j)}$ is clear. For all $k \in \mathbb{Z}$, $(L_k \cap W^{(j)})^\# = L_k^\# + (W^{(j)})^\# = L_{d-k} + W^{(1)} + \cdots + W^{(j-1)} + W^{(j+1)} + \cdots + W^{(f)}$. Restricting the duality operation to $W^{(j)}$, this says that $(L_k \cap W^{(j)})^\#$ is the projection in $W^{(j)}$ of L_{d-k}, which is $L_{d-k} \cap W^{(j)}$. ■

Note that all but at most 2 of the $\mathfrak{L}^j$ have period 2. If $d = -1$ then $e_{d+1} = 1$ and put $V_{d+1} = W^{(d+1)}$, otherwise $V_{d+1} = 0$. If $2f = e+1-d$ then $e_{d+f} = 1$ and put $V_\infty = W^{(d+f)}$, otherwise $V_\infty = 0$.

Now consider the case $e_j = 2$. Then $\mathcal{V}^{(j-1)}$ and $\mathcal{V}^{(d-j)}$ are totally isotropic residual subspaces of $\mathfrak{L}^j$ so, by (3.1.11), there are totally isotropic subspaces V_j and V_{-j} of $W^{(j)}$ such that $\mathfrak{L}(V_j) = \mathcal{V}^{(j-1)}$ and $\mathfrak{L}(V_{-j}) = \mathcal{V}^{(d-j)}$. Then clearly $W^{(j)} = V_j \oplus V_{-j}$.

Altogether we have

$$V = V_\infty \oplus V_{-r} \oplus \cdots \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus \cdots \oplus V_r$$

as required, $e = 2r$, $2r+1$ or $2r+2$. So this completes the proof of (3.2.2). ■

We think of $\Omega \subset \{\infty, -r, ..., r\}$ as an ordered set of indices. Also, $\mathbb{Z}/e\mathbb{Z}$ acts on Ω by cyclic permutation. For $\omega \in \Omega$, $i \in \mathbb{Z}$, we will write $\omega + i$ for the translate by $i + e\mathbb{Z}$ of ω.

We can push this further. We choose $\mathfrak{o}_F$-bases for $V_\infty \cap L_{d-r-1}$, $V_{-j} \cap L_{d-j}$ for $r \geq j \geq 1$, and $V_0 \cap L_d$ and $\mathfrak{o}_F$-bases for $V_j \cap L_{j-1}$ for $1 \leq j \leq r$. Then, with respect to this basis $\mathcal{B}$, h has matrix

$$\begin{pmatrix} J_\infty & 0 & \cdots & & & & \cdots & 0 \\ 0 & 0 & \cdots & & & \cdots & 0 & I \\ \vdots & \vdots & & & & \iddots & \iddots & 0 \\ & & & & & I & \iddots & \vdots \\ & & & & J_0 & & & \\ & \vdots & \iddots & \epsilon I & & & & \\ \vdots & 0 & \iddots & \iddots & & & & \vdots \\ 0 & \epsilon I & 0 & \cdots & & & \cdots & 0 \end{pmatrix} \qquad \text{where} \qquad I = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

J_ω is the matrix of $h|_{V_\omega \times V_\omega}$, for $\omega = \infty$, 0, and $\mathfrak{A} = \mathfrak{A}(\mathfrak{L})$ has the form

$$\textbf{(3.2.5)} \qquad \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \cdots & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{o}_F & \ddots & & \mathfrak{o}_F \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathfrak{p}_F & & \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \cdots & \mathfrak{p}_F & \mathfrak{o}_F \end{pmatrix}.$$

So we have chosen a basis of $\mathfrak{L}$ in the sense of **[BK]** (1.1.7). Note that the above matrices are $\boldsymbol{n}$-block matrices, where $\boldsymbol{n} = (n_\infty, n_{-r}, \ldots, n_r)$ is a vector of positive integers and $n_\omega = \dim V_\omega$, for $\omega \in \Omega$.

Putting $A^{(\omega\omega')} = \mathrm{Hom}(V_{\omega'}, V_\omega)$, for $\omega, \omega' \in \Omega$, we get a block decomposition $A = \coprod_{\omega,\omega'} A^{(\omega\omega')}$ with respect to which $\mathfrak{A}$ has the form above. Then, for $a \in A^{(\omega\omega')}$, we have $h(V_{v'}, \overline{a}V_v) = h(aV_{v'}, V_v)$ which is non-zero if and only if $v' = \omega'$ and $v = -\omega$. Hence $\overline{a} \in A^{(-\omega', -\omega)}$. In particular, the involution on A fixes diagonal "bands" of blocks in A.

We now define a set of representatives for the cosets $\mathfrak{P}^m/\mathfrak{P}^{m+1}$, for any integer m, as in **[BK]** (2.5.5). We define integers l, k by

$$\textbf{(3.2.6)} \qquad 0 \leq l = m - ek \leq e - 1.$$

Consider the set $\boldsymbol{B}_l = \boldsymbol{B}_l(\mathfrak{A})$ of e-tuples of matrices

$$\boldsymbol{b} = (b_\omega : \omega \in \Omega)$$

where $\boldsymbol{b}_\omega$ has entries in $\mathfrak{o}_F$ and dimensions $n_\omega \times n_{\omega+l}$. For $\boldsymbol{b} \in \boldsymbol{B}_l$, we define an $\boldsymbol{n}$-block matrix $\boldsymbol{r}_m(\boldsymbol{b})$ over F by

$$\boldsymbol{r}_m(\boldsymbol{b})_{\omega,\omega'} = \begin{cases} \pi_F^k b_\omega & \text{if } \omega \leq r - l \text{ and } \omega = \omega' - l, \\ \pi_F^{k+1} b_\omega & \text{if } \omega > r - l \text{ and } \omega = \omega' - l, \\ 0 & \text{otherwise.} \end{cases}$$

Then the map $\boldsymbol{b} \to \boldsymbol{r}_m(\boldsymbol{b})$ induces a bijection between $\boldsymbol{B}_l(\mathfrak{A})$ (modulo $\mathfrak{p}_F$) and $\mathfrak{P}^m/\mathfrak{P}^{m+1}$.

Moreover, since the involution fixes bands of blocks, $\overline{\boldsymbol{r}_m(\boldsymbol{b})} = \boldsymbol{r}_m(\boldsymbol{b}')$, for some $\boldsymbol{b}' \in \boldsymbol{B}_l(\mathfrak{A})$. Then we can define an involution $^-$ on $\boldsymbol{B}_l(\mathfrak{A})$ by putting $\overline{\boldsymbol{b}} = \boldsymbol{b}'$. (Note that, in the case $F \neq F_0$, the involution on $\boldsymbol{B}_l(\mathfrak{A})$ is dependent on m.)

(3.2.7) Definition *Let* $[\mathfrak{A}, n, n-1, b]$ *be a stratum in* A. *It is in* band form *if* $b = \boldsymbol{r}_{-n}(\boldsymbol{b})$, *for some* $\boldsymbol{b} \in \boldsymbol{B}_l(\mathfrak{A})$, *where* $l, k \in \mathbb{Z}$ *are such that* $0 \leq l = -n - ek \leq e-1$.

Any stratum is equivalent to a stratum in band form and further, since the involution on A fixes bands of blocks, any skew stratum is equivalent to a skew stratum in band form.

It will sometimes be more convenient to choose a slightly different basis, which is a basis for $\mathfrak{L}$ in the sense of (3.1.1). We can choose bases for V_∞ and V_0 such that

$$\textbf{(3.2.8)} \qquad J_\infty = \begin{pmatrix} & & \pi_F^{-1} I \\ & K_\infty & \\ \epsilon \pi_F^{-1} I & & \end{pmatrix}, \qquad J_0 = \begin{pmatrix} & & I \\ & K_0 & \\ \epsilon I & & \end{pmatrix},$$

where K_ω is the matrix of the anisotropic part of $h|_{V_\omega \times V_\omega}$, $\omega = \infty, 0$. Then, rearranging the bases for $V_\infty, V_{-r}, \ldots, V_r$, the form h has matrix

$$\textbf{(3.2.9)} \qquad \begin{pmatrix} K_\infty & & & \\ & & & I \\ & & K_0 & \\ & \epsilon I & & \end{pmatrix}.$$

Moreover, the matrix

$$K = \begin{pmatrix} K_\infty & \\ & K_0 \end{pmatrix}$$

is at most 2×2 since the anisotropic part of V with respect to h is at most 2-dimensional. From **[M1]** (1.8), the only possibilities for K are

$$K = (\,\delta\,), \quad (\,\delta\varepsilon\,) \quad \text{or} \quad \begin{pmatrix} -\delta\varepsilon & \\ & \delta \end{pmatrix}$$

where $\varepsilon \in F_0$, $\varepsilon \notin N_{F/F_0}(F)$ and $\delta = 1$, if h is hermitian, $\delta \in F_-$, if h is skew-hermitian.

(3.3) Residual maps

In this section we look at maps between residual subspaces.

Let $\mathfrak{L}$ be an $\mathfrak{o}_F$-lattice chain in V and $\mathfrak{A}$ the associated hereditary order in A. Let $x \in A$ be such that $\nu_{\mathfrak{A}}(x) = n$. Then x induces maps

$$\begin{aligned} \widetilde{x}_k : L_k/L_{k+1} &\to L_{k+n}/L_{k+n+1} \\ v + L_{k+1} &\mapsto xv + L_{k+n+1} \end{aligned}$$

for $k \in \mathbb{Z}$. Let $\mathcal{V}$ be the residual subspace $\mathfrak{L}(V)$ of $\mathfrak{L}$, so $\mathcal{V}_k = L_k/L_{k+1}$, for $k \in \mathbb{Z}$. Then we say that $\widetilde{x} : \mathcal{V} \to \mathcal{V}$ is of *degree n*.

(3.3.1) Lemma *Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A. Then $x \in \mathfrak{K}(\mathfrak{A})$ if and only if $\widetilde{x}_k$ is an isomorphism for all $k \in \mathbb{Z}$.*

Proof: Suppose $x \in \mathfrak{K}(\mathfrak{A})$ and put $n = \nu_{\mathfrak{A}}(x)$; then $\nu_{\mathfrak{A}}(x^{-1}) = -n$ so x^{-1} induces a map of degree $-n$, $\widetilde{x^{-1}} : \mathcal{V} \to \mathcal{V}$. Then, for $k \in \mathbb{Z}$, we see that $(\widetilde{x^{-1}})_{k+n} \circ \widetilde{x}_k$ is the identity on L_k/L_{k+1}, since $\widetilde{x}_k(v + L_{k+1}) = xv + L_{k+n+1} = x(v + L_{k+1})$.

Conversely, suppose $x \notin \mathfrak{K}(\mathfrak{A})$ and put $n = \nu_{\mathfrak{A}}(x)$. We have that $xL_k \subsetneqq L_{k+n}$, for some $k \in \mathbb{Z}$. Then $\widetilde{x}_k$ is not surjective; in particular, it is not an isomorphism. ■

In particular, π_F induces an isomorphism $\widetilde{\pi_F} : \mathcal{V} \to \mathcal{V}$ of degree $e = e(\mathfrak{L})$, the $\mathfrak{o}_F$-period of $\mathfrak{L}$. Indeed, if $\mathcal{W}$ is any residual subspace of $\mathfrak{L}$ then $(\widetilde{\pi_F})_k\mathcal{W}_k = \mathcal{W}_{k+e}$, for all $k \in \mathbb{Z}$, so we have an isomorphism $\widetilde{\pi_F} : \mathcal{W} \to \mathcal{W}$.

Let $\mathcal{W}^1$, $\mathcal{W}^2$ be residual subspaces of $\mathfrak{L}$ and let $\varrho : \mathcal{W}^1 \to \mathcal{W}^2$ be of degree n. i.e. ϱ is a set of k_F-linear maps $\{\varrho_k : k \in \mathbb{Z}\}$, where $\varrho_k : \mathcal{W}^1_k \to \mathcal{W}^2_{k+n}$. Then we say ϱ is a *residual map* if $\varrho \circ \widetilde{\pi_F} = \widetilde{\pi_F} \circ \varrho$, i.e. for all $k \in \mathbb{Z}$

$$\varrho_{k+e} \circ (\widetilde{\pi_F})_k = (\widetilde{\pi_F})_{k+n} \circ \varrho_k.$$

(3.3.2) Lemma *Let $\varrho : \mathcal{V} \to \mathcal{V}$ be a residual map of degree n. Then there exists $x \in A$, $\nu_{\mathfrak{A}}(x) = n$ such that $\varrho = \widetilde{x}$.*

Proof: We may assume that $0 \le n \le e - 1$ by scaling by $\widetilde{\pi_F}$. For $0 \le k \le e - 1$ choose a basis $\{v_{ik} : 1 \le i \le d_i\}$ of $\mathcal{V}_k$ and choose $\hat{v}_{ik} \in V$ such that $\hat{v}_{ik} + L_{k+1} = v_{ik}$. Taking the index k modulo e, we have

$$\varrho(v_{ik}) = \sum_{1 \le i \le d_{n+k}} \varrho_{ik} v_{i,n+k}$$

for some $\varrho_{ik} \in k_F$. Choose $x_{ik} \in \mathfrak{o}_F$ such that $x_{ik} + \mathfrak{p}_F = \varrho_{ik}$ for each i, k and define x by

$$x\hat{v}_{ik} = \sum_{1 \le i \le d_{n+k}} x_{ik}\hat{v}_{i,n+k}.$$

Clearly $x \in A$, $\nu_{\mathfrak{A}}(x) = n$ and $\widetilde{x} = \varrho$. ■

We write $\mathrm{End}^n_{k_F}\mathcal{V}$ for the k_F-module of residual maps of degree n.

(3.3.3) Corollary *For each $n \in \mathbb{Z}$, the natural map $\mathfrak{P}^n \to \mathrm{End}^n_{k_F}\mathcal{V} : x \mapsto \widetilde{x}$ induces an isomorphism of k_F-spaces $\mathfrak{P}^n/\mathfrak{P}^{n+1} \to \mathrm{End}^n_{k_F}\mathcal{V}$.*

Proof: The natural map is surjective by (3.3.2) and clearly the kernel is $\mathfrak{P}^n$. ■

Note also that if ϱ is a residual isomorphism (i.e. ϱ_k is an isomorphism for each $k \in \mathbb{Z}$) then $x \in \mathfrak{K}(\mathfrak{A})$.

Let now $h : V \times V \to F$ be an ϵ-hermitian form as before and let $\mathfrak{L}$ be a self-dual lattice chain in V. Then, by (3.1.6) we have nondegenerate sesquilinear forms $\boldsymbol{h}_k : \mathcal{V}_k \times \mathcal{V}_{d-k-1} \to k_F$ for all $k \in \mathbb{Z}$.

Let $\varrho : \mathcal{V} \to \mathcal{V}$ be a residual map of degree n. Then we define the *adjoint* residual map $\overline{\varrho}$, also of degree n, by

$$\boldsymbol{h}_{k+n}(\varrho_k v, v') = \boldsymbol{h}_k(v, \overline{\varrho}_{d-k-1-n} v')$$

for $v \in \mathcal{V}_k$, $v' \in \mathcal{V}_{d-k-1-n}$. This is well-defined as $\boldsymbol{h}_k$ is nondegenerate for all $k \in \mathbb{Z}$.

(3.3.4) Lemma *Let $x \in A$; then $\overline{\widetilde{x}} = \widetilde{\overline{x}}$.*

Proof: Put $n = \nu_{\mathfrak{A}}(x)$ and choose any $v \in \mathcal{V}_k$, $v' \in \mathcal{V}_{d-k-1-n}$. Choose also $\hat{v}, \hat{v}' \in V$ such that $\hat{v} + L_{k+1} = v$ and $\hat{v}' + L_{d-k-n} = v'$. Then

$$\begin{aligned} \boldsymbol{h}_{k+n}(\widetilde{x}v, v') &= h(x\hat{v}.\hat{v}') + \mathfrak{p}_F \\ &= h(\hat{v}, \overline{x}\hat{v}') + \mathfrak{p}_F \\ &= \boldsymbol{h}_k(v, \widetilde{\overline{x}}v'). \end{aligned}$$

■

A residual map ϱ is called *skew* if $\varrho = -\overline{\varrho}$.

(3.3.5) Lemma *Let ϱ be a skew residual map of degree n. Then there exists $x \in A_-$, $\nu_{\mathfrak{A}}(x) = n$, such that $\varrho = \widetilde{x}$.*

Proof: Let $y \in A$, $\nu_{\mathfrak{A}}(y) = n$ be such that $\widetilde{y} = \varrho$, by (3.3.2). Then $\widetilde{(-\overline{y})} = -\overline{\widetilde{y}} = \varrho$ so, by (3.3.3), $-\overline{y} \in y + \mathfrak{P}^{1+n}$. Then, by (1.2.2), there exists $u \in \mathfrak{P}^{1+n}$ such that $y + \overline{y} = u + \overline{u}$ and we put $x = y - u$. ■

(3.3.6) Corollary *Writing $\mathrm{End}^n_{k_F,-}\mathcal{V}$ for the skew residual maps of degree n, we have an isomorphism $\mathfrak{P}^n_-/\mathfrak{P}^{n+1}_- \to \mathrm{End}^n_{k_F,-}\mathcal{V}$.*

Put $\mathcal{A} = \mathrm{End}^0_{k_F}\mathcal{V}$. Then the map

$$\begin{aligned} \mathcal{A} &\to \prod_{i=0}^{e-1} \mathrm{End}_{k_F} L_i/L_{i+1} \\ \varrho &\mapsto (\varrho_0, \ldots, \varrho_{e-1}) \end{aligned}$$

is an isomorphism of k_F-algebras since $\varrho_{re+i} = \widetilde{\pi_F}^{\,r} \varrho_i \widetilde{\pi_F}^{\,-r}$ for all $r \in \mathbb{Z}$, $0 \le i \le e-1$. So we have notions of minimal polynomial (over k_F) and Jordan decomposition in $\mathcal{A}$.

4
REFINEMENTS

In this chapter, h is an alternating form, i.e. this chapter is only for the symplectic group.

In this chapter we look at the refinement process for skew simple strata (cf. **[BK]** (2.2)). In order to do this we must first adapt the notion of a (W,E)-decomposition **[BK]** (1.2) to our situation. We now recall this.

Let E/F be a subfield of A and write $A(E) = \mathrm{End}_F(E)$, $B = \mathrm{End}_E(V)$. Let W be the F-span of an E-basis of our vector space V. Then the choice of W induces an embedding of algebras $\iota_W : A(E) \hookrightarrow A$ and an isomorphism

(4.0.1) $$A(E) \otimes_E B \simeq A$$

of $(A(E), B)$-bimodules (**[BK]** (1.2.6)). Moreover, if $\mathfrak{A}$ is a hereditary order in A normalised by E then, by **[BK]** (1.2.8) we can choose W such that the isomorphism of (4.0.1) restricts to an isomorphism

(4.0.2) $$\mathfrak{A}(E) \otimes_{\mathfrak{o}_E} \mathfrak{B},$$

where $\mathfrak{A}(E)$ is the unique hereditary $\mathfrak{o}_F$-order in A normalised by E (**[BK]** (1.2.7)) and $\mathfrak{B} = \mathfrak{A} \cap B$.

In our situation, E/F is a subfield of A stable under the involution but not fixed pointwise by it. Then E, considered as an F-vector space, has a nondegenerate alternating form on it (indeed we will need two different forms in general). Then $A(E)$ has an involution and we would like $A(E)_-$ to be embedded in A_- by ι_W. In general, we will need to take not the space $A(E)_-$ but the space of elements skew according to both involutions (induced from the two different forms on E).

This will allow us to prove that if $[\mathfrak{A}, n, r, \beta]$ is a skew simple stratum and $b \in \mathfrak{P}_-^{-r}$ such that $[\mathfrak{B}, r, r-1, s(b)]$ is equivalent to a skew simple stratum then the stratum $[\mathfrak{A}, n, r-1, \beta + b]$ is equivalent to a skew simple stratum (cf. **[BK]** (2.2.8)).

(4.1) (W,E)-decomposition

Let E/F be a subfield of A such that $E = \overline{E}$ and let E_0 be the fixed field of the involution, $E_0 \neq E$. Let $e = e(E|F)$ be the ramification degree of E/F and let $\mathfrak{D}^{-1}$ be the inverse different of E/F. Let $B = \mathrm{End}_E V$ be the A-centraliser of E so $B = \overline{B}$. There exists a hermitian form $f : V \times V \to E$ by

$$h(ev, w) = \mathrm{tr}_{E/F}(\delta e f(v,w)) \quad \text{for all } e \in E, \qquad v, w \in V$$

for some $\delta \in \mathfrak{p}_E{}^{e-1}\mathfrak{D}^{-1}$, $\delta \notin \mathfrak{p}_E{}^{e+1}\mathfrak{D}^{-1}$, such that $\delta + \overline{\delta} = 0$. This is well-defined as $\mathrm{tr}_{E/F}$ is nondegenerate, and f itself is nondegenerate by the nondegeneracy of

h. Then f determines an adjoint involution on B, which is in fact the involution $\bar{\ }$ on A restricted to B since, for $x \in B$ and $v, w \in V$, $e \in E$, we have

$$\begin{aligned} \mathrm{tr}_{E/F}(\delta e f(xv, w)) = h(exv, w) &= h(xev, w) \\ &= h(ev, \overline{x}w) = \mathrm{tr}_{E/F}(\delta e f(v, \overline{x}w)). \end{aligned}$$

As a direct consequence of this we have:

(4.1.1) Lemma *The centraliser in G of E, $B \cap G$, is the unitary group of E-automorphisms of V which fix the hermitian form f.*

From the theory of hermitian forms (see **[Sch]** (1.5.11), (7.1)), V then has a Witt decomposition as an E-space, $V = V^{an} \perp V^{sp}$ where $f|_{V^{an}\times V^{an}}$ is anisotropic, $\dim_E V^{an} \leq 2$, and $f|_{V^{sp}\times V^{sp}}$ is split (i.e. it is a sum of hyperbolic planes).
Let $\mathfrak{A}$ be an $\mathfrak{o}_F$-order in A, normalised by E, such that $\mathfrak{A} = \overline{\mathfrak{A}}$ and let $\mathfrak{L} = \mathfrak{L}(\mathfrak{A})$ be the self-dual $\mathfrak{o}_F$-lattice chain associated to it. Let $\mathfrak{B} = \mathfrak{A} \cap B$ and $\mathfrak{Q} = \mathfrak{P} \cap B$, where $\mathfrak{P}$ is the Jacobson radical of $\mathfrak{A}$, as usual. Then $\mathfrak{L}$ is in fact a self-dual $\mathfrak{o}_E$-lattice chain in B since, for $L \in \mathfrak{L}$,

$$\begin{aligned} L^{\#} = \{v \in V : h(x, L) \subset \mathfrak{p}_F\} &= \{v \in V : \delta f(v, L) \subset \mathfrak{D}^{-1}\mathfrak{p}_F\} \\ &= \begin{cases} \{v \in V : f(v, L) \subset \mathfrak{p}_E\} & \text{if } \delta \notin \mathfrak{p}_E^e \mathfrak{D}^{-1}; \\ \{v \in V : f(v, L) \subset \mathfrak{o}_E\} & \text{if } \delta \in \mathfrak{p}_E^e \mathfrak{D}^{-1}. \end{cases} \end{aligned}$$

Then by **[M1]** (1.7) we can choose a basis for the lattice chain such that, with respect to this basis, the hermitian form f has matrix:

$$\begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & C_2 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}, \qquad \text{where} \quad I = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} \qquad \text{and} \quad C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$$

and C is the matrix of $f|_{V_{an}\times V_{an}}$.

The possibilities for C are:

$$\begin{aligned} &\dim V^{an} = 1 \qquad C = (\,1\,) \quad \text{or} \quad (\,\varepsilon\,); \\ &\dim V^{an} = 0 \qquad C = \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix}; \end{aligned}$$

where $\varepsilon \in E_0$, $\varepsilon \notin \mathrm{N}_{E/E_0}(E)$. (In fact, in the case $\dim V^{an} = 1$, we may always assume $C = (1)$ by changing δ to $\delta\varepsilon$ and choosing a basis for the form given by this.) Note also that we may change ε by any norm so we will choose it carefully.

(4.1.2) Lemma *In the situation above there exists $\varepsilon \in E_0$, $\varepsilon \notin N_{E/E_0}(E)$ such that $\varepsilon^n \in F$ for some $n \in \mathbb{Z}$, $(n, p) = 1$.*

Proof: We first consider the case E/E_0 ramified. Then let ε be a root of unity in E_0 which is not a square. Then $\varepsilon^m = 1$ for some $m \in \mathbb{Z}$, $(m, p) = 1$. Then this ε will do. For suppose ε is a norm; then any element of $\mathfrak{o}_0^\times$ is a norm since

ε generates the group of roots of unity while every element of $U^1(\mathfrak{o}_0)$ is a square as $p \neq 2$. Further π_0 is also a norm, for some prime element π_0, since E/E_0 is ramified. So every element of E_0 is a norm, which is absurd.

Now consider the case E/E_0 unramified. Let E_{tr} be the maximal tamely ramified extension of F contained in E_0 and let $m = e(E_{tr}|F)$ be the ramification degree of E_{tr}/F, $(m, p) = 1$. Then we can choose a uniformizer π_{tr} of E_{tr} such that $\pi_{tr}^m = \xi\pi_F$, for ξ a (p prime) root of unity in E_{tr}. We let $\varepsilon = \pi_{tr}$ and observe that $\nu_E(\varepsilon)$ is a power of p, since E_0/E_{tr} is wildly ramified, which is odd. However, all norms have even valuation so ε cannot be a norm. ■

So we will choose ε according to (4.1.2) and fix it.

Let W be the F-subspace of V spanned by the given E-basis so that $V \cong E \otimes_F W$. Let W_1 be the subspace of W spanned by the vector v in the basis such that $f(v, v) = \pm\varepsilon$ (if it exists, $W_1 = 0$ otherwise) and let W_2 be the subspace spanned by the remaining basis vectors. Set $V_i = E \otimes_F W_i$, $i = 1, 2$. Then, for $i = 1, 2$, we define symmetric bilinear forms $f_i : W_i \times W_i \to F$ by

$$\begin{aligned} f_1(v_1, w_1) &= \varepsilon^{-1} f(v_1, w_1) \qquad \text{for } v_1, w_1 \in W_1 \\ f_2(v_2, w_2) &= f(v_2, w_2) \qquad \text{for } v_2, w_2 \in W_2. \end{aligned}$$

We also define two skew-symmetric bilinear forms $g_i : E \times E \to F$, $i = 1, 2$ by

$$\begin{aligned} g_1(e, e') &= \mathrm{tr}_{E/F}\ \varepsilon\delta e\overline{e'} \qquad \text{for } e, e' \in E \\ g_2(e, e') &= \mathrm{tr}_{E/F}\ \delta e\overline{e'} \qquad \text{for } e, e' \in E \end{aligned}$$

giving rise to involutions ${}^{\sim 1}$ and ${}^{\sim 2}$ on $A(E) = \mathrm{End}_F(E)$ respectively. These involutions both restrict to ${}^{-}$ on E for $E \hookrightarrow A(E)$. Note that, in general, the two involutions are not the same, i.e. there exists $a \in A(E)$ such that $\widetilde{a}^1 \neq \widetilde{a}^2$. In fact, for $a \in A(E)$, $\widetilde{\widetilde{a}^1}^2 = \varepsilon a \varepsilon^{-1}$.

Now define $\bigoplus_{i=1}^2 g_i \otimes f_i : V \times V \to F$ by

$$\bigoplus_{i=1}^{2} g_i \otimes f_i(e \otimes (w_1 + w_2), e' \otimes (w_1' + w_2')) = g_1(e, e')\ f_1(w_1, w_1') + g_1(e, e')\ f_2(w_2, w_2')$$

for $e, e' \in E$, $w_i, w_i' \in W_i$, $i = 1, 2$. Then

$$\begin{aligned} &\bigoplus_{i=1}^{2} g_i \otimes f_i(e \otimes (w_1 + w_2), e' \otimes (w_1' + w_2')) \\ &\qquad = \mathrm{tr}_{E/F}(\varepsilon\delta e\overline{e'})\ f_1(w_1, w_1') + \mathrm{tr}_{E/F}(\delta e\overline{e'})\ f_2(w_1, w_1') \\ &\qquad = \mathrm{tr}_{E/F}\delta e\overline{e'}[\varepsilon f_1(w_1, w_1') + f_2(w_2, w_2')] \\ &\qquad = \mathrm{tr}_{E/F}\delta e\overline{e'} f(w_1 + w_2, w_1' + w_2'), \qquad \text{since } W_1 \perp W_2, \\ &\qquad = h(e \otimes (w_1 + w_2), e' \otimes (w_1' + w_2')). \end{aligned}$$

So $\bigoplus_{i=1}^2 g_i \otimes f_i = h$.

Let $A^{ij} = \mathrm{Hom}(V_i, V_j)$, $i,j = 1,2$, so $A = \bigoplus_{i,j} A^{ij}$. Let $\iota_i : A(E) \hookrightarrow A^{ii} = A(E) \otimes_F \mathrm{End}_F(W_i)$ be the embedding given by $\iota_i(a) = a \otimes 1$, as in **[BK]** (1.2.5), and let $\iota : A(E) \hookrightarrow A^{11} \oplus A^{22} \subset A$ be given by $\iota(a) = \iota_1(a) + \iota_2(a)$. Then for $a \in A(E)$ we have

$$\begin{aligned} h(\iota(a)(e(w_1+w_2)), e'(w_1'+w_2')) &= g_1(ae, e')f_1(w_1, w_1') + g_2(ae, e')f_2(w_2, w_2') \\ &= g_1(e, \widetilde{e'}^1)f_1(w_1, w_1') + g_2(e, \widetilde{e'}^2)f_2(w_2, w_2') \\ &= h(e(w_1+w_2), (\iota_1(\widetilde{a}^1) + \iota_2(\widetilde{a}^2))(e'(w_1'+w_2'))). \end{aligned}$$

So $\overline{\iota(a)} = \iota_1(\widetilde{a}^1) + \iota_2(\widetilde{a}^2)$.

We also have, from (4.0.1), that the isomorphism $V \cong E \otimes_F W$ gives rise to an isomorphism of $(A(E), B)$-bimodules,

(4.1.3) $$A(E) \otimes_E B \cong A$$

and, as in **[BK2]** (5.3), this restricts to isomorphisms

(4.1.4) $$\mathfrak{A}(E) \otimes_{\mathfrak{o}_E} \mathfrak{B} \cong \mathfrak{A} \qquad \text{and} \qquad \mathfrak{A}(E) \otimes_{\mathfrak{o}_E} \mathfrak{Q}^n \cong \mathfrak{P}^n \qquad \text{for } n \in \mathbb{Z}$$

where $\mathfrak{A}(E) = \mathrm{End}^0_{\mathfrak{o}_E}\{\mathfrak{p}_E^i : i \in \mathbb{Z}\}$ is the unique hereditary $\mathfrak{o}_F$-order in $A(E)$ normalised by E and $\mathfrak{P}(E)$ is its radical.

(4.1.5) Lemma *The lattice chain $\{\mathfrak{p}_E^i : i \in \mathbb{Z}\}$ is self-dual with respect to both alternating forms, g_1 and g_2. So $\widetilde{\mathfrak{A}(E)}^1 = \widetilde{\mathfrak{A}(E)}^2 = \mathfrak{A}(E)$.*

Proof: Consider the form g_1 (for g_2 it will be similar and simpler). Then

$$\begin{aligned} (\mathfrak{p}_E^i)^{\#} = \{x \in E : g_1(x, \mathfrak{p}_E^i) \subset \mathfrak{p}_E\} &= \{x \in E : \mathrm{tr}_{E/F}\delta\varepsilon x \mathfrak{p}_E^i \subset \mathfrak{p}_F\} \\ &= \{x \in E : \delta\varepsilon x \mathfrak{p}_E^i \subset \mathfrak{D}^{-1}\mathfrak{p}_F\} \\ &= \begin{cases} \varepsilon^{-1}\mathfrak{o}_E & \text{if } \delta \notin \mathfrak{p}_E^e\mathfrak{D}^{-1} \\ \varepsilon^{-1}\mathfrak{p}_E & \text{if } \delta \in \mathfrak{p}_E^e\mathfrak{D}^{-1} \end{cases} \end{aligned}$$

so $\{\mathfrak{p}_E^i : i \in \mathbb{Z}\}$ is self-dual. The second assertion now follows from (1.1.2). ■

In particular, we will be able to look at the exact sequences of (2.1). First we must check that the tame corestriction found in (2.1.1) will commute with both involutions.

(4.1.6) Lemma *Let s_0 be a tame corestriction on $A(E)$ relative to E/F which commutes with $\widetilde{\ }^1$. Then it also commutes with $\widetilde{\ }^2$.*

Proof: We have $s(\widetilde{a}^2) = s(\varepsilon\widetilde{a}^1\varepsilon^{-1}) = \varepsilon s(\widetilde{a}^1)\varepsilon^{-1}$, since $\varepsilon \in B(E) = E$, so $s(\widetilde{a}^2) = \varepsilon\widetilde{s(a)}^1\varepsilon^{-1} = \widetilde{s(a)}^2$. ■

(4.1.7) Proposition (cf. [BK] (1.3.9)) *Let s_0 be a tame corestriction on $A(E)$ relative to E/F which commutes with the involutions on $A(E)$. Then the map $s = s_0 \otimes 1_B$ is a tame corestriction on A relative to E/F which commutes with the involution on A.*

Proof: By **[BK]** (1.3.9), s is certainly a tame corestriction on A relative to E/F. Moreover, for $a \in A(E)$, $b \in B$, we have

$$\begin{aligned} s(\overline{a \otimes b}) &= s(\overline{\iota(a)b}) = s(\bar{b}\ \overline{\iota(a)}) \\ &= \bar{b}s(\iota_1(\widetilde{a}^1) + \iota_2(\widetilde{a}^2)) = \bar{b}(\iota_1(s_0(\widetilde{a}^1)) + \iota_2(s_0(\widetilde{a}^2))) \\ &= \bar{b}(\iota_1(\widetilde{s_0(a)}^1) + \iota_2(\widetilde{s_0(a)}^2)) = \bar{b}(\overline{\iota(s_0(a))}) \\ &= \bar{b}(\overline{s(\iota(a))}) = \overline{s(\iota(a))b} = \overline{s(\iota(a)b)} = \overline{s(a \otimes b)}. \end{aligned} \qquad \blacksquare$$

Note then that any tame corestriction s on A relative to E/F which commutes with the involution on A takes the form $s_0 \otimes 1_B$ as above, since s and s_0 are uniquely determined up to multiplication by $u \in \mathfrak{o}_E^\times$ such that $u = \overline{u}$.

Now let E_1/E be a subfield of B such that $\overline{E_1} = E_1$. Then, as above, we have two involutions on $A(E_1)$, $\widetilde{\ }^1$ and $\widetilde{\ }^2$, and elements δ_1, $\varepsilon_1 \in E_1$.

(4.1.8) Lemma *Let s' be a tame corestriction on $A(E_1)$ relative to E/F which commutes with $\widetilde{\ }^1$. Then it also commutes with $\widetilde{\ }^2$.*

Proof: Since $\varepsilon_1 \in E_1 \subset \mathrm{End}_E(E_1)$, the proof is the same as for (4.1.5). $\blacksquare$

(4.1.9) Lemma *Let s be a tame corestriction on A relative to E/F which commutes with $^-$. Then $s = s' \otimes 1_{B_1}$ for some tame corestriction s' on $A(E_1)$ relative to E/F which commutes with $\widetilde{\ }^1$ and $\widetilde{\ }^2$.*

Proof: As in the proof of **[BK]** (2.2.8), for s' as above $s' \otimes 1_{B_1}$ is a tame corestriction on A relative to E/F and, as in (4.1.7), it commutes with the involution. Then, changing s' by some $u \in \mathfrak{o}_E^\times$ such that $u = \overline{u}$, we have $s = s' \otimes 1_{B_1}$. $\blacksquare$

Now let $\beta \in A_-$ be such that $E = F[\beta]$ is a field and put $B = \mathrm{End}_E(V)$ as usual. Let $c \in B_-$ be such that $E_1 = E[c]$ is a field. Let $\widetilde{\ }^1$, $\widetilde{\ }^2$ be the involutions on $A(E_1) = \mathrm{End}_F(E_1)$ as above; recall that $\widetilde{\widetilde{a}^1}^2 = \varepsilon_1 a \varepsilon_1^{-1}$, where ε_1 satisfies $\overline{\varepsilon}_1 = \varepsilon_1$ and $\varepsilon^{n_1} \in F$ for some $n_1 \in \mathbb{Z}$, $(n_1, p) = 1$. Then, writing $\sigma_1(a) = \widetilde{\widetilde{a}^1}^2$, we have $\sigma_1^{n_1} \equiv 1$. Set $\Gamma_1 = \langle \sigma_1 \rangle$, the cyclic group generated by σ_1.

The extension E/F is a subfield of $A(E_1)$ so we may consider $B(E_1)$, the centraliser of E in $A(E_1)$. Then the tame corestriction s' on $A(E_1)$ relative to E/F given by (2.1.1) commutes with σ_1 by (4.1.8), as does a_β since $\sigma_1(\beta) = \beta$.

Recall that $\mathfrak{A}(E_1)$ is the unique hereditary order in $A(E_1)$ normalised by E_1, with Jacobson radical $\mathfrak{P}(E_1)$. We put $\mathfrak{B}(E_1) = \mathfrak{A}(E_1) \cap B(E_1)$ and $\mathfrak{Q}(E_1) = \mathfrak{P}(E_1) \cap B(E_1)$, the Jacobson radical of the hereditary $\mathfrak{o}_E$-order $\mathfrak{B}(E_1)$. Put $\mathfrak{N}_k = \mathfrak{N}_k(\beta, \mathfrak{A}(E_1))$. Then the exact sequence (from **[BK]** (1.4.10))

$$0 \to \mathfrak{Q}(E_1)^m \mathfrak{N}_k / \mathfrak{Q}(E_1)^m \to \mathfrak{P}(E_1)^{m+k} \to \mathfrak{Q}(E_1)^{m+k} \to 0, \qquad k \geq k_0(\beta, \mathfrak{A}(E_1)),$$

is Γ_1-equivariant. So we have a long exact sequence in cohomology

$$\begin{aligned} 0 \to (\mathfrak{Q}(E_1)^m \mathfrak{N}_k / \mathfrak{Q}(E_1)^m)^{\Gamma_1} &\to (\mathfrak{P}(E_1)^{m+k})^{\Gamma_1} \to (\mathfrak{Q}(E_1)^{m+k})^{\Gamma_1} \\ &\to H^1(\Gamma_1, \mathfrak{Q}(E_1)^m \mathfrak{N}_k / \mathfrak{Q}(E_1)^m) \to H^1(\Gamma_1, \mathfrak{P}(E_1)^{m+k}) \to \dots \end{aligned}$$

Since $|\Gamma_1| = n_1$ and $(n_1, p) = 1$, we have $H^1(\Gamma_1, \mathfrak{Q}(E_1)^m\mathfrak{N}_k/\mathfrak{Q}(E_1)^m) = 0$, by **[Bn]** (10.2), and hence an exact sequence

(4.1.10) $$(\mathfrak{P}(E_1)^{m+k})^{\Gamma_1} \to (\mathfrak{Q}(E_1)^{m+k})^{\Gamma_1} \to 0$$

in $A^{\Gamma_1} = \{a \in A(E_1) : \widetilde{a}^1 = \widetilde{a}^2\}$. Set

$$A_-^{\Gamma_1} = \{a \in A^{\Gamma_1} : \widetilde{a}^1 = -a\}$$
$$A_+^{\Gamma_1} = \{a \in A^{\Gamma_1} : \widetilde{a}^1 = a\}.$$

Then $A^{\Gamma_1} = A_-^{\Gamma_1} \oplus A_+^{\Gamma_1}$ and (4.1.10) splits into two sequences to give

(4.1.11) $$(\mathfrak{P}(E_1)^{m+k})_-^{\Gamma_1} \to (\mathfrak{Q}(E_1)^{m+k})_-^{\Gamma_1} \to 0$$

Note that $c \in E_1 \hookrightarrow A(E_1)$ satisfies $\widetilde{c}^1 = \widetilde{c}^2 = \overline{c} = -c$ so $c \in A_-^{\Gamma_1}$.

(4.2) Refinements

Let $[\mathfrak{A}, n, r, \beta]$ be a skew simple stratum in A. As usual we write $E = F[\beta]$, B for the A-centraliser of E, $\mathfrak{B} = \mathfrak{A} \cap B$, and $\mathfrak{Q} = \mathfrak{P} \cap B$, the Jacobson radical of $\mathfrak{B}$. We also fix a tame corestriction s on A relative to E/F such that s commutes with the involution on A, as in (2.1.1). As in **[BK]** (2.2), a *refinement* of our given stratum is a stratum of the form $[\mathfrak{A}, n, r-1, \beta + b]$, where $b \in \mathfrak{P}^{-r}$. We can then form the *derived stratum* $[\mathfrak{B}, r, r-1, s(b)]$, which is stratum in B.

A *skew refinement* is a refinement where we also have $b + \overline{b} = 0$. Then the derived stratum is a skew stratum.

(4.2.1) Lemma (cf. [BK] (2.2.1)) *Let $[\mathfrak{A}, n, r, \beta]$ be a skew simple stratum in A. Let $[\mathfrak{A}, n, r-1, \beta+b]$, $[\mathfrak{A}, n, r-1, \beta+b']$ be skew refinements of it. Write $k = k_0(\beta, \mathfrak{A})$, $\mathfrak{N} = \mathfrak{N}_k(\beta, \mathfrak{A})$. Then the derived strata $[\mathfrak{B}, r, r-1, s(b)]$, $[\mathfrak{B}, r, r-1, s(b')]$ in B are equivalent if and only if there exists $y \in (\mathfrak{Q}^{-(r+k)}\mathfrak{N})_-$ such that*

$$[\mathfrak{A}, n, r-1, C(y)^{-1}(\beta+b)C(y)] \sim [\mathfrak{A}, n, r-1, \beta+b'].$$

Proof: We have $[\mathfrak{B}, r, r-1, s(b)] \sim [\mathfrak{B}, r, r-1, s(b')]$ if and only if $s(b-b') \in \mathfrak{Q}_-^{1-r}$. By (2.1.5), this happens if and only if there exists $y \in (\mathfrak{Q}^{-(r+k)}\mathfrak{N})_-$ such that $b - b' \equiv a_\beta(y) \pmod{\mathfrak{P}_-^{1-r}}$. Then, since $-(r+k) \geq 1$, we have

$$C(y)^{-1}(\beta+b)C(y) \equiv \beta + b + a_\beta(y) \pmod{\mathfrak{P}^{1-r}},$$

since $(C(y) - y - 1) \in \mathfrak{Q}^{-2(r+k)}\mathfrak{N}$ so $a_\beta(C(y) - y - 1) \in \mathfrak{P}^{-(r+k)-r} \subset \mathfrak{P}^{1-r}$. ■

(4.2.2) Proposition (cf. [BK] (2.2.8)) *Let $[\mathfrak{A}, n, r, \beta]$ be a skew simple stratum in A. Let B be the A-centraliser of $E = F[\beta]$ and $\mathfrak{B} = \mathfrak{A} \cap B$. Let $b \in A$ with $\nu_{\mathfrak{A}}(b) = -r$ and let s be a tame corestriction on A relative to E/F which commutes with the involution on A. Suppose that the stratum $[\mathfrak{B}, r, r-1, s(b)]$ is equivalent to a skew simple stratum $[\mathfrak{B}, r, r-1, c]$ in B. Then $[\mathfrak{A}, n, r-1, \beta+b]$*

is equivalent to a skew simple stratum $[\mathfrak{A}, n, r-1, \beta_1]$. *Moreover,* $k_0(\beta_1, \mathfrak{A}) = \max\{k_0(\beta, \mathfrak{A}), k_0(c, \mathfrak{B})\}$.

Proof: We follow the proof of **[BK]** (2.2.8). Let $E_1 = F[\beta, c]$, so $E_1 = \overline{E}_1$, and let B_1 denote the A-centraliser of the field E_1, $\mathfrak{B}_1 = \mathfrak{A} \cap B_1$. Then we have a decomposition

$$\mathfrak{A} = \mathfrak{A}(E_1) \otimes_{\mathfrak{o}_{E_1}} \mathfrak{B}_1$$

as in (4.1.4). By (4.1.9) our tame corestriction s on A relative to E/F takes the form $s' \otimes 1_{B_1}$ for some tame corestriction s' on $A(E_1)$ relative to E/F, which commutes with the involutions on $A(E_1)$.

Let $\mathfrak{C}$ denote the hereditary $\mathfrak{o}_E$-order $\mathfrak{A}(E_1) \cap \mathrm{End}_E(E_1)$. Then we have decompositions $\mathfrak{A}(E_1) = \mathfrak{A}(E) \otimes_{\mathfrak{o}_E} \mathfrak{C}$ and $\mathfrak{B} = \mathfrak{C} \otimes_{\mathfrak{o}_{E_1}} \mathfrak{B}_1$.

Put $e_1 = e(\mathfrak{B}_1|\mathfrak{o}_{E_1})$. Then the strata $[\mathfrak{A}(E_1), n/e_1, r/e_1, \beta]$, $[\mathfrak{C}, r/e_1, r/e_1 - 1, c]$ are simple and $k_0(\beta, \mathfrak{A}(E_1)) = k_0(\beta, \mathfrak{A})/e_1$ by **[BK]** (1.4.13). Now $c \in A(E_1)_{-}^{\Gamma_1}$ so, by (4.1.11), choose $b_1 \in \mathfrak{A}(E_1)_{-}^{\Gamma_1}$ with $\nu_{\mathfrak{A}(E_1)}(b_1) = -r/e_1$ such that $s'(b_1) = c$. By **[BK]** (2.2.3), the stratum $[\mathfrak{A}(E_1), n/e_1, r/e_1 - 1, \beta + b_1]$ is simple. Consider the stratum $[\mathfrak{A}, n, r-1, \iota(\beta + b_1)]$; this is a skew refinement of $[\mathfrak{A}, n, r, \beta]$, since $\overline{\iota(\beta + b_1)} = \iota_1(\widetilde{\beta + b_1}^{\,1}) + \iota_2(\widetilde{\beta + b_1}^{\,2}) = \iota_1(-(\beta+b_1)) + \iota_2(-(\beta+b_1)) = -\iota(\beta+b_1)$, and

$$[\mathfrak{B}, r, r-1, \iota(s(b_1))] = [\mathfrak{B}, r, r-1, c]$$

as in **[BK]** (2.2.8). Then, by (4.2.1), we can find a conjugate of $[\mathfrak{A}, n, r-1, \iota(\beta+b_1)]$ equivalent to $[\mathfrak{A}, n, r-1, \beta+b]$. The rest of the proof is exactly the same as **[BK]** (2.2.8). ■

(4.2.3) Remark If $E_1 = E[c]$ is a maximal subfield of B then we can use **[BK]** (2.2.2) to deduce that $[\mathfrak{B}, r, r-1, s(b)]$ is itself simple, with $E[s(b)]$ a maximal subfield of B, and **[BK]** (2.2.3) to conclude that $[\mathfrak{A}, n, r-1, \beta+b]$ is itself simple.

5
STANDARD FORM

The main result of this chapter is the following (cf. **[BK]** (2.4.1)), which we prove in (5.4):

Let $[\mathfrak{A}, n, n-1, \beta]$ *be a skew pure stratum in* A*. Then it is equivalent to a skew simple stratum.*

This then allows us to use induction "along β" (i.e. on $k_0(\beta, \mathfrak{A})$) as in **[BK]** to define simple characters and prove results concerning them (see chapter 6).

In order to prove this result we need several results in the generality of unitary groups attached to a hermitian form so h will be:

ϵ-hermitian in sections (5.1–2);
skew-hermitian in section (5.3).

(5.1) Lifting

In this section $h : V \times V \to F$ will be an ϵ-hermitian form.

Let $\mathfrak{A}$ be a maximal order associated to a self-dual lattice chain $\mathfrak{L} = \{L_i : i \in \mathbb{Z}\}$ in V and we put $\mathfrak{P}$ =rad $\mathfrak{A}$, as usual. Let $\mathcal{V} = \mathfrak{L}(V)$ be the standard residual subspace of $\mathfrak{L}$ and put $\mathcal{A} = \mathrm{End}^0_{k_F}\mathcal{V}$ ($\simeq \mathfrak{A}/\mathfrak{p}_F\mathfrak{A}$ by (3.3.2)). For each $k \in \mathbb{Z}$, $\mathcal{V}$ inherits from V a nondegenerate ϵ-sesquilinear form $\boldsymbol{h}_k : \mathcal{V}_k \times \mathcal{V}_{d-k-1} \to k_F$ as in (3.1.6), where d is the unique integer such that $L_k^{\#} = L_{d-k}$ for all $k \in \mathbb{Z}$. Then we have an adjoint involution $^-$ on $\mathcal{A}$ so we can define $\mathcal{A}_-, \mathcal{A}_+$ to be the skew elements, the symmetric elements respectively. In fact $\mathcal{A}_\pm \simeq \mathfrak{A}_\pm/(\mathfrak{p}_F\mathfrak{A})_\pm$ by (1.2.5).

(5.1.1) Lemma *Let* $\eta = \pm$ *be a sign. The natural map* $\mathfrak{A}_\eta \to \mathcal{A}_\eta$ *is surjective.*

For $f(X) \in F[X]$, $f(X) = a_nX^n + \cdots + a_0$, define $\overline{f}(X) \in F[X]$ to be the polynomial $\overline{f}(X) = \overline{a_n}X^n + \cdots + \overline{a_0}$. Also, for $f(X) \in \mathfrak{o}_F[X]$, $f(X) = a_nX^n + \cdots + a_0$, define $\widetilde{f}(X) \in k_F[X]$ to be the polynomial obtained by reducing the coefficients modulo $\mathfrak{p}_F$.

(5.1.2) Lemma *Let* $\gamma, \gamma' \in U(\mathfrak{A})$*. Suppose* γ, γ' *have the same minimal polynomial, that the reduction modulo* $\mathfrak{p}_F$ *of this minimal polynomial is irreducible and that* $\gamma \equiv \gamma' (\mathrm{mod}\ \mathfrak{P}^n)$*. Then there exists* $v \in U^n(\mathfrak{A})$ *such that* $\gamma = v\gamma'v^{-1}$*.*

Proof: The elements γ, γ' generate unramified extensions over F; in particular, they are minimal over F. Write B_γ for the centralizer of γ in A and $\mathfrak{B}_\gamma = \mathfrak{A} \cap B_\gamma$. By **[BH1]** (1.6), there exists $u \in U(\mathfrak{A})$ such that $\gamma = u\gamma'u^{-1}$. Since $\gamma - \gamma' \in \mathfrak{P}^n$, we have $\gamma u - u\gamma \in \mathfrak{P}^n$. i.e. $u \in \mathfrak{N}_n(\gamma, \mathfrak{A})$. But γ is minimal so $\mathfrak{N}_n(\gamma, \mathfrak{A}) = \mathfrak{B} + \mathfrak{P}^n$. Hence $u = wv$, for $w \in U(\mathfrak{B})$ and $v \in U^n(\mathfrak{A})$, and $\gamma = v\gamma'v^{-1}$ as required. ■

We fix an irreducible polynomial $\phi(X) \in k_F[X]$ of degree N, $\phi(X) \neq X$. The following is a Hensel's Lemma-type result.

(5.1.3) Proposition *Let $\eta = \pm$ be a sign and let $\varrho \in \mathcal{A}_\eta$. Suppose ϱ has irreducible characteristic polynomial $\phi(X) \in k_F[X]$. Let $\Phi(X) \in \mathfrak{o}_F[X]$ be a polynomial such that $\widetilde{\Phi}(X) = \phi(X)$ and such that $\overline{\Phi}(\eta X) = \Phi(X)$. Then there exists $\gamma \in \mathfrak{A}_\eta$ such that $\widetilde{\gamma} = \varrho$ and $\Phi(\gamma) = 0$.*

Proof: By Hensel's Lemma, there exists $\gamma_1 \in U(\mathfrak{A})$ such that $\widetilde{\gamma}_1 = \varrho$ and $\Phi(\gamma_1) = 0$. Also, $\overline{\gamma_1} \equiv \eta\gamma_1 \pmod{\mathfrak{P}}$ since $\widetilde{\gamma}_1 \in \mathcal{A}_\eta$.

For $n \geq 1$, we will find, by induction, $\gamma_n \in U(\mathfrak{A})$ such that $\widetilde{\gamma}_n = \varrho$, γ_n is conjugate to γ_1 (in particular, $\Phi(\gamma_n) = 0$), $\overline{\gamma_n} \equiv \eta\gamma_n \pmod{\mathfrak{P}^n}$ and $\gamma_n \equiv \gamma_{n-1} \pmod{\mathfrak{P}^{n-1}}$. Granting this, let γ be the limit of the sequence $\{\gamma_n\}$. Then $\widetilde{\gamma} = \varrho$, $\overline{\gamma} = \eta\gamma$ and $\Phi(\gamma) = 0$ as required.

We have already found γ_1 so assume we have found γ_n as required, for some $n \geq 1$.

We have that $\Phi(\gamma_n) = 0$ and also $\Phi(\eta\overline{\gamma_n}) = \overline{\overline{\Phi}(\eta\gamma_n)} = \overline{\Phi(\gamma_n)} = 0$ so γ_n and $\eta\overline{\gamma_n}$ are conjugate. Moreover, $\gamma_n \equiv \eta\overline{\gamma_n} \pmod{\mathfrak{P}^n}$ hence $\eta\overline{\gamma_n} = u_n\gamma_n u_n^{-1}$, for some $u_n \in U^n(\mathfrak{A})$, by (5.1.2).

Write $u_n = 1 + x_n$, $x_n \in \mathfrak{P}^n$, so that $\eta\overline{\gamma_n} \equiv \gamma_n + x_n\gamma_n - \gamma_n x_n \pmod{\mathfrak{P}^{2n}}$. We put $y_n = \frac{x_n + \overline{x_n}}{2}$ and $v_n = 1 + y_n \in U^n(\mathfrak{A})$; then $v_n = \overline{v}_n$ and $\eta\overline{\gamma_n} \equiv v_n\gamma_n v_n^{-1} \pmod{\mathfrak{P}^{2n}}$. By (1.2.4)(*ii*), there exists $w_n \in U^n(\mathfrak{A})$ such that $v_n = \overline{w_n}w_n$. Then we put $\gamma_{n+1} = w_n\gamma_n w_n^{-1}$. ■

We finish this section with some results which will prove important later.

(5.1.4) Lemma *Let $\eta = \pm$ be a sign. Let $\gamma, \gamma' \in \mathfrak{A}_\eta^\times$ be such that $\gamma \equiv \gamma'$ mod $\mathfrak{P}^r$, $r \geq 1$. Then there exists $u \in U^r(\mathfrak{A})$ such that $\gamma \equiv u\gamma'\overline{u} \pmod{\mathfrak{P}^{2r}}$.*

Proof: Let $a = \gamma - \gamma' \in \mathfrak{P}^r$ and put $u = (1 + \frac{1}{2}a\gamma'^{-1})$. ■

(5.1.5) Proposition *With hypotheses as in (5.1.4), there exists $u \in U^r(\mathfrak{A})$ such that $\gamma = u\gamma'\overline{u}$.*

Proof: We find, by induction, $u_n \in U^r(\mathfrak{A})$ such that

$$\textbf{(5.1.6)} \qquad \gamma \equiv u_n\gamma'\overline{u_n} \pmod{\mathfrak{P}^{rn}}$$

and $u_{n+1} \equiv u_n \pmod{\mathfrak{P}^{rn}}$. Then $\{u_n\}$ converges to $u \in U^r(\mathfrak{A})$ such that $\gamma = u\gamma'\overline{u}$.

We can take $u_1 = 1$ so suppose we have found $u_n \in U^r(\mathfrak{A})$ satisfying (5.1.6). Then γ and $u_n\gamma'\overline{u_n}$ satisfy the conditions of (5.1.4) (with r replaced by rn). So there exists $u' \in U^{rn}(\mathfrak{A})$ such that $\gamma \equiv u'u_n\gamma'\overline{u_n}\overline{u'}$ mod $\mathfrak{P}^{2rn}$ and, in particular, this congruence holds (mod $\mathfrak{P}^{(r+1)n}$) since $r \geq 1$. So we set $u_{n+1} = u'u_n$. ■

(5.1.7) Lemma *Let $\Phi(X) \in \mathfrak{o}_F[X]$ be an irreducible polynomial of degree N such that $\overline{\Phi}(X) = \Phi(X)$ and such that its reduction modulo $\mathfrak{p}_F$, $\widetilde{\Phi}$ is also irreducible. Let $\gamma \in U(\mathfrak{A})$ have characteristic polynomial $\Phi(X)$. Let $\partial : \mathfrak{A} \to \mathfrak{A}$ be the map given by $\partial(x) = x\overline{\gamma} - \gamma x$, for $x \in \mathfrak{A}$. Suppose $y \in \mathrm{Im}(\partial) \cap \mathfrak{P}_-$; then there exists $\xi \in \mathfrak{P}_+$ such that $y = \partial(\xi)$.*

Proof: We need only show that for $y \in \mathrm{Im}\,(\partial) \cap \mathfrak{P}$ there exists $\xi \in \mathfrak{P}$ such that $y = \partial(\xi)$, since ∂ maps $\mathfrak{A}_+$ to $\mathfrak{A}_-$ and vice versa.

Let E/F be a splitting field for $\Phi(X)$ and let $\partial_E : \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_E \to \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_E$ be the extended map. The result then holds for ∂ if and only if it holds for ∂_E.

Let λ_i, $i = 1, ..., n$, be the eigenvalues of γ in E; these are also the eigenvalues of $\overline{\gamma}$. Further, the λ_i are distinct modulo $\mathfrak{p}_F$ since γ generates a maximal extension of F in A. Hence the non-zero eigenvalues of ∂_E, which are $\lambda_i - \lambda_j$ for $1 \le i \ne j \le n$, are $\not\equiv 0 \pmod{\mathfrak{p}_F}$ and the result follows. ■

(5.1.8) Proposition *Let $\alpha, \beta \in U(\mathfrak{A}) \cap A_+$ be such that $\gamma := \alpha\beta$ generates a maximal (unramified) extension of F in A. Let $\phi(X) \in k_F[X]$ be the reduction modulo $\mathfrak{p}_F$ of the characteristic polynomial of γ and suppose that $\overline{\phi}(X) = \phi(X)$. Let $\Phi(X) \in \mathfrak{o}_F[X]$ be such that $\overline{\Phi}(X) = \Phi(X)$ and such that $\widetilde{\Phi}(X) = \phi(X)$. Let $\gamma' \in U(\mathfrak{A})$ be such that $\gamma' \equiv \gamma \pmod{\mathfrak{p}_F}$ and such that $\Phi(\gamma') = 0$. (There exists such a γ' by Hensel's Lemma.) Then there exist $\alpha' \in \alpha + \mathfrak{P}_+$, $\beta' \in \beta + \mathfrak{P}_+$ such that $\gamma' = \alpha'\beta'$.*

Proof: Write $\gamma' = \gamma + \delta$ with $\delta \in \mathfrak{P}$. Note that $\alpha\overline{\gamma} = \alpha\beta\alpha = \gamma\alpha$ and also that γ' generates a maximal unramified extension of F.

We have $\alpha\overline{\gamma'} - \gamma'\alpha = \alpha\overline{\delta} - \delta\alpha \in \mathfrak{P}_-$ so, by (5.1.7), there exists $\xi \in \mathfrak{P}_+$ such that $\xi\overline{\gamma'} - \gamma'\xi = \alpha\overline{\gamma'} - \gamma'\alpha$. Then we put $\alpha' = \alpha - \xi$ and $\beta' = {\alpha'}^{-1}\gamma'$. ■

(5.2) Jordan form

Let $\mathcal{V}$ be an N-dimensional k_F-vector space, equipped with a nondegenerate ϵ-hermitian form $\boldsymbol{h} : \mathcal{V} \times \mathcal{V} \to k_F$. We put $\mathcal{A} = \mathrm{End}_{k_F}\mathcal{V}$. Recall also that k_0 is the residue class field of F_0.

For $\mathcal{W}$ a subspace of $\mathcal{V}$ we define its orthogonal complement to be

$$\mathcal{W}^\perp = \{v \in \mathcal{V} : \boldsymbol{h}(v, \mathcal{W}) \equiv 0\}.$$

Then, by the nondegeneracy of $\boldsymbol{h}$, $\mathcal{W}^{\perp\perp} = \mathcal{W}$.

(5.2.1) Lemma *Let $\mathcal{W}$ be a totally isotropic residual subspace of $\mathfrak{L}$. Then $\boldsymbol{h}$ induces an ϵ-hermitian form $\boldsymbol{h}'$ on $\mathcal{W}^\perp/\mathcal{W}$ which is nondegenerate.*

Proof: For $v \in \mathcal{W}^\perp$, $v' \in \mathcal{W}^\perp$, we define $\boldsymbol{h}'$ by $\boldsymbol{h}'(v+\mathcal{W}, v'+\mathcal{W}) = \boldsymbol{h}(v, v')$. This is well defined since $v, v' \in \mathcal{W}^\perp$ and $\mathcal{W}$ is totally isotropic. Suppose now there exists $v \in \mathcal{W}^\perp$ such that $\boldsymbol{h}'(v + \mathcal{W}, \mathcal{W}^\perp/\mathcal{W}) \equiv 0$; then $\boldsymbol{h}(v, \mathcal{W}^\perp) \equiv 0$ so $v \in \mathcal{W}^{\perp\perp} = \mathcal{W}$ as required. ■

Let $y \in \mathcal{A}$ be such that $y = \eta\overline{y}$, $\eta = \pm$. Let the characteristic polynomial of y be $\phi(X)^t \in k_F[X]$, with $\phi(X)$ irreducible of degree N/t. Let $y = y_{ss} + y_{np}$ be the Jordan decomposition of y. Then, by uniqueness of the decomposition, we have $y_{ss} = \eta\overline{y}_{ss}$ and $y_{np} = \eta\overline{y}_{np}$. Then y_{ss} is elliptic with irreducible minimal polynomial $\phi(X)$ and $\mathfrak{l} = k_F[y_{ss}]$ is a field with $y_{np}, y \in \mathrm{End}_{\mathfrak{l}}\mathcal{V}$. We also set $\mathfrak{l}_0 = \{l \in \mathfrak{l} : \overline{l} = l\}$; note that $\mathfrak{l} = \mathfrak{l}_0$ if, and only if, $k_F = k_0$ and $\eta = +$. As $\mathrm{tr}_{\mathfrak{l}/k_F}$

and $\boldsymbol{h}$ are nondegenerate, there exists a nondegenerate form $\boldsymbol{f} : \mathcal{V} \times \mathcal{V} \to \mathfrak{l}$ such that, for $v, v' \in \mathcal{V}$,

$$\boldsymbol{h}(lv, v') = \mathrm{tr}_{\mathfrak{l}/k_F}(l\,\boldsymbol{f}(v, v')), \qquad \text{for all } l \in \mathfrak{l}.$$

The adjoint involution on $\mathrm{End}_{\mathfrak{l}}\mathcal{V}$ defined by $\boldsymbol{f}$ is precisely that defined by $\boldsymbol{h}$.

If $\mathfrak{l} = \mathfrak{l}_0$ then $\boldsymbol{f}$ is ϵ-bilinear while if $\mathfrak{l} \neq \mathfrak{l}_0$ then $\boldsymbol{f}$ is ϵ-hermitian. In either case, $\mathcal{V} = \mathcal{V}^{sp} \perp \mathcal{V}^{an}$, where $\boldsymbol{f}|_{\mathcal{V}^{an}\times\mathcal{V}^{an}}$ is anisotropic and $\boldsymbol{f}|_{\mathcal{V}^{sp}\times\mathcal{V}^{sp}}$ is split. In fact, if $\mathfrak{l} = \mathfrak{l}_0$ and $\epsilon = -1$ then $\boldsymbol{f}$ is split so $\dim_{\mathfrak{l}}\mathcal{V}^{an} = 0$; if $\mathfrak{l} = \mathfrak{l}_0$ and $\epsilon = +1$ then $\dim_{\mathfrak{l}}\mathcal{V}^{an} \leq 2$; while if $\mathfrak{l} \neq \mathfrak{l}_0$ then $N_{\mathfrak{l}/\mathfrak{l}_0}$ is surjective so $\dim_{\mathfrak{l}}\mathcal{V}^{an} \leq 1$.

(5.2.2) Lemma *(i) With notation as above, there exists a flag of $\mathfrak{l}$-subspaces of $\mathcal{V}$, $\mathcal{V} = \mathcal{V}^0 \supset \mathcal{V}^1 \supset \cdots \supset \mathcal{V}^s = 0$ such that $\mathcal{V}^{q\perp} = \mathcal{V}^{s-q}$ and $y_{np}\mathcal{V}^{q-1} \subset \mathcal{V}^q$ and such that, moreover, $\dim_{\mathfrak{l}}\mathcal{V}^q/\mathcal{V}^{q+1} = 1$ for all q (so $s = t$) except in the case $\mathfrak{l} = \mathfrak{l}_0$, $\epsilon = +1$ and t even when, possibly, $\dim_{\mathfrak{l}}\mathcal{V}^{t/2-1}/\mathcal{V}^{t/2} = 2$ and $s = t - 1$.*
(ii) Given such a flag, there exist $\mathfrak{l}$-subspaces $\mathcal{W}^q$, $1 \leq q \leq s$, of $\mathcal{V}$ such that $\mathcal{V}^{q-1} = \bigoplus_{j=q}^{s} \mathcal{W}^j$ and $\mathcal{W}^{q\perp} = \bigoplus_{j\neq s-q+1} \mathcal{W}^j$ for $1 \leq q \leq s$.

Proof: (i) We proceed by induction on t, the cases $t = 0$ and $t = 1$ being obvious. So assume $t \geq 2$ and that we have the result for $t-2$. Since y_{np} is nilpotent there exists $v \in \mathcal{V}$ such that $y_{np}v = 0$. Then for $v' \in \mathcal{V}$, $\boldsymbol{f}(y_{np}v', v) = \eta\boldsymbol{f}(v', y_{np}v) = 0$ so $y_{np}\mathcal{V} \subset \langle v\rangle_{\mathfrak{l}}^{\perp}$.

Suppose first there exists such a v such that $\boldsymbol{f}(v, v) = 0$; then let $\mathcal{V}^{s-1} = \langle v\rangle_{\mathfrak{l}}$, $\mathcal{V}^1 = \langle v\rangle_{\mathfrak{l}}^{\perp}$. Then, replacing $\mathcal{V}$ with $\mathcal{V}^1/\mathcal{V}^{s-1}$ we have, by the inductive hypothesis, a flag

$$\mathcal{V}^1/\mathcal{V}^{s-1} = \mathcal{V}'^0 \supset \mathcal{V}'^1 \supset \cdots \supset \mathcal{V}'^{s-2} = 0$$

which we can lift to a flag as required. (Note that the form induced by $\boldsymbol{f}$ on $\mathcal{V}^1/\mathcal{V}^{s-1}$ is indeed nondegenerate by (5.2.1).)

Now suppose that for all $v \in \mathcal{V}$ such that $y_{np}v = 0$ we have $\boldsymbol{f}(v, v) \neq 0$. Since $y_{np}\mathcal{V} \subset \langle v\rangle_{\mathfrak{l}}^{\perp}$ and $t \geq 2$, there exists $v' \in \langle v\rangle_{\mathfrak{l}}^{\perp}$ such that $y_{np}v' = 0$.

We deal first with the case $\mathfrak{l} \neq \mathfrak{l}_0$. Since $N_{\mathfrak{l}/\mathfrak{l}_0}\mathfrak{l} = \mathfrak{l}_0$, we may assume $\boldsymbol{f}(v, v) = -\boldsymbol{f}(v', v') = 1$. But then $y_{np}(v + v') = 0$ and $\boldsymbol{f}(v + v', v + v') = 0$, contradicting our assumption.

Suppose now $\mathfrak{l} = \mathfrak{l}_0$; then, since $\boldsymbol{f}(v, v) \neq 0$, we must have $\epsilon = +1$. If $t = 2$ then we put $\mathcal{V}_0 = \mathcal{V}$, $\mathcal{V}_1 = 0$ and we are done. Otherwise we have, as before, that $y_{np}\mathcal{V} \subset \langle v, v'\rangle_{\mathfrak{l}}^{\perp}$ so there exists $v'' \in \langle v, v'\rangle_{\mathfrak{l}}^{\perp}$ such that $y_{np}v'' = 0$. But then $\boldsymbol{f}(v'', v'') = -l^2\boldsymbol{f}(v, v) - l'^2\boldsymbol{f}(v', v')$ for some $l, l' \in \mathfrak{l}$ so, putting $w = lv + l'v' + v'$ we have $\boldsymbol{f}(w, w) = 0$ but $y_{np}w = 0$, contradicting the assumption.

(ii) We proceed by induction on s. If $s = 1$ we are done, putting $\mathcal{W}^1 = \mathcal{V}$ so we assume $s \geq 2$. Choose a non-zero $v_1 \in \mathcal{V}^{s-1}$ and $w \in \mathcal{V}$ such that $\boldsymbol{f}(v_1, w) = 1$. Put $w_1 = w - \frac{1}{2}\boldsymbol{f}(w, w)v_1$; then we put $\mathcal{W}^1 = \langle w_1\rangle_{\mathfrak{l}}$ and $\mathcal{W}^s = \langle v_1\rangle_{\mathfrak{l}}$.

Suppose now that we have chosen $\mathcal{W}^j$ for $1 \leq j \leq q-1$, $s-q+2 \leq j \leq s$. If $s-q+2 = q$ then we are done. Otherwise, choose $v \in \mathcal{V}^{s-q-1}$ such that $v \notin \mathcal{V}^{s-q}$. We then put $v_q = v - \sum_{j=1}^{q-1}\boldsymbol{f}(v, w_j)v_j$ and set $\mathcal{W}^{s-q+1} = \langle v_q\rangle_{\mathfrak{l}}$. If $s-q+1 = q$ then we are done. Otherwise, choose $w \in \mathcal{V}^q$ such that $\boldsymbol{f}(v_q, w) = 1$. Put $w' = w - \frac{1}{2}\boldsymbol{f}(w, w)v_q$ and $w_q = w' - \sum_{j=1}^{q-1}\boldsymbol{f}(w', w_j)v_j$. Then we put $\mathcal{W}^q = \langle w_q\rangle_{\mathfrak{l}}$. ■

Before continuing, we examine the case $\dim_{\mathfrak{l}}\mathcal{W}^{t/2-1} = 2$ of (5.2.2) in more detail. So $k_F = k_0$, $\eta = +$, $\epsilon = +1$.

Choose $u', u'' \in \mathcal{W}^{t/2-1}$ such that $\boldsymbol{f}(u', u'') = 0$ and put $\mathcal{U}' = \langle u' \rangle_{\mathfrak{l}}$, $\mathcal{U}'' = \langle u'' \rangle_{\mathfrak{l}}$; then we have

$$\mathcal{U}'^{\perp} = \mathcal{U}'' \oplus \bigoplus_{i \neq \frac{t}{2}-1} \mathcal{W}^i, \qquad y_{np}\mathcal{U}' \subset \bigoplus_{i > \frac{t}{2}-1} \mathcal{W}^i;$$

$$\mathcal{U}''^{\perp} = \mathcal{U}' \oplus \bigoplus_{i \neq \frac{t}{2}-1} \mathcal{W}^i, \qquad y_{np}\mathcal{U}'' \subset \bigoplus_{i > \frac{t}{2}-1} \mathcal{W}^i.$$

Now suppose $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, where $\mathcal{V}_1, \mathcal{V}_2$ are maximal isotropic subspaces of $\mathcal{V}$; in particular, $\boldsymbol{f}$ is split. Suppose further that $y\mathcal{V}_i \subset \mathcal{V}_i$, for $i = 1, 2$; then y_{ss} and y_{np} also fix $\mathcal{V}_i$, $i = 1, 2$. Also, we can write $t = 2t'$, for $t' \in \mathbb{Z}$.

Let $b \in \mathcal{A}$ be such that $b\mathcal{V}_1 = \mathcal{V}_2$, $b\mathcal{V}_2 = 0$, $b = -\overline{b}$, and b commutes with y. Then b also commutes with y_{ss} and y_{np} so $b \in \mathrm{End}_{\mathfrak{l}}\mathcal{V}$.

(5.2.3) Lemma *(i) With notation as above, there exist flags of $\mathfrak{l}$-subspaces*

$$\mathcal{V}_1 = \mathcal{V}_1^0 \supset \mathcal{V}_1^1 \supset \cdots \supset \mathcal{V}_1^s = 0$$
$$\mathcal{V}_2 = \mathcal{V}_2^0 \supset \mathcal{V}_2^1 \supset \cdots \supset \mathcal{V}_2^s = 0$$

such that $(\mathcal{V}_1^q \oplus \mathcal{V}_2)^{\perp} = \mathcal{V}_2^{s-q}$, $b\mathcal{V}_1^q = \mathcal{V}_2^q$ and $y_{np}\mathcal{V}_1^{q-1} \subset \mathcal{V}_1^q$ for $1 \leq q \leq s$ and such that, moreover, $\dim_{\mathfrak{l}}\mathcal{V}_1^q/\mathcal{V}_1^{q+1} = 1$ for all q (so $s = t'$) except in the case $\mathfrak{l} = \mathfrak{l}_0$, $\epsilon = -1$, and t' even when, possibly $\dim_{\mathfrak{l}}\mathcal{V}_1^{t'/2-1}/\mathcal{V}_1^{t'/2} = 2$ and $s = t' - 1$.
(ii) Given such flags, there exist residual $\mathfrak{l}$-subspaces $\mathcal{W}_i^q$, $1 \leq q \leq s$, of $\mathcal{V}_i$, $i = 1, 2$, such that $\mathcal{V}_i^{q-1} = \bigoplus_{j=q}^{s} \mathcal{W}_i^j$, $\mathcal{W}_1^{q\perp} = \mathcal{V}_1 \oplus \bigoplus_{j \neq s-q+1} \mathcal{W}_2^j$ and $\mathcal{W}_2^q = b\mathcal{W}_1^q$ for $1 \leq q \leq s$, $i = 1, 2$.

Proof: We define a nondegenerate $-\epsilon$-hermitian form $\boldsymbol{f}'$ on $\mathcal{V}_1$ by $\boldsymbol{f}'(v, v') = \boldsymbol{f}(v, bv')$. Then we apply (5.2.2) to $(\mathcal{V}_1, \boldsymbol{f}')$ and put $\mathcal{V}_2^q = b\mathcal{V}_1^q$. ■

We now examine the case $\dim_{\mathfrak{l}}\mathcal{W}_1^{t'/2-1} = 2$ of (5.2.3) in more detail. So $k_F = k_0$, $\eta = +$, $\epsilon = -1$.

Choose $u', u'' \in \mathcal{W}_1^{t'/2-1}$ such that $\boldsymbol{f}(bu', u'') = 0$ and put $\mathcal{U}' = \langle u' \rangle_{\mathfrak{l}}$, $\mathcal{U}'' = \langle u'' \rangle_{\mathfrak{l}}$; then we have

$$\mathcal{U}'^{\perp} = \mathcal{V}_1 \oplus b\mathcal{U}'' \oplus \bigoplus_{i \neq \frac{t'}{2}-1} \mathcal{W}_2^i, \qquad y_{np}\mathcal{U}' \subset \bigoplus_{i > \frac{t'}{2}-1} \mathcal{W}_1^i;$$

$$\mathcal{U}''^{\perp} = \mathcal{V}_1 \oplus b\mathcal{U}' \oplus \bigoplus_{i \neq \frac{t'}{2}-1} \mathcal{W}_2^i, \qquad y_{np}\mathcal{U}'' \subset \bigoplus_{i > \frac{t'}{2}-1} \mathcal{W}_1^i.$$

(5.3) Non-split fundamental strata

In this section, h will be a skew-hermitian form.

Let $[\mathfrak{A}, m, m-1, b]$ be a stratum in A and set

$$y = y_b = \pi_F^{m/g} b^{e/g} \in \mathfrak{A}$$

where $e = e(\mathfrak{A})$ is the $\mathfrak{o}_F$-period of $\mathfrak{A}$ and $g = (m, e)$.

Let $\Phi(X) \in \mathfrak{o}_F[X]$ be the characteristic polynomial of y and let $\phi_b(X) \in k_F[X]$ be its reduction modulo $\mathfrak{p}_F$. This polynomial $\phi_b(X)$ depends only on the equivalence class of the stratum, since it is also the characteristic polynomial of y considered as an element of $\mathfrak{A}/\mathfrak{P}$, and is called the *characteristic polynomial* of the stratum.

(5.3.1) Definition *Let $[\mathfrak{A}, m, m-1, b]$ be a stratum in A, $\phi_b(X)$ its characteristic polynomial.*
(i) The stratum is called fundamental *if $\phi_b(X)$ is not a power of X.*
(ii) The stratum is called split *if $\phi_b(X)$ has (at least) two distinct prime factors.*

In particular, a non-split fundamental stratum has characteristic polynomial $\phi(X)^t$ for $\phi(X) \in k_F[X]$ an irreducible monic polynomial of degree N/t, $\phi(X) \neq X$.

Now let $[\mathfrak{A}, m, m-1, b]$ be a skew non-split fundamental stratum in A. We recall that π_F is a fixed uniformizer of F such that $\pi_F = \overline{\pi_F}$ if F/F_0 is unramified, while $\pi_F = -\overline{\pi_F}$ if F/F_0 is ramified. In particular, we have $y = \eta\overline{y}$, for $\eta = \pm$, and $\overline{\phi}(\eta X) = \phi(X)$.

The main result of this section is the following:

(5.3.2) Theorem (cf. [BK] (2.3.4)) *Let $[\mathfrak{A}, m, m-1, b]$ be a skew non-split fundamental stratum in A. Then there exists a skew simple stratum $[\mathfrak{A}', m', m'-1, b']$ in A such that*

$$b + \mathfrak{P}^{1-m} \subset b' + \mathfrak{P}'^{1-m'}.$$

Moreover, $m'/e(\mathfrak{A}') = m/e(\mathfrak{A})$ and the lattice chain defining $\mathfrak{A}'$ contains that which defines $\mathfrak{A}$. In particular, $[\mathfrak{A}', m', m'-1, b]$ is a skew stratum and it is equivalent to $[\mathfrak{A}', m', m'-1, b']$.

Proof: Put $e = e(\mathfrak{A})$, $g = (m, e)$ and $y = \pi_F^{m/g} b^{e/g}$, as above; so $\overline{y} = \eta y$, $\eta = \pm$. Let $\phi(X) \in k_F[X]$ be the unique (monic) irreducible factor of the characteristic polynomial of the stratum $[\mathfrak{A}, n, n-1, b]$. Let $\Phi(X) \in \mathfrak{o}_F[X]$ be such that $\widetilde{\Phi}(X) = \phi(X)$ and such that $\overline{\Phi}(\eta X) = \Phi(X)$. Then we will show that we can choose b' such that the element $y' = \pi_F^{m/g} b'^{e/g}$ has minimal polynomial $\Phi(X)$.

We now reduce to the case where the stratum takes a *standard form* (see (5.3.10) below).

Let $\mathfrak{L} = \{L_k : k \in \mathbb{Z}\}$ be the (self-dual) lattice chain associated to $\mathfrak{A}$ and, after renumbering, let $d = 0$ or -1 be the integer such that $L_k^\# = L_{d-k}$ for all $k \in \mathbb{Z}$, given by (1.1.2). We define residual subspaces $\mathcal{W}^{(i)}$ of $\mathfrak{L}$, $0 \leq i \leq g-1$, by

$$\mathcal{W}_k^{(i)} = \begin{cases} L_k/L_{k+1} & \text{if } k \equiv i \pmod{g} \\ 0 & \text{otherwise} \end{cases}$$

Let $f = [\frac{g+1-d}{2}]$; for $1 \le j \le f$, set

$$\mathcal{V}^{(j)} = \mathcal{W}^{(j+d-1)} + \mathcal{W}^{(g-j)}$$

Note that if $2j \not\equiv 1-d \pmod{g}$ then the sum is direct, while if $2j \equiv 1-d \pmod{g}$ then $\mathcal{V}^{(j)} = \mathcal{W}^{(j+d-1)}$. Set

$$g_j = \begin{cases} 1 & \text{if } 2j \equiv 1-d \pmod{g} \\ 2 & \text{otherwise.} \end{cases}$$

(5.3.3) Lemma *The residual subspaces $\mathcal{V}^{(j)}$ are nondegenerate.*

Proof: This is identical to (3.2.3). ■

The residual subspaces $\mathcal{V}^{(j)}$ are clearly orthogonal and $L_k/L_{k+1} = \mathcal{V}_k^{(1)} \oplus \cdots \oplus \mathcal{V}_k^{(f)}$ for all k so we can apply (3.1.9) to get an orthogonal decomposition

$$V = V^{(1)} \perp \cdots \perp V^{(f)}$$

such that $\mathcal{V}^{(j)} = \mathfrak{L}(V^{(j)})$ for $1 \le j \le f$. Then h decomposes as $h = h_1 \perp \cdots \perp h_f$, where h_j is a nondegenerate ϵ-hermitian form on $V^{(j)}$, $1 \le j \le f$.

Write $L_k^{(j)} = L_k \cap V^{(j)}$ for $k \in \mathbb{Z}$, $1 \le j \le f$.

(5.3.4) Lemma *The set $\mathfrak{L}^{(j)} = \{L_k^{(j)} : k \in \mathbb{Z}\}$ is a self-dual lattice chain in $V^{(j)}$, $1 \le j \le f$. Moreover, the $\mathfrak{o}_F$-period $e_j = e(\mathfrak{L}^{(j)})$ is $\frac{e}{g} g_j$.*

Proof: This is identical to (3.2.4). ■

Putting $A^{(ij)} = \mathrm{Hom}(V^{(j)}, V^{(i)})$, we get a "block decomposition":

$$A = \coprod_{1 \le i,j \le f} A^{(ij)}.$$

Write $\mathbf{1}^{(j)}$ for the projection $V \to V^{(j)}$ with kernel $\coprod_{i \ne j} V^{(i)}$ so that $\mathbf{1}^{(j)}$ is in fact the identity element of the algebra $A^{(jj)}$ and $A^{(ij)} = \mathbf{1}^{(i)}.A.\mathbf{1}^{(j)}$. Then $\mathbf{1}^{(j)} L_k = L_k^{(j)} \subset L_k$ for all $k \in \mathbb{Z}$ so we have $\mathbf{1}^{(j)} \in \mathfrak{A}$, $1 \le j \le f$.

We denote ${}^{-j}$ the adjoint involution on $A^{(jj)}$ induced by the form h_j, $1 \le j \le f$. Then set $A_-^{(jj)} = \{x \in A^{(jj)} : \overline{x}^j = -x\}$.

(5.3.5) Lemma *Write $b = \sum b_{ij}$ with $b_{ij} \in A^{(ij)}$. Then*
(i) $b_{jj} \in A_-^{(jj)}$ for $1 \le j \le f$;
(ii) $\mathfrak{P}^{-m} \cap A^{(ij)} = \mathfrak{P}^{1-m} \cap A^{(ij)}$ for $i \ne j$;
(iii) $b \equiv \sum b_{jj} \pmod{\mathfrak{P}^{1-m}}$.

Proof: (*i*) Let $w, w' \in V^{(j)}$. Then we have

$$h_j(b_{jj} w, w') = h(bw, w') = h(w, -bw') = h_j(w, -b_{jj} w')$$

so $\overline{b_{jj}}^j = -b_{jj}$.

(ii) Let $x \in \mathfrak{P}^{-m} \cap A^{(ij)}$, $i \neq j$ so $xL_k^{(j)} \subset L_{k-m}^{(i)}$. Note that

$$\begin{aligned} &L_k^{(j)} \neq L_{k+1}^{(j)} \text{ if and only if } L_{k-m}^{(j)} \neq L_{k-m+1}^{(j)} \\ \text{and} \qquad &L_k^{(j)} \neq L_{k+1}^{(j)} \text{ implies } L_k^{(i)} = L_{k+1}^{(i)} \end{aligned}$$

for $k \in \mathbb{Z}$. Then if $L_k^{(j)} \neq L_{k+1}^{(j)}$ we have $x(L_k^{(j)}) \subset L_{k-m}^{(i)} = L_{k-m+1}^{(i)}$ while if $L_k^{(j)} = L_{k+1}^{(j)}$ then $x(L_k^{(j)}) = x(L_{k+1}^{(j)}) = L_{k-m+1}^{(i)}$. So $x \in \mathfrak{P}^{1-m} \cap A^{(ij)}$.
(iii) is immediate from (ii). ■

Let $\mathfrak{A}^{(j)} = \mathfrak{A}(\mathfrak{L}^{(j)})$ be the hereditary $\mathfrak{o}_F$-order in $A^{(jj)}$ corresponding to $\mathfrak{L}^{(j)}$.

(5.3.6) Lemma *With the notation above, we have* $\mathfrak{A}^{(j)} = \mathfrak{A} \cap A^{(jj)}$.

Proof: Let $x \in \mathfrak{A} \cap A^{(jj)}$; then $xL_k \subset L_k$ for $k \in \mathbb{Z}$ so $xL_k^{(j)} \subset L_k^{(j)}$. i.e. $x \in \mathfrak{A}^{(j)}$. Conversely, let $x \in \mathfrak{A}^{(j)}$; then $xL_k^{(j)} \subset L_k^{(j)}$. But $x = \mathbf{1}^{(j)} x \mathbf{1}^{(j)}$ so $xL_k = \mathbf{1}^{(j)} x \mathbf{1}^{(j)} L_k = \mathbf{1}^{(j)} x L_k^{(j)} \subset L_k^{(j)} \subset L_k$. ■

Let $\mathfrak{P}^{(j)}$ be the Jacobson radical of $\mathfrak{A}^{(j)}$.

(5.3.7) Lemma (i) $\mathfrak{P} \cap A^{(jj)} = \mathfrak{P}^{(j)}$*;*
(ii) $\mathfrak{P}^{-m} \cap A^{(jj)} = (\mathfrak{P}^{(j)})^{-\frac{m}{g} g_j}$*;*
(iii) $\mathfrak{P}^{1-m} \cap A^{(jj)} = (\mathfrak{P}^{(j)})^{1-\frac{m}{g} g_j}$*.*

Proof: The proof of (i) is similar to (5.3.6), as are the proofs of (ii) and (iii), having observed that $L_k^{(j)} \neq L_{k+1}^{(j)}$ if and only if $k \equiv -j$ or $j + d - 1 \pmod g$, so that

$$\begin{aligned} (\mathfrak{P}^{(j)})^{g_j} L_k^{(j)} &= L_{k+g}^{(j)} && \text{for } k \in \mathbb{Z}; \\ (\mathfrak{P}^{(j)})^{1+g_j} L_k^{(j)} &= L_{k+g+1}^{(j)} && \text{for } k \equiv -j \text{ or } j+d-1 \pmod g. \end{aligned}$$ ■

In particular, we have $\nu_{\mathfrak{A}^{(j)}}(b_{jj}) = -\frac{m}{g} g_j = -m_j$. We consider the stratum $[\mathfrak{A}^{(j)}, m_j, m_j - 1, b_{jj}]$ in $A^{(jj)}$. Set $y_j = \pi_F^{m_j/g_j} b_{jj}^{e_j/g_j} = \pi_F^{m/g} b_{jj}^{e/g}$. Then we have

$$y \equiv \sum_j y_j \pmod{\mathfrak{P}}$$

from (5.3.5)(iii), since $b_{jj} b_{ii} = 0$ for $i \neq j$. The characteristic polynomial of y as an element of $\mathfrak{A}/\mathfrak{P} = \coprod \mathfrak{A}^{(j)}/\mathfrak{P}^{(j)}$ is the product of the characteristic polynomials of the y_j as elements of $\mathfrak{A}^{(j)}/\mathfrak{P}^{(j)}$. In particular, $[\mathfrak{A}^{(j)}, m_j, m_j - 1, b_{jj}]$ is a non-split skew fundamental stratum with characteristic polynomial a power of $\phi(X)$.
Each lattice chain $\mathfrak{L}^{(j)}$ is given by

$$\cdots \supsetneqq \pi_F^{-1} L_{r_j}^{(j)} \supset L_{r_j}^{(j)\#} \supsetneqq \cdots \supsetneqq L_0^{(j)\#} \supset L_0^{(j)} \supsetneqq \cdots \supsetneqq L_{r_j}^{(j)} = \pi_F L_{r_j}^{(j)\#} \supsetneqq \cdots$$

so $e_j = 2r_j$, $2r_j + 1$ or $2r_j + 2$. Note further that if $g_j = 2$ then, for $k = j + d - 1$, $(L_k^{(j)})^\# = L_{d-k}^{(j)} = L_{-j+1}^{(j)} = L_k^{(j)}$ as $-j + 1 > -j$. So if we renumber the lattices of $\mathfrak{L}^j$ then $(L_0^{(j)})^\# = L_0^{(j)}$.

Putting this together, we split the possibilities into five cases:

(5.3.8) **(*i*)** $(e_j, m_j) = 2,\ L_0^{(j)} = L_0^{(j)\#}$ *and* $L_{r_j}^{(j)} = \pi_F L_{r_j}^{(j)\#}$*; so* $e_j = 2r_j$*;*
(*ii*) $(e_j, m_j) = 1,\ L_0^{(j)} \neq L_0^{(j)\#}$ *and* $L_{r_j}^{(j)} = \pi_F L_{r_j}^{(j)\#}$*; so* $e_j = 2r_j + 1$*;*
(*iii*) $(e_j, m_j) = 1,\ L_0^{(j)} \neq L_0^{(j)\#}$ *and* $L_{r_j}^{(j)} \neq \pi_F L_{r_j}^{(j)\#}$*; so* $e_j = 2r_j + 2$*;*
(*iv*) $(e_j, m_j) = 1,\ L_0^{(j)} = L_0^{(j)\#}$ *and* $L_{r_j}^{(j)} \neq \pi_F L_{r_j}^{(j)\#}$*; so* $e_j = 2r_j + 1$*;*
(*v*) $(e_j, m_j) = 1,\ L_0^{(j)} = L_0^{(j)\#}$ *and* $L_{r_j}^{(j)} = \pi_F L_{r_j}^{(j)\#}$*; so* $e_j = 2r_j + 1$.

Suppose, in each of these cases, we can find a stratum $[\mathfrak{A}'^{(j)}, m'_j, m'_j - 1, b'_{jj}]$ such that $\mathfrak{A}'^{(j)} \subset \mathfrak{A}^{(j)}$, $y'_{jj} = \pi_F^{m'_j/g'_j} b'^{e'_j/g'_j}_{jj}$ has minimal polynomial $\Phi(X)$ and

$$b_{jj} + (\mathfrak{P}^{(j)})^{1-m_j} \subset b'_{jj} + (\mathfrak{P}'^{(j)})^{1-m'_j}.$$

Then put $b' = \sum b'_j$, $m' = \sum m'_j$ and $e' = \sum e'_j$. Set

$$\begin{aligned}\mathfrak{L}' = \{\mathfrak{o}_F\text{-lattices } L \in V :\ & L_i \supseteq L \supseteq L_{i+1} \text{ for some } i \in \mathbb{Z} \text{ and,} \\ & \text{for } 1 \le j \le f,\ L \cap V^{(j)} = L'^{(j)}_{k_j} \text{ for some } k_j \in \mathbb{Z}\}.\end{aligned}$$

(5.3.9) Lemma *With notation as above, $\mathfrak{L}'$ is a self-dual lattice chain in V and we have $e(\mathfrak{L}') = e'$ and $b'\mathfrak{A}' = \mathfrak{P}'^{m'}$, where $\mathfrak{A}' = \mathfrak{A}(\mathfrak{L}')$ and $\mathfrak{P}'$ is the radical of $\mathfrak{A}'$.*

Proof: For each $i \in \mathbb{Z}$ there exists a unique j, $1 \le j \le f$, such that $L_i^{(j)} \neq L_{i+1}^{(j)}$. Hence $\mathfrak{L}'$ is a lattice chain and $e(\mathfrak{L}') = e'$. It is self-dual since each $\mathfrak{L}'^{(j)}$ is. ■

Then $F[b'] = F[b'_{jj}]$ for any j, $1 \le j \le f$ since $b'_{ii}b'_{jj} = 0$ for $i \neq j$ so $F[b']$ is a field and b' is minimal over F. In particular, $[\mathfrak{A}', m', m' - 1, b']$ is a skew simple stratum in A. Moreover, we have

$$\begin{aligned} b + \mathfrak{P}^{1-m} &= \coprod_{j=1}^{f} (b_{jj} + \mathfrak{P}^{(j)^{1-m_j}}) \coprod_{i \neq j} \mathfrak{P}^{1-m} \cap A^{(ij)} \\ &\subset \coprod_{j=1}^{f} (b'_{jj} + \mathfrak{P}'^{(j)^{1-m'_j}}) \coprod_{i \neq j} \mathfrak{P}'^{1-m'} \cap A^{(ij)} = b' + \mathfrak{P}'^{1-m'}, \end{aligned}$$

since $\mathfrak{P}^{1-m} \cap A^{(ij)} = \mathfrak{P}'^{1-m'} \cap A^{(ij)}$, for $i \neq j$, by counting lattices. Hence we are done.

Before completing the proof in the cases (*i*–*v*) of (5.3.8) we describe what we mean by *standard form*. We choose a splitting of $\mathfrak{L}$, $V_\infty \oplus V_{-r} \oplus \cdots \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus \cdots \oplus V_r$ (see (3.2.2)) with, possibly, $V_\infty = 0$ and/or $V_0 = 0$. Let Ω be the set of indices $\omega \in \{\infty, -r, ..., r\}$ such that V_ω is non-trivial; then, as in (3.2), $\mathbb{Z}/e\mathbb{Z}$ acts on Ω by translation in the obvious way. For $\omega \in \Omega$, $i \in \mathbb{Z}$, we will write $\omega + i$ for the translate by $i + e\mathbb{Z}$ of ω. We also choose a basis $\mathcal{B}$ as in (3.2.5), such that the matrices J_∞, J_0 take the form described in (3.2.8). Put $\delta = \deg \phi(X)$.

(5.3.10) Definition *Let $[\mathfrak{A}, m, m-1, b]$ be a non-split fundamental stratum in band form, $b = \boldsymbol{r}_{-m}(\boldsymbol{b})$, where $\boldsymbol{b} = (b_\omega : \omega \in \Omega)$.*
(a) Suppose first we are in one of (i–v) of (5.3.8). Then the stratum $[\mathfrak{A}, m, m-1, b]$ is in standard form *with respect to $\mathcal{B}$ if the following holds:*
Each b_ω is block diagonal (in blocks of size $\delta \times \delta$ or, in cases (iv), (v), possibly one middle block of size $2\delta \times 2\delta$), except for the following which are upper block triangular modulo $\mathfrak{p}_F$:

case (i) b_1 *and* b_{-1-m};
case (ii) $b_{[\frac{e}{2}]m}$;
case (iii) b_∞ *and* $b_{\infty-m}$;
case (iv) $b_{\infty+[\frac{e}{2}]m}$;
case (v) $b_{\frac{e-m+1}{2}}$.

(b) Suppose otherwise. Then the stratum $[\mathfrak{A}, m, m-1, b]$ is in standard form *with respect to $\mathcal{B}$ if each of the strata $[\mathfrak{A}^{(j)}, m_j, m_j - 1, b_{jj}]$ (as described above) is in standard form.*

(5.3.11) Proposition *Let $[\mathfrak{A}, m, m-1, b]$ be a non-split fundamental stratum which takes the form of one of cases (i–v) of (5.3.8) and is in band form, $b = \boldsymbol{r}_{-m}(\boldsymbol{b})$, where $\boldsymbol{b} = (b_\omega : \omega \in \Omega)$. Then there exists a self-dual basis $\mathcal{B}$ which matches the splitting such that $[\mathfrak{A}, m, m-1, b]$ is in standard form with respect to $\mathcal{B}$.*

Proof: We treat the five cases separately, although there are many similarities.

Case (*i*)

The maps $\widetilde{b}_k$ and $(\widetilde{\pi_F})_k$ are isomorphisms for all $k \in \mathbb{Z}$ by (3.3.1) so, in particular, we have $\dim_{k_F} \mathcal{V}_k = \dim_{k_F} \mathcal{V}_{k+2}$ for all $k \in \mathbb{Z}$. Further, duality via $\boldsymbol{h}$ gives $\dim_{k_F} \mathcal{V}_k = \dim_{k_F} \mathcal{V}_{-1-k}$ for all $k \in \mathbb{Z}$, so $\dim_{k_F} \mathcal{V}_k = \frac{N}{e}$ for all $k \in \mathbb{Z}$.

Form $y = \pi_F^{m/2} b^{e/2}$ and consider $\widetilde{y}_0$, an automorphism of $\mathcal{V}_0$. It has characteristic polynomial $\phi(X)^t$, for some t, so, using Jordan canonical form, we can find a flag of k_F-subspaces

$$\mathcal{V}_0 = \mathcal{V}_0^0 \supset \mathcal{V}_0^1 \supset \cdots \supset \mathcal{V}_0^s = 0$$

such that $\phi(\widetilde{y}_0)\mathcal{V}_0^{q-1} \subset \mathcal{V}_0^q$ for $1 \le q \le s = t$ and which is stable under $\widetilde{y}_0$. So, as an automorphism of $\mathcal{V}_0^q / \mathcal{V}_0^{q+1}$, $\widetilde{y}_0$ has (irreducible) characteristic polynomial $\phi(X)$.

Then we obtain a flag in each $\mathcal{V}_k$ by translation by $\tilde{b}$, $\tilde{\pi}_F$ and by duality:

$$\mathcal{V}_k = \mathcal{V}_k^0 \supset \mathcal{V}_k^1 \supset \cdots \supset \mathcal{V}_k^s = 0$$

with

$$\mathcal{V}_k^q = \begin{cases} \tilde{\pi}_F^u \tilde{b}^\nu \mathcal{V}_0^q & k \equiv 0 \pmod 2 \\ \tilde{\pi}_F^u \tilde{b}^\nu (\mathcal{V}_0^{s-q})^\perp & k \equiv 1 \pmod 2, \end{cases}$$

where $u, \nu \in \mathbb{Z}$ are such that

$$k = \begin{cases} ue + \nu m & \text{if } k \equiv 0 \pmod 2 \\ ue + \nu m - 1 & \text{if } k \equiv 1 \pmod 2 \end{cases}$$

Note that this is well defined since the flags are $\widetilde{y}$-stable and that $(\mathcal{V}_k^q)^\perp = \mathcal{V}_{-1-k}^{s-q}$.
Now we must lift this structure from the residual level to V. Choose a decomposition $\mathcal{V}_0 = \bigoplus_{j=1}^s \mathcal{W}_0^j$ such that $\mathcal{V}_0^{q-1} = \bigoplus_{j=q}^s \mathcal{W}_0^j$ for $1 \leq q \leq s$. Put $\mathcal{W}_{-1}^q = \bigcap_{j\neq s-q+1} (\mathcal{W}_0^j)^\perp$ for $1 \leq q \leq s$; then $\mathcal{V}_{-1}^{q-1} = \bigoplus_{j=q}^s \mathcal{W}_{-1}^j$. Then for $i = 0, -1$, $1 \leq q \leq s$, let $\mathcal{W}^{i,q}$ be the residual subspace of $\mathfrak{L}$ given by

$$(\mathcal{W}^{i,q})_k = \begin{cases} \widetilde{\pi_F}^u \mathcal{W}_i^q & \text{if } k = ue + i \\ 0 & \text{otherwise.} \end{cases}$$

Recall that we have a splitting $V_{-r} \oplus \cdots \oplus V_{-1} \oplus V_1 \oplus \cdots \oplus V_r$ of $\mathfrak{L}$ (so $\Omega = \{-r, \ldots, -1, 1, \ldots, r\}$ in this case). For $1 \leq q \leq s$, there exist (totally isotropic) subspaces W_1^q of V_1 such that

$$\mathfrak{L}(W_1^q) = \mathcal{W}^{0,q}.$$

We also put

$$\mathfrak{L}(W_{-1}^q) = \bigcap_{j\neq s-q+1} (W_1^q)^\perp;$$

then $\mathfrak{L}(W_{-1}^q) = \mathcal{W}^{-1,q}$ and $(W_1^q)^\perp \cap V_{-1} = \bigoplus_{j\neq s-q+1} W_{-1}^j$.
Now, for $1 \leq \nu \leq \frac{e}{2} - 1$ and $1 \leq q \leq s$, we put

$$\begin{aligned} W_{1-\nu m}^q &= b^\nu W_1^q, \\ W_{-1+\nu m}^q &= b^{-\nu} W_{-1}^q, \end{aligned}$$

so that $V_\omega = W_\omega^1 \oplus \cdots \oplus W_\omega^q$ for all $\omega \in \Omega$. Then b_ω is block diagonal for all $\omega \in \Omega$, except b_1 and b_{-1-m} which are upper block triangular modulo $\mathfrak{p}_F$ since y is.

Case *(ii)*

As in case (i), $\dim_{k_F} \mathcal{V}_k = \frac{N}{e}$ for all $k \in \mathbb{Z}$.
Form $y = \pi_F^m b^e$ and consider $\widetilde{y}_{-1}$ as an automorphism of $\mathcal{V}_{-1}$. It has characteristic polynomial $\phi(X)^t$, for some t. We also have the nondegenerate form $\boldsymbol{h}_{-1} : \mathcal{V}_{-1} \times \mathcal{V}_{-1} \to k_F$ and, for $^-$ the involution associated to $\boldsymbol{h}_{-1}$, we have $\overline{\widetilde{y}_{-1}} = \pm\widetilde{y}_{-1}$. By (5.2.2)$(i)$, applied to $(\mathcal{V}_{-1}, \boldsymbol{h}_{-1})$, we can find a flag of k_F-subspaces

$$\mathcal{V}_{-1} = \mathcal{V}_{-1}^0 \supset \mathcal{V}_{-1}^1 \supset \cdots \supset \mathcal{V}_{-1}^s = 0$$

such that $(\mathcal{V}_{-1}^q)^\perp = \mathcal{V}_{-1}^{s-q}$ for $0 \leq q \leq s = t$, $\phi(\widetilde{y}_{-1})\mathcal{V}_{-1}^{q-1} \subset \mathcal{V}_{-1}^q$ for $1 \leq q \leq s$ and which is stable under $\widetilde{y}_{-1}$.
From this flag we can obtain a flag in each $\mathcal{V}_k$ by translating by $\tilde{b}$ and $\tilde{\pi}_F$:

$$\mathcal{V}_k = \mathcal{V}_k^0 \supset \mathcal{V}_k^1 \supset \cdots \supset \mathcal{V}_k^s = 0$$

with

$$\mathcal{V}_k^q = \tilde{\pi}_F^u \tilde{b}^\nu \mathcal{V}_{-1}^q$$

where $u, \nu \in \mathbb{Z}$ are such that $k + 1 = ue + \nu m$. Note that this is well defined as the flag is $\widetilde{y}$-stable and that $(\mathcal{V}_k^q)^\perp = \mathcal{V}_{-2-k}^{s-q}$.

By (5.2.2)(ii), there exists a decomposition $\mathcal{V}_{-1} = \bigoplus_{j=1}^{s} \mathcal{W}_{-1}^{j}$ such that $\mathcal{V}_{-1}^{q-1} = \bigoplus_{j=q}^{s} \mathcal{W}_{-1}^{j}$ and $(\mathcal{W}_{-1}^{j})^{\perp} = \bigoplus_{j \neq s-q+1} \mathcal{W}_{-1}^{j}$, for $1 \leq q \leq s$. Then for $1 \leq q \leq s$, let $\mathcal{W}^{-1,q}$ be the residual subspace of $\mathfrak{L}$ given by

$$(\mathcal{W}^{-1,q})_k = \begin{cases} \widetilde{\pi_F}^{u} \mathcal{W}_i^q & \text{if } k = ue-1 \\ 0 & \text{otherwise.} \end{cases}$$

Recall that we have a splitting $V_{-r} \oplus \cdots \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus \cdots \oplus V_r$ of $\mathfrak{L}$ (so $\Omega = \{-r, \ldots, 0, \ldots, r\}$). For $1 \leq q \leq s$, we will now choose subspaces W_0^q of V_0 such that $\mathfrak{L}(W_0^q) = \mathcal{W}^{-1,q}$.

If s is odd then $\mathcal{W}^{-1,[\frac{s+1}{2}]}$ is a nondegenerate residual subspace so there exists a nondegenerate subspace $W_0^{[\frac{s+1}{2}]}$ of V_0 such that $\mathfrak{L}(W_0^{[\frac{s+1}{2}]}) = \mathcal{W}^{-1,[\frac{s+1}{2}]}$. Then we replace V_0 with $(W_0^{[\frac{s+1}{2}]})^{\perp} \cap V_0$ to reduce to the case where s is even.

If s is even then we first choose totally isotropic subspaces W_0^1, W_0^s of V_0 such that $\mathfrak{L}(W_0^q) = \mathcal{W}^{-1,q}$, for $q = 1, s$. The subspace $W_0^1 \oplus W_0^s$ is nondegenerate and, replacing V_0 by $(W_0^1 \oplus W_0^s)^{\perp} \cap V_0$, we may continue inductively to choose W_0^q, for $1 \leq q \leq s$.

With these choices, W_0^q is totally isotropic if, and only if, $\mathcal{W}^{-1,q}$ is and, further, $(W_0^q)^{\perp} \cap V_0 = \bigoplus_{j \neq s-q+1} W_0^j$.

Now, for $1 \leq \nu \leq [\frac{e}{2}]$ and $1 \leq q \leq s$, we put

$$\begin{aligned} W_{0-\nu m}^q &= b^{\nu} W_0^q, \\ W_{0+\nu m}^q &= b^{-\nu} W_0^q, \end{aligned}$$

so that $V_\omega = W_\omega^1 \oplus \cdots \oplus W_\omega^q$ for all $\omega \in \Omega$. Then b_ω is block diagonal for all $\omega \in \Omega$, except $b_{m[\frac{e}{2}]}$, which is upper block triangular modulo $\mathfrak{p}_F$.

Case (iii)

We proceed as in case (ii) to obtain the k_F-spaces $\mathcal{V}_k^q$, $1 \leq q \leq s$, $k \in \mathbb{Z}$, and hence subspaces W_0^q of V_0. (Note that here the splitting of $\mathfrak{L}$ takes the form $V_\infty \oplus V_{-r} \oplus \cdots \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus \cdots \oplus V_r$, so $\Omega = \{\infty, -r, \ldots, 0, \ldots, r\}$.) Again, for $1 \leq \nu \leq \frac{e}{2} - 1$ and $1 \leq q \leq s$, we put

$$\begin{aligned} W_{0-\nu m}^q &= b^{\nu} W_0^q; \\ W_{0+\nu m}^q &= b^{-\nu} W_0^q. \end{aligned}$$

(Note that we have not yet chosen subspaces of V_∞.) Then b_ω is block diagonal for $\omega \in \Omega \setminus \{\infty, \infty - m\}$.

Now we must choose subspaces W_∞^q of V_∞ such that b_∞ is upper block triangular modulo $\mathfrak{p}_F$; for then $b_{\infty-m}$ must also be upper block triangular modulo $\mathfrak{p}_F$, since y is. (In fact, $b_{\infty-m} = (\pi_F^{-1} J_\infty^{-1} I)\, {}^{\dagger}b_\infty$, where J_∞, I are as in (3.2.5) and † represents transposition in the off-diagonal; so $b_{\infty-m}$ is upper block triangular modulo $\mathfrak{p}_F$ if, and only if, b_∞ is also, regardless of the element y.)

We consider the k_F-space $\mathcal{V}_r$, with form $\boldsymbol{h}_r' : \mathcal{V}_r \times \mathcal{V}_r \to k_F$ given by $\boldsymbol{h}_r'(v, v') = \boldsymbol{h}_r(v, \widetilde{\pi_F}^{-1} v')$, for $v, v' \in \mathcal{V}_r$. This form is still skew-hermitian if F/F_0 is unramified, but is symmetric if F/F_0 is ramified.

We have subspaces $\mathcal{V}_r^q \subset \mathcal{V}_r$ such that $(\mathcal{V}_r^q)^\perp = \widetilde{\pi_F}^{-1}\mathcal{V}_r^{s-q}$ (for $\perp$ relative to $\boldsymbol{h}_r$). So, for $\perp$ relative to the form $\boldsymbol{h}'_r$, we have $(\mathcal{V}_r^q)^\perp = \mathcal{V}_r^{s-q}$. We may now apply (5.2.2)(ii) to obtain subspaces $\mathcal{W}_r^q$ of $\mathcal{V}_r$ such that $\mathcal{V}_r^{q-1} = \bigoplus_{j=q}^s \mathcal{W}_r^j$ and $(\mathcal{W}_r^j)^\perp = \bigoplus_{j\neq s-q+1}\mathcal{W}_r^j$, for $1 \leq q \leq s$.

Now we define residual subspaces $\mathcal{W}^{r,q}$ as for $\mathcal{W}^{-1,q}$ and find subspaces W_∞^q of V_∞ as for W_0^q in case (ii). Then, since $\widetilde{b}_{r+m}\mathcal{V}_{r+m}^q = \mathcal{V}_r^q$, for $1 \leq q \leq s$, we have that $b_\infty : V_{\infty+m} \to V_\infty$ is upper block triangular modulo $\mathfrak{p}_F$, as required.

Case (iv)

We treat this case as case (ii) except that we take $\mathcal{V}_r$ instead of $\mathcal{V}_{-1}$, with form $\boldsymbol{h}'_r : \mathcal{V}_r \times \mathcal{V}_r \to k_F$ given by $\boldsymbol{h}'_r(v,v') = \overline{\pi_F}^{-1}h(v,v') + \mathfrak{p}_F$ $(= \boldsymbol{h}_r(v, \widetilde{\pi_F}^{-1}v'))$ for $v, v' \in \mathcal{V}_r$. As in case (iii), this form is skew-hermitian if F/F_0 is unramified, but symmetric if F/F_0 is ramified.

The rest is effectively identical to case (ii), with V_0 replaced by V_∞, although it is possible to have a double-sized block from (5.2.2)(i) (in the case F/F_0 ramified, m odd and t even). As in case (ii), there is only one b_ω which is upper block triangular modulo $\mathfrak{p}_F$, namely $b_{\infty+m[\frac{s}{2}]}$.

Case (v)

As in case (i), $\dim\mathcal{V}_k = \frac{N}{e}$ for all $k \in \mathbb{Z}$. Note that $-m$ must be odd so we put $-m = 2l+1$; then b induces the map $\widetilde{b}_{-l-1} : \mathcal{V}_{-l-1} \to \mathcal{V}_l$.

Form $y = \pi_F^m b^e$ and consider $\widetilde{y}$ as an automorphism of $\mathcal{V}_{-l-1}\oplus\mathcal{V}_l$, where $\mathcal{V}_{-l-1}$ and $\mathcal{V}_l$ are maximal isotropic subspaces. It has characteristic polynomial $\phi(X)^{2t}$, some t. We also have the form $\boldsymbol{h}'$ on $\mathcal{V}_{-l-1} \oplus \mathcal{V}_l$, given by $\boldsymbol{h}'(v_{-l-1} + v_l, v'_{-l-1} + v'_l) = \boldsymbol{h}_{-l-1}(v_{-l-1}, v'_l) + \boldsymbol{h}_l(v_l, v'_{-l-1})$, which is nondegenerate, since $\boldsymbol{h}_l$ is, and skew-hermitian. For $\bar{\ }$ the involution associated to this form, we have $\overline{\widetilde{y}} = \pm\widetilde{y}$. By (5.2.3)$(i)$, applied to $(\mathcal{V}_{-l-1} \oplus \mathcal{V}_l, \boldsymbol{h}')$, we can find flags of k_F-subspaces

$$\mathcal{V}_{-l-1} = \mathcal{V}_{-l-1}^0 \supset \mathcal{V}_{-l-1}^1 \supset \cdots \supset \mathcal{V}_{-l-1}^s = 0$$
$$\mathcal{V}_l = \mathcal{V}_l^0 \supset \mathcal{V}_l^1 \supset \cdots \supset \mathcal{V}_l^s = 0$$

such that $(\mathcal{V}_{-l-1}^q)^\perp = \mathcal{V}_l^{s-q}$, $\widetilde{b}\mathcal{V}_{-l-1}^q = \mathcal{V}_l^q$, each flag is $\widetilde{y}$-stable and $\phi(\widetilde{y})\mathcal{V}_l^{q-1} \subset \mathcal{V}_l^q$ for $1 \leq q \leq s$, $s = t$ or $t-1$. (Note that $s = t-1$ can only occur if $F = F_0$; for it occurs only if $k_F = k_0$ and, if $F \neq F_0$, then F/F_0 must be ramified. But then $\overline{y} = (-1)^{m/g}(-1)^{e/g}y = -y$ so $\eta = -$.)

From this flag we can obtain a flag in each $\mathcal{V}_k$ by translating by $\tilde{b}$ and $\tilde{\pi}_F$:

$$\mathcal{V}_k = \mathcal{V}_k^0 \supset \mathcal{V}_k^1 \supset \cdots \supset \mathcal{V}_k^s = 0$$

with

$$\mathcal{V}_k^q = \tilde{\pi}_F^u \tilde{b}^\nu \mathcal{V}_{-1}^q$$

where $u, \nu \in \mathbb{Z}$ are such that $k + l + 1 = ue + \nu m$. Note that this is well defined as the flag is $\widetilde{y}$-stable and that $(\mathcal{V}_k^q)^\perp = \mathcal{V}_{-1-k}^{s-q}$.

By (5.2.3)(ii) there exist decompositions $\mathcal{V}_i = \bigoplus_{j=1}^s \mathcal{W}_i^j$, $i = -l-1, l$, such that $\mathcal{V}_i^{q-1} = \bigoplus_{j=q}^s \mathcal{W}_i^j$, $(\mathcal{W}_{-l-1}^q)^\perp = \mathcal{V}_{-l-1} \oplus \bigoplus_{j\neq s-q+1}\mathcal{W}_l^j$ and $\mathcal{W}_l^q = \widetilde{b}\mathcal{W}_{-l-1}^q$ for

$1 \leq q \leq s, i = -l-1, l$. Then for $i = -l-1, l$, $1 \leq q \leq s$, let $\mathcal{W}^{i,q}$ be the residual subspace of $\mathfrak{L}$ given by

$$(\mathcal{W}^{i,q})_k = \begin{cases} \widetilde{\pi_F}^u \mathcal{W}_i^q & \text{if } k = ue + i \\ 0 & \text{otherwise.} \end{cases}$$

Recall that we have a splitting $V_{-r} \oplus \cdots \oplus V_{-1} \oplus V_1 \oplus \cdots \oplus V_r$ of $\mathfrak{L}$ (so $\Omega = \{-r, \ldots, -1, 1, \ldots, r\}$). Then there exist (totally isotropic) subspaces W^q_{-l-1} of V_{-l-1} such that

$$\mathfrak{L}(W^q_{-l-1}) = \mathcal{W}^{-l-1,q}.$$

We also put

$$W^q_{l+1} = \bigcap_{j \neq s-q+1} (W^j_{-l-1})^{\perp};$$

then $\mathfrak{L}(W^q_{l+1}) = \mathcal{W}^{l,q}$ and $(W^q_{l+1})^{\perp} \cap V_{-l-1} = \bigoplus_{j \neq s-q+1} W^j_{-l-1}$.
Now, for $1 \leq \nu \leq \frac{e}{2} - 1$ and $1 \leq q \leq s$, we put

$$\begin{aligned} W^q_{l+1-\nu m} &= b^{\nu} W^q_{l+1}, \\ W^q_{-l-1+\nu m} &= b^{-\nu} W^q_{-l-1}, \end{aligned}$$

so that $V_\omega = W^1_\omega \oplus \cdots \oplus W^q_\omega$ for all $\omega \in \Omega$. Then b_ω is block diagonal for all $\omega \in \Omega$, except $b_{\frac{e+m-1}{2}}$ which is upper block triangular modulo $\mathfrak{p}_F$.
This completes the proof of (5.3.11) in all cases. ■

(5.3.12) Remark *In cases $(i), (ii), (iii), (iv)$, it is possible for us to choose the bases for the W^q_ω such that all the b_ω are diagonal with 1s and -1s on the diagonal, excepting those blocks which are upper triangular by blocks modulo $\mathfrak{p}_F$. In case (v), we may choose the bases for the W^q_ω such that all the b_ω are 1 or -1, excepting $b_{\frac{e+m-1}{2}}$ and $b_{\frac{1-m}{2}}$.*

We now prove a result similar to **[BK]** (2.5.8) about the shapes of the blocks of a stratum in standard form which is equivalent to a simple stratum.

(5.3.13) Proposition *Let $[\mathfrak{A}, m, m-1, b]$ be a non-split fundamental stratum which takes the form of one of cases (i–v) of (5.3.8) and is in band form, $b = \boldsymbol{r}_{-m}(\boldsymbol{b})$, where $\boldsymbol{b} = (b_\omega : \omega \in \Omega)$. Suppose $[\mathfrak{A}, m, m-1, b]$ is equivalent to a simple stratum. Then there exists a self-dual basis $\mathcal{B}$ which matches the splitting such that $[\mathfrak{A}, m, m-1, b]$ is in standard form with respect to $\mathcal{B}$ and b_ω is block diagonal modulo $\mathfrak{p}_F$ for all $\omega \in \Omega$.*

Proof: Put $y = \pi_F^{m/g} b^{e/g}$ as usual; then $y = \boldsymbol{r}_0(\boldsymbol{y})$, for some $\boldsymbol{y} = (y_\omega) \in \boldsymbol{B}_0(\mathfrak{A})$.
By **[BK]** (2.5.8), $y \pmod{\mathfrak{p}_F}$ is semisimple so y_ω is block diagonal (mod $\mathfrak{p}_F$), for all $\omega \in \Omega$. Then, cases (ii), (iv), (v), where only one b_ω is upper triangular modulo $\mathfrak{p}_F$, we are done, as indeed we are in case (i) since 1 and -1 are incongruent modulo 2 in Ω.

Consider now case (iii) where b_∞ and $b_{\infty+m}$ are upper block triangular modulo $\mathfrak{p}_F$. Write

$$b_\infty = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \theta & \zeta \\ 0 & 0 & \xi \end{pmatrix} \pmod{\mathfrak{p}_F},$$

where the block sizes are $[\frac{s}{2}]\delta$, 0 (respectively δ), $[\frac{s}{2}]\delta$ if s is even (respectively odd) and we have chosen a self-dual basis matching the decomposition (as in (3.2.9)). We also have that α and ξ are upper block triangular modulo $\mathfrak{p}_F$.

A simple matrix calculation shows that

$$b_{\infty+m} = \begin{pmatrix} \xi^* & -\zeta^* K & -\gamma^* \\ 0 & -\theta^* K & -\beta^* \\ 0 & 0 & -\alpha^* \end{pmatrix} \pmod{\mathfrak{p}_F},$$

where x^* is the conjugate transpose of x (for transposition in the off-diagonal) and K is the central block of IJ_∞ (notation as in (3.2)).

Let $u = \boldsymbol{r}_0(\boldsymbol{u})$, with $\boldsymbol{u} = (u_\omega) \in \boldsymbol{B}_0(\mathfrak{A})$, be the unipotent block diagonal matrix all of whose blocks are the identity except u_∞ which is given by

$$\begin{pmatrix} 1 & (\zeta\xi^{-1})^* K & \frac{1}{2}(\zeta\xi^{-1})^* K \zeta\xi^{-1} \\ 0 & 1 & \zeta\xi^{-1} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $(u^{-1}bu)_\infty$ has matrix

$$\begin{pmatrix} \alpha & \beta' & \gamma' \\ 0 & \theta & 0 \\ 0 & 0 & \xi \end{pmatrix} \pmod{\mathfrak{p}_F},$$

for some $\beta', \gamma,$. So, changing the basis $\mathcal{B}$, we may assume $\zeta = 0$.

We have $y = \boldsymbol{r}_0(\boldsymbol{y})$, with $\boldsymbol{y} = (y_\omega) \in \boldsymbol{B}_0(\mathfrak{A})$, and

$$y_{\infty+m} = \begin{pmatrix} \xi^*\alpha & \xi^*\beta & \xi^*\gamma - \gamma^*\xi \\ 0 & -\delta^* K \delta & -\beta * \xi \\ 0 & 0 & -\alpha^*\xi \end{pmatrix} Z \pmod{\mathfrak{p}_F},$$

where $Z \in \mathrm{GL}(sd, \mathfrak{o}_F)$ is block diagonal. Since $y_{\infty+m}$ is block diagonal modulo $\mathfrak{p}_F$, we have $-\xi^*\beta \equiv 0 \pmod{\mathfrak{p}_F}$ and $\xi^*\gamma - \gamma^*\xi \equiv 0 \pmod{\mathfrak{p}_F}$. In particular, $\beta \equiv 0 \pmod{\mathfrak{p}_F}$ and $(\gamma\xi^{-1})^* \equiv \gamma\xi^{-1} \pmod{\mathfrak{p}_F}$.

Put $y = \frac{1}{2}(\gamma^*\xi + \xi^*\gamma)$. Let $u' = \boldsymbol{r}_0(\boldsymbol{u}')$, with $\boldsymbol{u}' = (u'_\omega) \in \boldsymbol{B}_0(\mathfrak{A})$, be the unipotent block diagonal matrix all of whose blocks are the identity except u'_∞ which is given by

$$\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $(u'^{-1}bu')_\infty$ has matrix

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \xi \end{pmatrix} \pmod{\mathfrak{p}_F}.$$

Hence, again changing the basis $\mathcal{B}$, we may assume $\beta = 0$ and $\gamma = 0$.
Let $u'' = \boldsymbol{r}_0(\boldsymbol{u}'')$, with $\boldsymbol{u}'' = (u''_\omega) \in \boldsymbol{B}_0(\mathfrak{A})$, be the block diagonal matrix all of whose blocks are the identity except u_∞ which is given by

$$\begin{pmatrix} \alpha^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\alpha^{-1})^* \end{pmatrix}.$$

Then $(u''^{-1}bu'')_\infty$ has matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & (\alpha^{-1})^*\xi \end{pmatrix} \pmod{\mathfrak{p}_F}.$$

So, changing the basis $\mathcal{B}$, we may assume $\alpha = 1$; then ξ must be block diagonal modulo $\mathfrak{p}_F$ since y is. ■

(5.3.14) Corollary *Let $[\mathfrak{A}, m, m-1, b]$ be a non-split fundamental stratum in band form, $b = \boldsymbol{r}_{-m}(\boldsymbol{b})$, where $\boldsymbol{b} = (b_\omega : \omega \in \Omega)$. Suppose $[\mathfrak{A}, m, m-1, b]$ is equivalent to a simple stratum. Then there exists a self-dual basis $\mathcal{B}$ which matches the splitting such that $[\mathfrak{A}, m, m-1, b]$ is in standard form with respect to $\mathcal{B}$ and b_ω is block diagonal modulo $\mathfrak{p}_F$ for all $\omega \in \Omega$.*

We now return to the proof of (5.3.2), in cases (*i*–*v*) of (5.3.8), using notation from the proof of (5.3.11). We first define a self-dual lattice chain $\mathfrak{L}'$, using the residual subspaces $\mathcal{V}_k^q$ in the proof of (5.3.11).
In all cases we have a flag of k_F-subspaces in each $\mathcal{V}_k$ and they satisfy the relation

$$(\mathcal{V}_k^q)^\perp = \mathcal{V}_{d-k-1}^{s-q}.$$

Then for each $k \in \mathbb{Z}$, $0 \le q \le s-1$ there exists a unique lattice L'_{sk+q} such that

$$L'_{sk+q}/L_{k+1} = \mathcal{V}_k^q.$$

Then $\mathfrak{L}' = \{L'_i; i \in \mathbb{Z}\}$ is a lattice chain in V, of period $e' = es$.

(5.3.15) Lemma *The lattice chain $\mathfrak{L}'$ is self-dual.*

Proof: We show that $(L'_{sk+q})^\# = L'_{s(d-k)-q}$.
Since $L_k \supset L'_{sk+q} \supset L_{k+1}$, we have $L_{d-k} \subset (L'_{sk+q})^\# \subset L_{d-k-1}$. Then, for $v \in L_{d-k-1}$, we have

$$\begin{aligned} h(v, L'_{sk+q}) \subset \mathfrak{p}_F &\iff \mathbf{h}(v + L_{d-k}, \mathcal{V}_k^q) \equiv 0 \\ &\iff v + L_{d-k} \subset (\mathcal{V}_k^q)^\perp = \mathcal{V}_{d-k-1}^{s-q} \\ &\iff v \in L'_{s(d-k-1)+(s-q)} = L'_{s(d-k)-q}. \end{aligned}$$

$\blacksquare$

Let $\mathfrak{A}' = \mathfrak{A}(\mathfrak{L}')$; then $\mathfrak{A}' \subset \mathfrak{A}$ and $\overline{\mathfrak{A}'} = \mathfrak{A}'$.

(5.3.16) Lemma *With notation as above, we have* $b\mathfrak{A}' = \mathfrak{P}'^{-m'}$, *where* $m' = ms$.

Proof: We have $b \in \mathfrak{K}(\mathfrak{A})$ by **[BK]** (2.5.6) so $L_{k-m} = bL_k \supset bL'_{sk+q} \supset bL_{k+1} = L_{k-m+1}$. Then, for $v \in L_{k-m}$, we have

$$\begin{aligned} v \in bL'_{sk+q} &\Longleftrightarrow v + L_{k-m+1} \in \widetilde{b}\mathcal{V}^q_k = \mathcal{V}^q_{k-m} \\ &\Longleftrightarrow v \in L'_{s(k-m)+q}. \end{aligned}$$

So $b \in \mathfrak{K}(\mathfrak{A}')$ and $b\mathfrak{A}' = \mathfrak{P}'^{-ms}$. $\blacksquare$

Now we find an element $b' \in b + \mathfrak{P}'^{1-m'}$ such that $\Phi(b') = 0$. We take a splitting for $\mathfrak{L}'$,

$$V = V'_\infty \oplus V'_{-r'} \oplus \cdots \oplus V'_{-1} \oplus V'_0 \oplus V'_1 \oplus \cdots \oplus V'_{r'}$$

with possibly $V'_\infty = 0$ and/or $V'_0 = 0$ and which is subordinate to the chosen splitting for $\mathfrak{L}$. We also put $\Omega' = \{\omega' : V'_{\omega'} \neq 0\}$.
We have $\dim_F V'_{\omega'} = \delta$ or 2δ for all $\omega' \in \Omega'$, where $\delta = \deg \phi(X)$. Moving to stratum equivalent to $[\mathfrak{A}', m', m'-1, b]$, we may assume b is in band form, $b = \boldsymbol{r}_{-m'}(\boldsymbol{b})$ for $\boldsymbol{b} \in \boldsymbol{B}_{-m'}(\mathfrak{A}')$. Then $y = \pi_F^{m'/g'} b^{e'/g'}$ is of the form $y = \boldsymbol{r}_0(\boldsymbol{y})$, $\boldsymbol{y} = (y_{\omega'}) \in \boldsymbol{B}_0(\mathfrak{A}')$, with

$$y_{\omega'} = b_{\omega'} b_{\omega'-m} \dots b_{\omega'-(e'/g'-1)m'} \quad \in \operatorname{Hom}(V'_{\omega'}, V'_{\omega'}).$$

Let C be a set of representatives for the equivalence classes of Ω' under the action of $\mathbb{Z}/g'\mathbb{Z}$, so $\operatorname{card}(C) = g'$. If $V'_0 \neq 0$ then we require that $0 \in C$; if $V'_\infty \neq 0$ and $\infty \not\equiv 0 \pmod{g'}$, then we require that $\infty \in C$. Also, if we are in case (v) and m' is odd, then we require that $\frac{1-m'}{2} \in C$. Then we have four cases:

(a) $\omega' \neq 0, \infty, \frac{1-m'}{2}$;
(b) $\omega' = 0$;
(c) $\omega' = \infty$;
(d) $\omega' = \frac{1-m'}{2}$.

Case *(a)*
In this case we have $\dim V'_{\omega'} = \delta$.
By Hensel's Lemma, there exists $x_{\omega'} \equiv y_{\omega'} \pmod{\mathfrak{p}_F}$ such that $\Phi(x_{\omega'}) = 0$. Then $x_{\omega'} = u_{\omega'} y_{\omega'}$ for some $u_{\omega'} \equiv 1 \pmod{\mathfrak{p}_F}$.

Case *(b)*
In this case we have $\dim V'_0 = \delta$.
By (5.1.3) there exists $x_{\omega'} \in A_-^{(\omega'\omega')}$ such that $x_{\omega'} \equiv y_{\omega'} \pmod{\mathfrak{p}_F}$ and $\Phi(x_{\omega'}) = 0$. Then, by (5.1.5), there exists $u_{\omega'} \equiv 1 \pmod{\mathfrak{p}_F}$ such that $x_{\omega'} = u_{\omega'} y_{\omega'} \overline{u_{\omega'}}$.

Case (c)

If $\dim V'_\infty = \delta$ then this is identical to case (b). So suppose we are in the case $\dim V'_\infty = 2\delta$.

By the discussion following (5.2.2), we have $\mathfrak{L}(V'_\infty)_{-r'-1} = \mathcal{U}' \perp \mathcal{U}''$, with $\widetilde{y}_{-r'-1}\mathcal{U}' = \mathcal{U}'$ and $\widetilde{y}_{-r'-1}\mathcal{U}'' = \mathcal{U}''$. Let U', U'' be orthogonal subspaces of V'_∞ such that $\mathfrak{L}(U')_{-r'-1} = \mathcal{U}'$ and $\mathfrak{L}(U'')_{-r'-1} = \mathcal{U}''$. Then $y_\infty \equiv y'_\infty + y''_\infty \pmod{\mathfrak{p}_F}$, where $y'_\infty \in \mathrm{Hom}(U', U')$ and $y''_\infty \in \mathrm{Hom}(U'', U'')$. Moving to an equivalent stratum if necessary, we assume that $y_\infty = y'_\infty + y''_\infty$.

The spaces U', U'' are equipped with nondegenerate skew-hermitian forms (by restriction of h). Then we apply case (b) to find x'_∞, x''_∞ and u'_∞, u''_∞. Then we put $x_\infty = x'_\infty + x''_\infty$ and $u_\infty = u'_\infty + u''_\infty$. In particular, we have $\Phi(x_\infty) = \Phi(x'_\infty) + \Phi(x''_\infty) = 0$ and $u_\infty \equiv 1 \pmod{\mathfrak{p}_F}$.

Case (d)

Following the remark (5.3.12), we assume that $b_{v'} = \pm 1$ for $v' \equiv \omega' \pmod{g'}$, $v' \neq \omega', \omega' - r' - 1$.

We first treat the case $\dim V'_{\omega'} = \delta$.

Write $\alpha = b_{\omega'}$ and $\beta = b_{\omega'-r'-1}$. A simple matrix calculation shows that $b + \overline{b} = 0$ if and only if $\alpha = \alpha^*$ and $\beta = \beta*$, where x^* is the conjugate transpose of x, for $x \in \mathbb{M}(\delta, F)$ (for transposition in the off-diagonal). The map $x \mapsto x^*$ is the adjoint involution of a hermitian form on $\mathbb{M}(\delta, F)$.

By Hensel's Lemma, there exists $x_{\omega'} \equiv y_{\omega'} \pmod{\mathfrak{p}_F}$ such that $\Phi(x_{\omega'}) = 0$. By (5.1.8), there exist $\alpha' \equiv \alpha \pmod{\mathfrak{p}_F}$, $\beta' \equiv \beta \pmod{\mathfrak{p}_F}$ such that $\alpha'^* = \alpha'$, $\beta'^* = \beta'$ and $x_{\omega'} = \alpha'\beta'$. By (5.1.5), there exist $u_{\omega'}, u_{\omega'-r'-1} \equiv 1 \pmod{\mathfrak{p}_F}$ such that $\alpha' = u_{\omega'}\alpha u^*_{\omega'}$ and $\beta' = u_{\omega'-r'-1}\beta u^*_{\omega'-r'-1}$.

Now consider the case $\dim V'_{\omega'} = 2\delta$

By the discussion following (5.2.3), we have $\mathfrak{L}(V'_{\omega'})_{\omega'-1} = \mathcal{U}' \oplus \mathcal{U}''$ such that $\mathfrak{L}(V'_{\omega'-r'-1})_{\omega'-r'-1} = \widetilde{b}\mathcal{U}' \oplus \widetilde{b}\mathcal{U}''$ and $\mathbf{h}(\widetilde{b}\mathcal{U}', \mathcal{U}'') \equiv 0$. Let U' be a subspace of $V_{\omega'} \oplus V_{\omega'-r'-1}$ such that $\mathfrak{L}(U')_{\omega'-1} = \mathcal{U}'$ and $\mathfrak{L}(U')_{\omega'-r'-1} = \widetilde{b}\mathcal{U}'$. Put $U'' = U'^{\perp} \cap (V_{\omega'} \oplus V_{\omega'-r'-1})$.

The spaces U', U'' are equipped with nondegenerate skew-hermitian forms (by restriction of h). Then we apply the case $\dim V'_{\omega'} = \delta$ to find $x'_{\omega'}, x''_{\omega'}$ and $u'_{\omega'}, u''_{\omega'}$ and $u'_{\omega'-r'-1}, u''_{\omega'-r'-1}$. Then we put $x_{\omega'} = x'_{\omega'} + x''_{\omega'}$, $u_{\omega'} = u'_{\omega'} + u''_{\omega'}$ and $u_{\omega'-r'-1} = u'_{\omega'-r'-1} + u''_{\omega'-r'-1}$. In particular, we have $\Phi(x_{\omega'}) = \Phi(x'_{\omega'}) + \Phi(x''_{\omega'}) = 0$, $u_{\omega'} \equiv 1 \pmod{\mathfrak{p}_F}$ and $u_{\omega'-r'-1} \equiv 1 \pmod{\mathfrak{p}_F}$.

We set $u_{\omega'} = 1$, if u'_ω is not defined, and put $u = \boldsymbol{r}_0(\boldsymbol{u})$ where $\boldsymbol{u} = (u_{\omega'}) \in \boldsymbol{B}_0(\mathfrak{A}')$. Put $b' = ub\overline{u}$ and $y' = \pi_F^{m'/g'} b'^{e'/g'} = \boldsymbol{r}_0(\boldsymbol{y}')$. Then $\boldsymbol{y}' = (y_{\omega'})$, with $y'_{\omega'}$ a conjugate of $x_{v'}$, where $v' \in C$ is such that $\omega' \equiv v' \pmod{g'}$. In particular, we have $\Phi(y') = 0$.

(5.3.17) Proposition *$[\mathfrak{A}', m', m'-1, b']$ is a skew simple stratum in A with $b + \mathfrak{P}^{1-m} \subset b' + \mathfrak{P}'^{1-m'}$.*

Proof: y' generates an unramified field extension over F and has normalized valuation 0 with respect to this extension. So, by **[Br2]** (3.2.11), $[\mathfrak{A}', m', m'-1, b']$ is simple. Then b' is skew, $\overline{\mathfrak{A}'} = \mathfrak{A}'$ and $b + \mathfrak{P}^{1-m} \subset b' + \mathfrak{P}'^{1-m'}$ by construction. ■

This completes the proof of (5.3.2). ■

(5.4) Pure is equivalent to simple

We first recall an important result from **[BK]** (2.4).

(5.4.1) Theorem [BK (2.4.1)] *(i) Let $[\mathfrak{A}, n, r, \beta]$ be a pure stratum in A. There exists a simple stratum $[\mathfrak{A}, n, r, \gamma]$ in A such that*

$$[\mathfrak{A}, n, r, \gamma] \sim [\mathfrak{A}, n, r, \beta].$$

For any simple stratum $[\mathfrak{A}, n, r, \gamma]$ satisfying this condition, $e(F[\gamma]|F)$ divides $e(F[\beta]|F)$ and $f(F[\gamma]|F)$ divides $f(F[\beta]|F)$.
In particular, among all the pure strata $[\mathfrak{A}, n, r, \beta']$ equivalent to the given stratum $[\mathfrak{A}, n, r, \beta]$, the simple ones are precisely those for which the field extension $F[\beta']/F$ has minimal degree.
(ii) Let $[\mathfrak{A}, n, r, \gamma_1]$, $[\mathfrak{A}, n, r, \gamma_2]$ be simple strata in A which are equivalent to each other. Then

(a) $k_0(\gamma_1, \mathfrak{A}) = k_0(\gamma_2, \mathfrak{A})$;

(b) $e(F[\gamma_1]|F) = e(F[\gamma_2]|F)$ and $f(F[\gamma_1]|F) = f(F[\gamma_2]|F)$;

(c) Let s_1 be a tame corestriction on A relative to $F[\gamma_1]/F$. Then there exists $\delta \in F[\gamma_1]$ such that

$$s_1(\gamma_1 - \gamma_2) \equiv \delta \pmod{\mathfrak{P}^{1-r}}$$

where $\mathfrak{P} = rad(\mathfrak{A})$.
(iii) Let $[\mathfrak{A}, n, r, \beta]$ be a pure stratum in A with $r = -k_0(\beta, \mathfrak{A})$. Let $[\mathfrak{A}, n, r, \gamma]$ be a simple stratum in A which is equivalent to $[\mathfrak{A}, n, r, \beta]$, let s_γ be a tame corestriction on A relative to $F[\gamma]/F$, let B_γ be the A-centralizer of γ, and $\mathfrak{B}_\gamma = \mathfrak{A} \cap B_\gamma$. Then $[\mathfrak{B}_\gamma, r, r-1, s_\gamma(\beta - \gamma)]$ is equivalent to a simple stratum in B_γ.

Let h be a skew-hermitian form.

(5.4.2) Theorem *Let $[\mathfrak{A}, n, n-1, b]$ be a skew non-split fundamental stratum in band form. Suppose also that it is equivalent to some simple stratum. Then it is equivalent to a skew simple stratum.*

Proof: We first choose a basis for V such that b is in standard form. By (5.3.13), each $b_{j,j-n}$ is block diagonal (mod $\mathfrak{p}_F$). Let b' be block diagonal such that $[\mathfrak{A}, n, n-1, b]$ is equivalent to $[\mathfrak{A}, n, n-1, b']$. Then, as in the discussion preceding (5.3.17), we can perturb to make b' minimal. ■

(5.4.3) Corollary *Let $[\mathfrak{A}, n, n-1, b]$ be a skew pure stratum. Then it is equivalent to a skew simple stratum.*

Proof: Let $[\mathfrak{A}, n, n-1, b']$ be a skew equivalent stratum in band form. Then the result follows immediately from (5.4.2). ■

We now return to the case where h is an alternating form, as in chapter 4.

(5.4.4) Proposition *Let $[\mathfrak{A}, n, r, \beta]$ be a simple stratum with β minimal and with $\beta + \overline{\beta} \in \mathfrak{P}^{-r}$. Then it is equivalent to a skew simple stratum $[\mathfrak{A}, n, r, \gamma]$.*

(5.4.5) Remark The element γ given by (5.4.4) will in fact be minimal by (5.4.1)$(ii)(a)$.

Proof: We prove by induction that for $t \geq r$ there exists a skew simple stratum $[\mathfrak{A}, n, t, \gamma_t]$ equivalent to $[\mathfrak{A}, n, t, \beta]$.

The case $t = n-1$ is given by (5.4.2) so assume we have a stratum $[\mathfrak{A}, n, t, \gamma_t]$ as required, $t > r$. We drop the index t. Let $E_\gamma = F[\gamma]$, let B_γ be the A-centralizer of γ, $\mathfrak{B}_\gamma = \mathfrak{A} \cap B_\gamma$, $\mathfrak{Q}_\gamma = \mathfrak{P} \cap B_\gamma$ and let s_γ be a tame corestriction relative to E_γ/F which commutes with the involution.

By (5.4.1)(ii), there exists $\delta \in E_\gamma$ such that $s_\gamma(\beta - \gamma) \equiv \delta \pmod{\mathfrak{Q}_\gamma^{1-t}}$. Now $\gamma + \overline{\gamma} = 0$ and $\beta + \overline{\beta} \in \mathfrak{P}^{-r} \subset \mathfrak{P}^{1-t}$ so $\delta + \overline{\delta} \in \mathfrak{Q}_\gamma^{1-t}$. Let $\epsilon = \frac{1}{2}(\delta + \overline{\delta}) \in \mathfrak{Q}_\gamma^{1-t} \cap E_\gamma$; then $\delta - \epsilon$ is skew. Setting $b = \beta - \gamma$, we have $s_\gamma(b) \equiv \delta - \epsilon \pmod{\mathfrak{Q}_\gamma^{1-t}}$. Then $[\mathfrak{B}_\gamma, t, t-1, s_\gamma(b)]$ is equivalent to $[\mathfrak{B}_\gamma, t, t-1, \delta - \epsilon]$ which is skew simple since $k_0(\delta - \epsilon, \mathfrak{A}) = -\infty$. Then, by (4.2.2), there exists γ_{t-1} such that $\gamma_{t-1} + \overline{\gamma_{t-1}} = 0$ and $[\mathfrak{A}, n, t-1, \gamma_{t-1}]$ is simple and equivalent to $[\mathfrak{A}, n, t-1, \beta]$. ■

(5.4.6) Proposition *Let $[\mathfrak{A}, n, r, \beta]$ be a simple stratum with $k_0(\beta, \mathfrak{A}) = -s$ and $\beta + \overline{\beta} \in \mathfrak{P}^{-r}$. Then there exists an equivalent skew simple stratum $[\mathfrak{A}, n, r, \gamma]$.*

Proof: We proceed by induction on s. The case $s = n$ is (5.4.4) so we assume we have the result for $k_0(\beta, \mathfrak{A}) \leq -(s+1)$.

By (5.4.1)(i), there exists a simple stratum $[\mathfrak{A}, n, s, \beta']$ equivalent to the pure stratum $[\mathfrak{A}, n, s, \beta]$. Further, $\beta' + \overline{\beta'} \in \mathfrak{P}^{-s}$ and $k_0(\beta', \mathfrak{A}) \leq -(s+1)$ so, by the induction hypothesis, there exists $\gamma' \in A_-$ such that $[\mathfrak{A}, n, s, \gamma']$ is simple and equivalent to $[\mathfrak{A}, n, s, \beta']$ and hence to $[\mathfrak{A}, n, s, \beta]$.

Let $E_{\gamma'} = F[\gamma']$, let $B_{\gamma'}$ be the A-centralizer of γ', $\mathfrak{B}_{\gamma'} = \mathfrak{A} \cap B_{\gamma'}$, $\mathfrak{Q}_{\gamma'} = \mathfrak{P} \cap B_{\gamma'}$ and let $s_{\gamma'}$ be a tame corestriction relative to $E_{\gamma'}/F$.

By (5.4.1)(iii), $[\mathfrak{B}_{\gamma'}, s, s-1, s_{\gamma'}(\beta - \gamma')]$ is equivalent to a simple stratum. Let $b = \beta - \gamma'$ so $b + \overline{b} \in \mathfrak{P}^{-r} \subset \mathfrak{P}^{1-s}$. Hence $s_{\gamma'}(b) + \overline{s_{\gamma'}(b)} \in \mathfrak{Q}_{\gamma'}^{1-s}$. So $[\mathfrak{B}_{\gamma'}, s, s-1, s_{\gamma'}(\beta - \gamma')]$ is equivalent to a skew simple stratum by (5.4.2). Then, by (4.2.2), there exists $\gamma_o \in A_-$ such that $[\mathfrak{A}, n, s-1, \gamma_o]$ is simple and equivalent to $[\mathfrak{A}, n, s-1, \gamma' + b] = [\mathfrak{A}, n, s-1, \beta]$.

We now prove, by induction on t, that for $s-1 \geq t \geq r$ there exists a skew simple stratum $[\mathfrak{A}, n, t, \gamma_t]$ equivalent to $[\mathfrak{A}, n, t, \beta]$. This will then complete the induction on s and the proof.

We have just done the case $t = s-1$ and the rest of the proof is identical to that of (5.4.4). ■

(5.4.7) Theorem *Let $[\mathfrak{A}, n, r, \beta]$ be a skew pure stratum in A. Then there exists a skew simple stratum $[\mathfrak{A}, n, r, \gamma]$ equivalent to it.*

Proof: By (5.4.1)(i) there exists a simple stratum $[\mathfrak{A}, n, r, \gamma']$ which is equivalent to $[\mathfrak{A}, n, r, \beta]$. Then $\gamma' + \overline{\gamma'} \equiv 0 \pmod{\mathfrak{P}^{-r}}$ and $k_0(\gamma', \mathfrak{A}) = -s$ for some $s > r$. So, by (5.4.6), there exists a skew simple stratum $[\mathfrak{A}, n, r, \gamma]$ as required. ■

6
SIMPLE CHARACTERS

From this chapter onwards, h is an alternating form, i.e. we consider only the symplectic group. In particular, N is even.

In this chapter we define some orders $\mathfrak{H}(\beta,\mathfrak{A})$, $\mathfrak{J}(\beta,\mathfrak{A})$ in A associated to a skew simple stratum $[\mathfrak{A},n,0,\beta]$ and hence some subgroups $H_-^{m+1}(\beta,\mathfrak{A})$, $J_-^{m+1}(\beta,\mathfrak{A})$ of G. We then define a set $\mathcal{C}_-(\mathfrak{A},m,\beta)$ of characters of $H_-^{m+1}(\beta,\mathfrak{A})$, called *simple characters*. These are defined analogously to the $\mathrm{GL}_N(F)$ case ([**BK**] (3.2)) and, in fact, the simple characters for G will be precisely the restrictions to G of simple characters for the groups $H^{m+1}(\beta,\mathfrak{A})$ in $\mathrm{GL}_N(F)$.

This chapter relies heavily on [**BK**] chapter 3; many results are proved using the analogous results there.

(6.1) The orders $\mathfrak{H}$ and $\mathfrak{J}$

Throughout this section $[\mathfrak{A},n,0,\beta]$ will be a skew simple stratum in A and we set $r=-k_0(\beta,\mathfrak{A})$.

(6.1.1) Definition ([BK] (3.1.7)) *(i) Suppose β is minimal over F; put*

$$\mathfrak{H}(\beta)=\mathfrak{H}(\beta,\mathfrak{A})=\mathfrak{B}_\beta+\mathfrak{P}^{[\frac{n}{2}]+1}.$$

(ii) Suppose that $r<n$ and let $[\mathfrak{A},n,r,\gamma]$ be a skew simple stratum equivalent to $[\mathfrak{A},n,r,\beta]$; put

$$\mathfrak{H}(\beta)=\mathfrak{H}(\beta,\mathfrak{A})=\mathfrak{B}_\beta+\mathfrak{H}(\gamma,\mathfrak{A})\cap\mathfrak{P}^{[\frac{r}{2}]+1}.$$

Note that, by [**BK**] (3.1.9), this inductive definition is independent of the choice of γ such that $[\mathfrak{A},n,r,\gamma]$ is a skew simple stratum equivalent to $[\mathfrak{A},n,r,\beta]$.

(6.1.2) Lemma *Let $[\mathfrak{A},n,0,\beta]$ be a skew simple stratum in A; then we have $\overline{\mathfrak{H}(\beta,\mathfrak{A})}=\mathfrak{H}(\beta,\mathfrak{A})$*

Proof: We proceed by induction along β. If β is minimal then $\mathfrak{H}(\beta)$ is the sum of $\mathfrak{B}_\beta$ and $\mathfrak{P}^{[\frac{n}{2}]+1}$, both of which are invariant under the involution. If $r<n$ and $[\mathfrak{A},n,r,\gamma]$ is a skew simple stratum equivalent to $[\mathfrak{A},n,r,\beta]$ then $\mathfrak{H}(\beta))$ is the sum of $\mathfrak{B}_\beta$ and $\mathfrak{H}(\gamma,\mathfrak{A})\cap\mathfrak{P}^{[\frac{r}{2}]+1}$, both of which are invariant under the involution, the latter by induction. ∎

(6.1.3) Definition ([BK] (3.1.8)) *(i) Suppose β is minimal over F; put*

$$\mathfrak{J}(\beta)=\mathfrak{J}(\beta,\mathfrak{A})=\mathfrak{B}_\beta+\mathfrak{P}^{[\frac{n+1}{2}]}.$$

(ii) Suppose that $r<n$ and let $[\mathfrak{A},n,r,\gamma]$ be a skew simple stratum equivalent to $[\mathfrak{A},n,r,\beta]$; put

$$\mathfrak{J}(\beta)=\mathfrak{J}(\beta,\mathfrak{A})=\mathfrak{B}_\beta+\mathfrak{J}(\gamma,\mathfrak{A})\cap\mathfrak{P}^{[\frac{r+1}{2}]}.$$

Again, this definition is independent of the choice of γ and, exactly as in (6.1.2) we have:

(6.1.4) Lemma *Let $[\mathfrak{A}, n, 0, \beta]$ be a skew simple stratum in A; then we have $\overline{\mathfrak{J}(\beta, \mathfrak{A})} = \mathfrak{J}(\beta, \mathfrak{A})$*

So we can define $\mathfrak{H}(\beta)_- = \mathfrak{H}(\beta) \cap A_-$ and $\mathfrak{J}(\beta)_- = \mathfrak{J}(\beta) \cap A_-$.
We now set

$$\mathfrak{H}^k_-(\beta) = \mathfrak{H}(\beta)_- \cap \mathfrak{P}^k, \quad \mathfrak{J}^k_-(\beta) = \mathfrak{J}(\beta)_- \cap \mathfrak{P}^k, \quad k \geq 0.$$

In particular, as in **[BK]** (3.1), we get that for β minimal over F,

$$\mathfrak{H}^k_-(\beta) = \begin{cases} \mathfrak{Q}^k_{\beta_-} + \mathfrak{P}_-^{[\frac{n}{2}]+1} & \text{for } 0 \leq k \leq [\frac{n}{2}] \\ \mathfrak{P}^k_- & \text{for } k \geq [\frac{n}{2}] + 1, \end{cases}$$

and that in the general case

$$\mathfrak{H}^k_-(\beta) = \begin{cases} \mathfrak{Q}^k_{\beta_-} + \mathfrak{H}_-^{[\frac{r}{2}]+1}(\gamma) & \text{for } 0 \leq k \leq [\frac{r}{2}] \\ \mathfrak{H}^k_-(\gamma) & \text{for } k \geq [\frac{r}{2}] + 1, \end{cases}$$

where $[\mathfrak{A}, n, r, \gamma]$ is a skew simple stratum equivalent to $[\mathfrak{A}, n, r, \beta]$. Similar remarks apply to $\mathfrak{J}$.
Now, as in **[BK]** (3.1.14), we define two families of compact open subgroups of $\mathrm{GL}_N(F)$ by

$$\left.\begin{aligned} H^m(\beta, \mathfrak{A}) &= \mathfrak{H}(\beta, \mathfrak{A}) \cap U^m(\mathfrak{A}) \\ J^m(\beta, \mathfrak{A}) &= \mathfrak{J}(\beta, \mathfrak{A}) \cap U^m(\mathfrak{A}) \end{aligned}\right\} \quad \text{for } m \geq 0.$$

Then we define two families of compact open subgroups of G by

$$\left.\begin{aligned} H^m_-(\beta, \mathfrak{A}) &= H^m(\beta, \mathfrak{A}) \cap G \\ J^m_-(\beta, \mathfrak{A}) &= J^m(\beta, \mathfrak{A}) \cap G \end{aligned}\right\} \quad \text{for } m \geq 0.$$

Then, by (1.2.4) we have a bijection between $\mathfrak{H}^m_-(\beta)$ and $H^m_-(\beta)$ given by the Cayley map.

(6.1.5) Proposition (cf. [BK] (3.1.15)) *(i) For $0 \leq m \leq [\frac{r}{2}] + 1$, we have*

$$H^m_-(\beta) = P^m(\mathfrak{B}_\beta) H_-^{[\frac{r}{2}]+1}(\beta),$$

and, for $0 \leq m \leq [\frac{r+1}{2}]$,

$$J^m_-(\beta) = P^m(\mathfrak{B}_\beta) J_-^{[\frac{r+1}{2}]}(\beta).$$

(ii) For $m \geq 0$, the groups $H^m_-(\beta)$, $J^m_-(\beta)$ are normalized by $\mathfrak{K}(\mathfrak{B}_\beta) \cap G$.
(iii) $J^m_-(\beta) \supset H^m_-(\beta)$ and $H^{m+1}_-(\beta)$ is a normal subgroup of $J_-(\beta)$, for all $m \geq 1$.
(iv) For $k, l \geq 1$, $[J^k_-(\beta), J^l_-(\beta)] \subset H^{k+l}_-(\beta)$.

Proof: By **[BK]** (3.1.15), $H^m_-(\beta) = U^m(\mathfrak{B}_\beta)H^{[\frac{r}{2}]+1}\beta) \cap G$. But $U^m(\mathfrak{B}_\beta) \cap H^{[\frac{r}{2}]+1}(\beta) = 1 + \mathfrak{Q}^m_\beta \cap \mathfrak{H}^{[\frac{r}{2}]+1}(\beta)$ so, by (1.2.7) we have $U^m(\mathfrak{B}_\beta)H^{[\frac{r}{2}]+1}\beta) \cap G = P^m(\mathfrak{B}_\beta)H^{[\frac{r}{2}]+1}(\beta)$. The same argument for J completes (i). Then (ii), (iii) and (iv) follow directly from **[BK]** (3.1.15). ■

(6.1.6) Lemma (cf. [BK] (3.1.19)) *For $m \geq -1$, we have*

$$(\mathfrak{H}^{m+1}_-(\beta))^* = a_\beta(\mathfrak{J}_-^{[\frac{r+1}{2}]}) + \mathfrak{P}_-^{-m}.$$

Proof: By **[BK]** (3.1.19) and (1.2.6). ■

We now choose a tame corestriction s_β on A relative to $F[\beta]/F$.

(6.1.7) Proposition (cf. [BK] (3.1.16)) *For $-1 \leq m \leq r-1$, we have an exact sequence*

$$(\mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta))_- + \mathfrak{J}_-^{[\frac{r+1}{2}]}(\beta) \xrightarrow{a_\beta^-} (\mathfrak{H}^{m+1}_-(\beta))^* \xrightarrow{s_\beta^-} \mathfrak{Q}_{\beta_-}^{-m} \to 0.$$

Proof: By **[BK]** (3.1.16) and (1.2.6). ■

(6.1.8) Corollary *For $0 \leq m \leq [\frac{r}{2}]+1$, we have an exact sequence*

$$0 \to \mathfrak{Q}_{\beta_-}^{[\frac{r+1}{2}]} \to \mathfrak{J}_-^{[\frac{r+1}{2}]}(\beta) \xrightarrow{a_\beta^-} (\mathfrak{H}^{m+1}_-(\beta))^* \xrightarrow{s_\beta^-} \mathfrak{Q}_{\beta_-}^{-m} \to 0.$$

There are similar results for $\mathfrak{J}$ (see **[BK]** (3.1.21), (3.1.22)).

(6.2) Simple characters

(6.2.1) Definition (cf. [BK] (3.2.1)) *Let β be skew and minimal over F, $E = F[\beta]$. For $0 \leq m \leq n-1$ let $\mathcal{C}_-(\mathfrak{A}, m, \beta)$ denote the set of characters θ of $H^{m+1}_-(\beta)$ such that*
(i) $\theta|_{H^{m+1}_-(\beta)\cap P^{[\frac{n}{2}]+1}(\mathfrak{A})} = \psi_\beta$
(ii) $\theta|_{H^{m+1}_-(\beta)\cap(B_\beta\cap G)}$ *factors through* $\det_{B_\beta} : B_\beta \cap G \to N_1(E)$
where $N_1(E) = \{e \in E : e\bar{e} = 1\}$.

(6.2.2) Proposition *In the situation of (6.2.1) we have*
(i) $\mathcal{C}_-(\mathfrak{A}, m, \beta) = \{\psi_\beta\}$ for $[\frac{n}{2}] \leq m \leq n-1$;
(ii) for all $\theta \in \mathcal{C}_-(\mathfrak{A}, m, \beta)$ there exists $\theta' \in \mathcal{C}(\mathfrak{A}, m, \beta)$ such that $\theta'|_{H^{m+1}_-(\beta)} = \theta$;
(iii) every $\theta \in \mathcal{C}_-(\mathfrak{A}, m, \beta)$ is normalized by $\mathfrak{K}(\mathfrak{B}_\beta) \cap G$.

Proof: (i) is clear from the definition and (iii) will follow from (ii) and **[BK]** (3.2.2)(ii).
If $m \geq [\frac{n}{2}]$ then $\theta = \psi_\beta$ which extends to the character $\psi_\beta \in \mathcal{C}(\mathfrak{A}, m, \beta)$.
Suppose then that $m < [\frac{n}{2}]$ and take $\theta \in \mathcal{C}_-(\mathfrak{A}, m, \beta)$; then $\theta|_{H^{m+1}_-(\beta)\cap(B_\beta\cap G)} = \theta|_{P^{m+1}(\mathfrak{B}_\beta)} = \chi \circ \det_{B_\beta}$, for some character χ of the closed subgroup

$\det_{B_\beta}(P^{m+1}(\mathfrak{B}_\beta))$ of $N_1(E)$. By the observation following **[BK]** (3.2.1), the character ψ_β of $U^{[\frac{n}{2}]+1}(\mathfrak{B}_\beta)$ also factors through the determinant, $\psi_\beta = \chi_1 \circ \det_{B_\beta}$ and, moreover, $\chi_1|_{\det P^{[\frac{n}{2}]+1}(\mathfrak{B}_\beta)} = \chi|_{\det P^{[\frac{n}{2}]+1}(\mathfrak{B}_\beta)}$ since $\theta|_{P^{[\frac{n}{2}]+1}(\mathfrak{B}_\beta)} = \psi_\beta$. Then we define a character χ' of $\det_{B_\beta}(P^{m+1}(\mathfrak{B}_\beta)U^{[\frac{n}{2}]+1}(\mathfrak{B}_\beta))$ by

$$\chi'(pu) = \chi(p)\chi_1(u) \qquad p \in \det_{B_\beta}(P^{m+1}(\mathfrak{B}_\beta)), u \in \det_{B_\beta}(U^{[\frac{n}{2}]+1}(\mathfrak{B}_\beta)).$$

Extend this to a character of $E^\times$, also denoted χ'. Now $H_-^{m+1}(\beta) = P^{m+1}(\mathfrak{B}_\beta)$. $P^{[\frac{n}{2}]+1}(\mathfrak{A})$, where the first factor normalizes the second, and further $H^{m+1}(\beta) = U^{m+1}(\mathfrak{B}_\beta).U^{[\frac{n}{2}]+1}(\mathfrak{A})$. So we define θ' by

$$\theta'(uh) = \chi'(\det_{B_\beta}(u)).\psi_\beta(h), \qquad u \in U^{m+1}(\mathfrak{B}_\beta),\ h \in U^{[\frac{n}{2}]+1}(\mathfrak{A}).$$

Since $U^{m+1}(\mathfrak{B}_\beta)$ normalizes ψ_β on $U^{[\frac{n}{2}]+1}(\mathfrak{A})$ by **[BK]** (3.2.2) and $\chi' \circ \det_{B_\beta}$, ψ_β agree on $U^{[\frac{n}{2}]+1}(\mathfrak{B}_\beta)$, this defines a character of $H^{m+1}(\beta)$ and clearly $\theta' \in \mathcal{C}(\mathfrak{A}, m, \beta)$. ∎

(6.2.3) Remark The fibres of the restriction map $\mathcal{C}(\mathfrak{A}, m, \beta) \to \mathcal{C}_-(\mathfrak{A}, m, \beta)$ are of the form $\theta.X$ where $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$ and X is the group of characters of the finite group $U^{m+1}(\mathfrak{B}_\beta)/\big(P^{m+1}(\mathfrak{B}_\beta)U^{[\frac{n}{2}]+1}(\mathfrak{B}_\beta)\big)$ which factor through $\det_{B_\beta}$.

(6.2.4) Definition (cf. [BK] (3.2.3)) *Let $[\mathfrak{A}, n, r, \gamma]$ be a skew simple stratum equivalent to the skew pure stratum $[\mathfrak{A}, n, r, \beta]$. Then, for $0 \le m \le r-1$, let $\mathcal{C}_-(\mathfrak{A}, m, \beta)$ be the set of characters θ of $H_-^{m+1}(\beta)$ such that*
(i) $\theta|_{H_-^{m+1}(\beta)\cap(B_\beta\cap G)}$ factors through $\det_{B_\beta}$;
(ii) θ is normalized by $\mathfrak{K}(\mathfrak{B}_\beta)\cap G$;
(iii) if $m' = \max\{m, [\frac{r}{2}]\}$, the restriction $\theta|_{H_-^{m'+1}(\beta)}$ is of the form $\theta_0\psi_c$ for some $\theta_0 \in \mathcal{C}_-(\mathfrak{A}, m', \gamma)$, $c = \beta - \gamma$.

For $m \ge r$ we set $\mathcal{C}_-(\mathfrak{A}, m, \beta) = \mathcal{C}_-(\mathfrak{A}, m, \gamma)$.

(6.2.5) Proposition (cf. [BK] (3.2.4)) *For $m \ge [\frac{n}{2}]$ we have $\mathcal{C}_-(\mathfrak{A}, m, \beta) = \{\psi_\beta\}$.*

(6.2.6) Proposition *In the situation of (6.2.4), for all $\theta \in \mathcal{C}_-(\mathfrak{A}, m, \beta)$ there exists $\theta' \in \mathcal{C}(\mathfrak{A}, m, \beta)$ such that $\theta'|_{H_-^{m+1}(\beta)} = \theta$.*

Proof: We proceed by induction along β. The case where β is minimal is just (6.2.2)(ii). Let $[\mathfrak{A}, n, r, \gamma]$ be a skew simple stratum equivalent to $[\mathfrak{A}, n, r, \beta]$.
If $m \ge [\frac{r}{2}]$ then $\theta = \theta_0\psi_c$ for some $\theta_0 \in \mathcal{C}_-(\mathfrak{A}, m, \gamma)$. By induction, there exists $\theta'_0 \in \mathcal{C}(\mathfrak{A}, m, \gamma)$ such that $\theta'_0|_{H_-^{m+1}(\gamma)} = \theta'_0|_{H_-^{m+1}(\beta)} = \theta_0$; also ψ_c extends to the character ψ_c of $H^{m+1}(\beta)$. Then, by **[BK]** (3.3.18), $\theta' = \theta'_0\psi_c$ is as required.
If $m < [\frac{r}{2}]$ then $\theta|_{H_-^{m'+1}(\beta)} = \theta_0\psi_c$ for some $\theta_0 \in \mathcal{C}_-(\mathfrak{A}, [\frac{r}{2}], \gamma)$. Then, by induction, there exists $\theta'_0 \in \mathcal{C}(\mathfrak{A}, [\frac{r}{2}], \gamma)$ such that $\theta'_0|_{H_-^{[\frac{r}{2}]+1}(\gamma)} = \theta_0$. Now $H_-^{m+1}(\beta) = P^{m+1}(\mathfrak{B}_\beta).H_-^{[\frac{r}{2}]+1}(\beta)$, with the first factor normalising the second, and also

$H^{m+1}(\beta) = U^{m+1}(\mathfrak{B}_\beta).H^{[\frac{r}{2}]+1}(\beta)$. Then $\theta_{H_-^{m+1}(\beta)\cap(B_\beta\cap G)} = \theta|_{P^{m+1}(\mathfrak{B}_\beta)} = \chi \circ \det_{B_\beta}$ for some character χ of the closed subgroup $\det_{B_\beta}(P^{m+1}(\mathfrak{B}_\beta))$ of $N_1(E)$. Further, $\theta_0'\psi_c \in \mathcal{C}(\mathfrak{A}, [\frac{r}{2}], \beta)$ by **[BK]** (3.3.18) so $\theta_0'\psi_c|_{U^{[\frac{r}{2}]+1}(\mathfrak{B}_\beta)} = \chi_1 \circ \det_{B_\beta}$ for some character χ_1 of $\det_{B_\beta}(U^{[\frac{r}{2}]+1}(\mathfrak{B}_\beta))$. Moreover, we have $\chi_1|_{\det P^{[\frac{r}{2}]+1}(\mathfrak{B}_\beta)} = \chi|_{\det P^{[\frac{r}{2}]+1}(\mathfrak{B}_\beta)}$ since $\theta_0\psi_c|_{P^{[\frac{r}{2}]+1}(\mathfrak{B}_\beta)} = \theta$. Then we define a character χ' of $\det_{B_\beta}(P^{m+1}(\mathfrak{B}_\beta)U^{[\frac{r}{2}]+1}(\mathfrak{B}_\beta))$ by

$$\chi'(pu) = \chi(p)\chi_1(u) \qquad p \in \det_{B_\beta}(P^{m+1}(\mathfrak{B}_\beta)), u \in \det_{B_\beta}(U^{[\frac{r}{2}]+1}(\mathfrak{B}_\beta)).$$

Extend this to a character of $E^\times$, also denoted χ'. Then we define θ' by

$$\theta'(uh) = \chi'(\det_{B_\beta}(u)).\theta_0'(h)\psi_c(h), \qquad u \in U^{m+1}(\mathfrak{B}_\beta),\ h \in H^{[\frac{r}{2}]+1}(\beta).$$

Then $U^{m+1}(\mathfrak{B}_\beta)$ normalizes $\theta_0'\psi_c$ on $H^{[\frac{r}{2}]+1}(\beta) = H^{[\frac{r}{2}]+1}(\gamma)$ since, as above, $\theta_0'\psi_c \in \mathcal{C}(\mathfrak{A}, [\frac{r}{2}], \beta)$ and $\chi' \circ \det_{B_\beta}$, $\theta_0'\psi_c$ agree on $U^{[\frac{n}{2}]+1}(\mathfrak{B}_\beta)$ so this defines a character of $H^{m+1}(\beta)$ and $\theta' \in \mathcal{C}(\mathfrak{A}, m, \beta)$. ∎

(6.2.7) Corollary (cf. [BK] (3.2.5)) *For $0 \le m \le [\frac{r}{2}]$, restriction induces a surjective map*

$$\mathcal{C}_-(\mathfrak{A}, m, \beta) \to \mathcal{C}_-(\mathfrak{A}, [\tfrac{r}{2}], \beta).$$

Note that, as in the proof of **[BK]** (3.2.5), the fibres are of the form $\theta.X$ for $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$ and X is the group of characters of $P^{m+1}(\mathfrak{B}_\beta)/P^{[\frac{r}{2}]+1}(\mathfrak{B}_\beta)$ which factor through $\det_{B_\beta}$.

By (6.2.6) we could have defined $\mathcal{C}_-(\mathfrak{A}, m, \beta)$ to be the set $\mathcal{C}(\mathfrak{A}, m, \beta)$ of characters of $H^{m+1}(\beta)$ restricted to $H_-^{m+1}(\beta)$. Indeed, from now on we will use this description.

(6.2.8) Corollary *In the situation of (6.2.4), $\mathcal{C}_-(\mathfrak{A}, m, \beta)$ is independent of the choice of the element γ.*

Proof: By **[BK]**(3.2.20)(i), $\mathcal{C}(\mathfrak{A}, m, \beta)$ is independent of the choice of γ and $\mathcal{C}_-(\mathfrak{A}, m, \beta) = \{\theta|H_-^{m+1}(\beta) : \theta \in \mathcal{C}(\mathfrak{A}, m, \beta)\}$. ∎

We now generalize (6.2.3) to describe the fibres of the restriction map $\mathcal{C}(\mathfrak{A}, m, \beta) \to \mathcal{C}_-(\mathfrak{A}, m, \beta)$ in all cases. First we give a result concerning the surjectivity of the determinant map.

We put $P(\mathfrak{o}_E) = N_1(E) \subset U(\mathfrak{o}_E)$, $P^n(\mathfrak{o}_E) = U^n(\mathfrak{o}_E) \cap N_1(E)$ for $n \ge 1$.

(6.2.9) Lemma *For $n \ge 0$, we have $\det_{B/E} P^{n+1}(\mathfrak{B}) = P^{[\frac{n}{e}]+1}(\mathfrak{o}_E)$.*

Proof: We have $\det_{B/E} U^{n+1}(\mathfrak{B}) = U^{[\frac{n}{e}]+1}(\mathfrak{o}_E)$ by **[BF2]**(2.8.3). Then certainly $\det_{B/E} P^{n+1}(\mathfrak{B}) \subset P^{[\frac{n}{e}]+1}(\mathfrak{o}_E)$ since $\det_{B/E}$ commutes with the involution $^-$. Suppose now we have $X \in P^{[\frac{n}{e}]+1}(\mathfrak{o}_E)$; then, by (1.2.4)$(i)$, $X = U\overline{U}^{-1}$ for some $U \in U^{[\frac{n}{e}]+1}(\mathfrak{o}_E)$. But we know $U = \det_{B/E} u$ for some $u \in U^{n+1}(\mathfrak{B})$. Then, putting $x = u\overline{u}^{-1}$, we have $x \in P^{n+1}(\mathfrak{B})$ and $\det_{B/E} x = X$ as required. ∎

(6.2.10) Proposition *Let* $[\mathfrak{A}, n, 0, \beta]$ *be a skew simple stratum with* $r = -k_0(\beta, \mathfrak{A})$ *and let* $[\mathfrak{A}, n, r, \gamma]$ *be a skew simple stratum equivalent to* $[\mathfrak{A}, n, r, \beta]$. *Let* $0 \leq m \leq r - 1$. *Then restriction induces a surjective map* $\mathcal{C}(\mathfrak{A}, m, \beta) \to \mathcal{C}_-(\mathfrak{A}, m, \beta)$. *If* $m \geq [\frac{r}{2}]$ *then the fibres are in bijection with the fibres of* $\mathcal{C}(\mathfrak{A}, m, \gamma) \to \mathcal{C}_-(\mathfrak{A}, m, \gamma)$; *if* $m < [\frac{r}{2}]$ *then the fibres are of the form* $\Theta.X \circ \det_{B_\beta/E_\beta}$, *where* X *is the group of characters of* $U^{[\frac{m}{e}]+1}(\mathfrak{o}_{E_\beta})/\big(P^{[\frac{m}{e}]+1}(\mathfrak{o}_{E_\beta})U^{[\frac{[\frac{r}{2}]}{e}]+1}(\mathfrak{o}_{E_\beta})\big)$ *and* Θ *is a fibre of* $\mathcal{C}(\mathfrak{A}, [\frac{r}{2}], \gamma) \to \mathcal{C}_-(\mathfrak{A}, [\frac{r}{2}], \gamma)$.

Proof: As usual, we will work by induction along β. For β minimal this follows from (6.2.3) and (6.2.8).

First we look at the case $m \geq [\frac{r}{2}]$. Let $\theta' \in \mathcal{C}(\mathfrak{A}, m, \beta)$; then $\theta' = \theta_0'\psi_c$, for $\theta_0' \in \mathcal{C}(\mathfrak{A}, m, \gamma)$, $c = \beta - \gamma$ and the result is clear.

Now suppose $m < [\frac{r}{2}]$ and let $\theta' \in \mathcal{C}(\mathfrak{A}, m, \beta)$; then $\theta'|_{H^{[\frac{r}{2}]+1}(\beta)} = \theta_0'\psi_c$, for $\theta_0' \in \mathcal{C}(\mathfrak{A}, [\frac{r}{2}], \gamma)$, $c = \beta - \gamma$. Then the result follows from (6.2.3) and the case $m = [\frac{r}{2}]$. ∎

We recall here a result of Glauberman (**[G]** or see **[BH2]** §A2). Let M be a finite group and let Γ be a subgroup of $\mathrm{Aut}M$ such that $|M|$, $|\Gamma|$ are relatively prime. We write ΓM for the semi-direct product $\Gamma \ltimes M$. The group Γ acts on the set $\mathrm{Irr}(M)$ of equivalence classes of irreducible representations of M; we denote the set of fixed points by $\mathrm{Irr}(M)^\Gamma$.

Let $\rho \in \mathrm{Irr}(M)^\Gamma$. Then, by **[G]** Theorem 1, there exists a unique (up to equivalence) representation $\widetilde{\rho}$ of ΓM such that $\widetilde{\rho}|_M \simeq \rho$ and $\det\widetilde{\rho}(\gamma) = 1$ for all $\gamma \in \Gamma$.

(6.2.11) Proposition ([G] Theorem 3) *Suppose, with notations as above, that the group* Γ *is cyclic. There is a canonical bijection*

$$\boldsymbol{g}_\Gamma : \mathrm{Irr}(M)^\Gamma \xrightarrow{\simeq} \mathrm{Irr}(M^\Gamma),$$

where M^Γ *is the centralizer of* Γ *in* M. *Explicitly, for* $\rho \in \mathrm{Irr}(M)^\Gamma$, *the representation* $\zeta = \boldsymbol{g}_\Gamma(\rho)$ *is given as follows. There is a sign* $\epsilon = \epsilon(\rho, \Gamma)$ *such that*

$$\mathrm{tr}\,\zeta(x) = \epsilon \mathrm{tr}\,\widetilde{\rho}(\gamma x),$$

for all generators γ *of* Γ *and* $x \in M^\Gamma$, *where* $\widetilde{\rho}$ *is the extension of* ρ *to* ΓM *as above.*

We now apply this to our situation. Let $[\mathfrak{A}, n, 0, \beta]$ be a skew simple stratum. All simple characters $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$ are trivial on the group $U^{n+1}(\mathfrak{A})$ so they are, essentially, characters of the finite p-group $H^{m+1}(\beta, \mathfrak{A})/U^{n+1}(\mathfrak{A})$.

Let $\sigma : A^\times \to A^\times$ be the involution given by $\sigma(x) = \overline{x}^{-1}$. Then σ acts on the group $H^{m+1}(\beta, \mathfrak{A})/U^{n+1}(\mathfrak{A})$, since both groups in this quotient are fixed by the involution $\bar{\ }$. We put $\Gamma = \langle\sigma\rangle$, a cyclic group of order 2. Then we have $H^{m+1}(\beta, \mathfrak{A})^\Gamma = H_-^{m+1}(\beta, \mathfrak{A})$ and $U^{n+1}(\mathfrak{A})^\Gamma = P^{n+1}(\mathfrak{A})$; in particular, we have $(H^{m+1}(\beta, \mathfrak{A})/U^{n+1}(\mathfrak{A}))^\Gamma = H_-^{m+1}(\beta, \mathfrak{A})/P^{n+1}(\mathfrak{A})$.

Since p is not 2, we apply Glauberman's correspondence (6.2.11) to the groups $M = H^{m+1}(\beta, \mathfrak{A})/U^{n+1}(\mathfrak{A})$, Γ; this gives us a bijection between those equivalence classes of representations of $H^{m+1}(\beta, \mathfrak{A})/U^{n+1}(\mathfrak{A})$ which are stable by σ and the equivalence classes of irreducible representations of $H_-^{m+1}(\beta, \mathfrak{A})/P^{n+1}(\mathfrak{A})$. Moreover, the relationship between the characters of these representations implies that we have a bijection

$$\mathcal{C}(\mathfrak{A}, m, \beta)^\Gamma \xrightarrow{\simeq} \mathcal{C}_-(\mathfrak{A}, m, \beta), \tag{6.2.12}$$

given by restriction of characters. (Note that the character of $H^{m+1}(\beta, \mathfrak{A})$ corresponding to a given simple character of $H_-^{m+1}(\beta, \mathfrak{A})$ is indeed a simple character, since we can construct it explicitly as in (6.2.2)(ii),(6.2.6).)

(6.3) Intertwining

(6.3.1) Proposition *Let $\theta \in \mathcal{C}_-(\mathfrak{A}, m, \beta)$, $0 \le m \le r-1$ and let $j \in J_-(\beta)$. Then $\theta(jhj^{-1}) = \theta(h)$ for all $h \in H_-^{m+1}(\beta)$.*

Proof: **[BK]** (3.3.1). ∎

(6.3.2) Theorem (cf. [BK] (3.3.2)) *Let $[\mathfrak{A}, n, 0, \beta]$ be a skew simple stratum in A, $r = -k_0(\beta, \mathfrak{A})$. Let $0 \le m \le r-1$ and $\theta \in \mathcal{C}_-(\mathfrak{A}, m, \beta)$. Then*

$$\begin{aligned} &I_G(\theta | H_-^{m+1}(\beta)) \\ &= (1 + \mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta) + \mathfrak{J}^{[\frac{r+1}{2}]}(\beta)) \cap G \ . \ B_\beta \cap G \ . \ (1 + \mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta) + \mathfrak{J}^{[\frac{r+1}{2}]}(\beta)) \cap G \end{aligned}$$

(6.3.3) Remark We can write

$$\begin{aligned} (1 + \mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta) + \mathfrak{J}^{[\frac{r+1}{2}]}(\beta)) \cap G &= \big((1 + \mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta)) \ . \ J^{[\frac{r+1}{2}]}(\beta)\big) \cap G \\ &= Q_{r-m} \ . \ J_-^{[\frac{r+1}{2}]}. \end{aligned}$$

Moreover, when $m \le [\frac{r}{2}]$, $\mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta) \subset \mathfrak{J}^{[\frac{r+1}{2}]}$ so the Theorem says

$$I_G(\theta | H_-^{m+1}(\beta)) = J_-^{[\frac{r+1}{2}]}(\beta) \ . \ B_\beta \cap G \ . \ J_-^{[\frac{r+1}{2}]}(\beta)$$

Proof: We proceed by induction along β, following the proof of **[BK]** (3.3.2). The case $r = \infty$ is trivial so we begin with $r = n$. Then β is minimal so $\mathfrak{J}^{[\frac{n+1}{2}]} = \mathfrak{P}^{[\frac{n+1}{2}]}$ and $\mathfrak{Q}_\beta^{n-m}\mathfrak{N}(\beta) = \mathfrak{P}^{n-m}$. Let $m' = \max\{m, [\frac{n}{2}]\}$; so $(1+\mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta)+\mathfrak{J}^{[\frac{r+1}{2}]}(\beta)) \cap G = P^{n-m'}(\mathfrak{A})$. Then

$$\begin{aligned} I_G(\theta | H_-^{m+1}(\beta)) \subset I_G(\theta | P^{m'+1}(\mathfrak{A})) &= \mathcal{I}_G[\mathfrak{A}, n, m', \beta] \\ &= P^{n-m'}(\mathfrak{A}) \ . \ B_\beta \cap G \ . \ P^{n-m'}(\mathfrak{A}) \end{aligned}$$

by (2.2.3). But we also have

$$\begin{aligned} I_G(\theta | H_-^{m+1}(\beta)) &\supset I_{GL_{2N}}(\theta | H^{m+1}(\beta)) \cap G \\ &= U^{n-m'}(\mathfrak{A}) B_\beta^\times U^{n-m'}(\mathfrak{A}), \qquad \text{by } \textbf{[BK]} \text{ (3.3.2)}, \\ &= P^{n-m'}(\mathfrak{A}) \ . \ B_\beta \cap G \ . \ P^{n-m'}(\mathfrak{A}), \qquad \text{by (2.2.3)}, \end{aligned}$$

so we in fact have the equality required.

Now we look at the general case where $r < n$. Let $[\mathfrak{A}, n, r, \gamma]$ be a skew simple stratum equivalent to $[\mathfrak{A}, n, r, \beta]$ and put $s = -k_0(\gamma, \mathfrak{A})$. We look first at the case $m \geq [\frac{r}{2}]$. Thus $H_-^{m+1}(\beta) = H_-^{m+1}(\gamma)$ and, for $\theta \in \mathcal{C}_-(\mathfrak{A}, m, \beta)$, we have $I_G(\theta) \subset I_G(\theta | H_-^{r+1}(\beta))$. Then, as in **[BK]** (3.3.2), the restriction of θ to $H_-^{r+1}(\beta) = H_-^{r+1}(\gamma)$ lies in $\mathcal{C}(\mathfrak{A}, r, \gamma)$. Put $I^r(\gamma) = I_G(\theta | H^{r+1})$ so, by induction,

$$I^r(\gamma) = (1 + \mathfrak{Q}_\gamma^{s-r}\mathfrak{N}(\gamma) + \mathfrak{J}^{[\frac{s+1}{2}]}(\gamma)) \cap G \ . \ B_\gamma \cap G \ . \ (1 + \mathfrak{Q}_\gamma^{s-r}\mathfrak{N}(\gamma) + \mathfrak{J}^{[\frac{s+1}{2}]}(\gamma)) \cap G$$

We define another set

$$I_m^-(\beta) = \{x \in G : x^{-1}(\beta + \mathfrak{H}_-^{m+1}(\beta)^*)x \cap (\beta + \mathfrak{H}_-^{m+1}(\beta)^*) \neq \emptyset\},$$

i.e. the formal intertwining of the coset $\beta + \mathfrak{H}_-^{m+1}(\beta)^*$. Note that this is just $I_m^+(\beta) \cap G$, where $I_m^+(\beta)$ is the formal GL_N-intertwining of the coset $\beta + \mathfrak{H}^{m+1}(\beta)^*$, as in **[BK]**.

(6.3.4) Lemma (cf. [BK] (3.3.5)) *$I_G(\theta) = I_m^-(\beta) \cap I^r(\gamma)$.*

Proof: We first show that

$$J_-^{[\frac{r+1}{2}]}(\beta) \ . \ I_m^-(\beta) \ . \ J_-^{[\frac{r+1}{2}]}(\beta) = I_m^-(\beta).$$

We clearly have the containment $\supset$. But

$$\begin{aligned} J_-^{[\frac{r+1}{2}]}(\beta) \ . \ I_m^-(\beta) \ . \ J_-^{[\frac{r+1}{2}]}(\beta) &\subset J^{[\frac{r+1}{2}]}(\beta) \ . \ I_m^+(\beta) \ . \ J^{[\frac{r+1}{2}]}(\beta) \ \cap G \\ &= I_m^+(\beta) \cap G, \qquad \text{by } \textbf{[BK]} \text{ (3.3.6)}, \\ &= I_m^-(\beta). \end{aligned}$$

We now take $x \in I^r(\gamma)$; then, as all the sets are bi-invariant by $J_-^{[\frac{r+1}{2}]}(\beta)$, we may assume that $x \in (1 + \mathfrak{Q}_\gamma^{s-r}\mathfrak{N}(\gamma)) \cap G \ . \ B_\gamma \cap G \ . \ (1 + \mathfrak{Q}_\gamma^{s-r}\mathfrak{N}(\gamma)) \cap G$. Let $h \in x^{-1}H_-^{m+1}(\beta)x \cap H_-^{m+1}(\beta)$; then, as in **[BK]** (3.3.5),

$$\theta^x(h)\theta^{-1}(h) = \psi_{x^{-1}\beta x - \beta}(h).$$

So x intertwines θ if and only if $\psi_{x^{-1}\beta x-\beta}(h) = 1$ for all $h \in x^{-1}H_-^{m+1}(\beta)x \cap H_-^{m+1}(\beta)$, i.e. if and only if x intertwines $\psi_\beta | H_-^{m+1}(\beta)$. But this is if and only if $x^{-1}\beta x - \beta \in (x^{-1}\mathfrak{H}_-^{m+1}(\beta)x \cap \mathfrak{H}_-^{m+1}(\beta))^* = x^{-1}(\mathfrak{H}_-^{m+1})^*x + (\mathfrak{H}_-^{m+1})^*$. i.e $x \in I_m^-(\beta)$. ∎

Now, as in **[BK]** (3.3.10), we have

$$I_m^-(\beta) \supset (1 + \mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta) + \mathfrak{J}^{[\frac{r+1}{2}]}(\beta)) \cap G \ . \ B_\beta \cap G \ . \ (1 + \mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta) + \mathfrak{J}^{[\frac{r+1}{2}]}(\beta)) \cap G.$$

Then, for $x \in I_m^-(\beta)$, we have

$$x^{-1}(\beta + (\mathfrak{H}_-^{m+1})^*)x \cap (\beta + (\mathfrak{H}_-^{m+1})^*) \neq \emptyset$$

so there exist $\delta_1, \delta_2 \in (\mathfrak{H}_-^{m+1})^*$ such that

$$\textbf{(6.3.5)} \qquad x^{-1}(\beta + \delta_1 + \mathfrak{P}_-^{-m})x \cap (\beta + \delta_2 + \mathfrak{P}_-^{-m}) \neq \emptyset.$$

(6.3.6) Lemma (cf. [BK] (3.3.12)) *Let $\delta \in (\mathfrak{H}_-^{m+1}(\beta))^*$. Then there exists $y \in \mathfrak{J}_-^{[\frac{r+1}{2}]}(\beta)$ such that*

$$C(y)^{-1}(\beta + \delta + \mathfrak{P}_-^{-m})C(y) = \beta + \mathfrak{P}_-^{-m}.$$

Proof: We show that for $k \in \mathbb{Z}$, $\delta \in (\mathfrak{H}_-^{m+1}(\beta)^* \cap \mathfrak{P}_-^k) + \mathfrak{P}_-^{-m}$, there exists $y \in \mathfrak{J}_-^{[\frac{r+1}{2}]}(\beta)$ such that

$$C(y)^{-1}(\beta + \delta)C(y) \equiv \beta \pmod{(\mathfrak{H}_-^{m+1}(\beta)^* \cap \mathfrak{P}_-^{k+1}) + \mathfrak{P}_-^{-m}}$$

and the result will follow by induction. Let $\delta \in (\mathfrak{H}_-^{m+1}(\beta)^* \cap \mathfrak{P}_-^k) + \mathfrak{P}_-^{-m}$. By (6.1.6), there exists $y \in \mathfrak{J}_-^{[\frac{r+1}{2}]}(\beta)$ such that $\delta + a_\beta(y) \in \mathfrak{P}_-^{-m}$; then $a_\beta(y) \in (\mathfrak{H}_-^{m+1}(\beta)^* \cap \mathfrak{P}_-^k) + \mathfrak{P}_-^{-m}$. We have $C(y) = 1 + y'$ for some $y' \in \mathfrak{J}^{[\frac{r+1}{2}]}$ so, as in the proof of **[BK]** (3.3.12),

$$C(y)^{-1}\delta C(y) \equiv \delta \pmod{(\mathfrak{H}^{m+1}(\beta)^* \cap \mathfrak{P}^{k+1}) + \mathfrak{P}^{-m}}.$$

Further $C(y)^{-1}\beta C(y) = \beta + (1+y')^{-1}a_\beta(y')$. Now y' is given by a power series in y and, for $n \geq 2$, $a_\beta(y^n) = \sum_{i=0}^n y^i a_\beta(y) y^{n-i} \in \mathfrak{J}^{[\frac{r+1}{2}]}(\mathfrak{H}^{m+1}(\beta)^* \cap \mathfrak{P}^k) + \mathfrak{P}^{-m}) + (\mathfrak{H}^{m+1}(\beta)^* \cap \mathfrak{P}^k) + \mathfrak{P}^{-m})\mathfrak{J}^{[\frac{r+1}{2}]} \subset \mathfrak{H}^{m+1}(\beta)^* \cap \mathfrak{P}^{k+1} + \mathfrak{P}^{-m}$. Hence $a_\beta(y') \equiv a_\beta(y) \pmod{(\mathfrak{H}^{m+1}(\beta)^* \cap \mathfrak{P}^{k+1}) + \mathfrak{P}^{-m}}$. Then $(1+y')^{-1} = 1 + y''$, for some $y'' \in \mathfrak{J}^{[\frac{r+1}{2}]}$, so

$$(1+y')^{-1}a_\beta(y') \equiv a_\beta(y) + y''a_\beta(y) \equiv a_\beta(y) \pmod{(\mathfrak{H}^{m+1}(\beta)^* \cap \mathfrak{P}^{k+1}) + \mathfrak{P}^{-m}}.$$

Then altogether we have

$$\begin{aligned} C(y)^{-1}(\beta + \delta)C(y) &\equiv \beta + \delta + a_\beta(y) \pmod{(\mathfrak{H}^{m+1}(\beta)^* \cap \mathfrak{P}^{k+1}) + \mathfrak{P}^{-m}} \\ &\equiv \beta \pmod{(\mathfrak{H}^{m+1}(\beta)^* \cap \mathfrak{P}^{k+1}) + \mathfrak{P}^{-m}}. \end{aligned}$$

Then, since the elements on both sides of this congruence are skew, we have the assertion. ∎

Then we can write in (6.3.5)

$$\beta + \delta_i + \mathfrak{P}_-^{-m} = C(y_i)^{-1}(\beta + \mathfrak{P}_-^{-m})C(y_i), \qquad i = 1, 2,$$

for some $y_1, y_2 \in \mathfrak{J}_-^{[\frac{r+1}{2}]}(\beta)$. Thus, if $z = C(y_1)xC(y_2)^{-1}$, we have

$$z^{-1}(\beta + \mathfrak{P}_-^{-m})z \cap (\beta + \mathfrak{P}_-^{-m}) \neq \emptyset.$$

Then, by (2.2.3), $z \in (1+\mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta)) \cap G \ . \ B_\beta \cap G \ . \ (1+\mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta)) \cap G$ and so x lies in

$$(1+\mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta)+\mathfrak{J}^{[\frac{r+1}{2}]}(\beta)) \cap G \ . \ B_\beta \cap G \ . \ (1+\mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta)+\mathfrak{J}^{[\frac{r+1}{2}]}(\beta)) \cap G$$

Finally we prove

$$I_m^-(\beta) \subset I^r(\gamma)$$

and this, together with (6.3.4), will conclude the proof of the theorem in the case $m \geq [\frac{r}{2}]$. Since $I^r(\gamma)$ is bi-invariant under multiplication by $J_-^{[\frac{r+1}{2}]}(\beta) = J_-^{[\frac{r+1}{2}]}(\gamma)$ it is enough to show that

$$I^r(\gamma) \supset (1+\mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta)) \cap G \ . \ B_\beta \cap G \ . \ (1+\mathfrak{Q}_\beta^{r-m}\mathfrak{N}(\beta)) \cap G.$$

This is just the formal intertwining of $\beta+\mathfrak{P}_-^{-m}$ so it is contained in the formal intertwining of $\beta+\mathfrak{P}_-^{-r} = \gamma+\mathfrak{P}_-^{-r}$, which is $(1+\mathfrak{Q}_\gamma^{s-r}\mathfrak{N}(\gamma)) \cap G \ . \ B_\gamma \cap G \ . \ (1+\mathfrak{Q}_\gamma^{s-r}\mathfrak{N}(\gamma)) \cap G$ and this is certainly contained in $I^r(\gamma)$.

Now we assume that $m \leq [\frac{r}{2}]$. Note that

$$\mathfrak{Q}_\beta^{r-[\frac{r}{2}]}\mathfrak{N}(\beta) = \mathfrak{Q}_\beta^{[\frac{r+1}{2}]}\mathfrak{N}(\beta) \subset \mathfrak{J}^{[\frac{r+1}{2}]}(\beta)$$

by **[BK]** (3.1.10) so the assertion is

$$I_G(\theta) = J_-^{[\frac{r+1}{2}]}(\beta) \ . \ B_\beta \cap G \ . \ J_-^{[\frac{r+1}{2}]}(\beta), \qquad m < [\tfrac{r}{2}].$$

We certainly have

$$I_G(\theta) \subset I_G(\theta|H_-^{[\frac{r}{2}]+1}) = J_-^{[\frac{r+1}{2}]}(\beta) \ . \ B_\beta \cap G \ . \ J_-^{[\frac{r+1}{2}]}(\beta),$$

by the case $m = [\frac{r}{2}]$ above. But also

$$\begin{aligned} I_G(\theta|H_-^{m+1}) \supset I_{GL_{2N}}(\theta|H^{m+1}) \cap G &= J^{[\frac{r+1}{2}]}(\beta) B_\beta^\times J^{[\frac{r+1}{2}]}(\beta) \cap G \\ &\supset J_-^{[\frac{r+1}{2}]}(\beta) \ . \ B_\beta \cap G \ . \ J_-^{[\frac{r+1}{2}]}(\beta) \end{aligned}$$

So we in fact have equality and we have completed the proof of (6.3.2). ■

(6.3.7) Corollary (of the proof) *In the situation of (6.3.2) let* $0 \leq m \leq [\frac{r}{2}]$. *Then*

$$J^m(\beta).B_\beta^\times.J^m(\beta) \ \cap G = J_-^m(\beta) \ . \ B_\beta \cap G \ . \ J_-^m(\beta).$$

(6.4) Heisenberg Representations

(6.4.1) Theorem (cf. [BK] (3.4.1)) *Let* $1 \leq m \leq r$ *and let* $\theta \in \mathcal{C}_-(\mathfrak{A}, m-1, \beta)$. *The pairing*

$$k_\theta : (j,k) \mapsto \theta[j,k] \qquad j,k \in J_-^m(\beta)$$

induces a nondegenerate alternating bilinear form

$$J_-^m(\beta)/H_-^m(\beta) \times J_-^m(\beta)/H_-^m(\beta) \to \mathbb{C}^\times.$$

Proof: We have $[J^m_-(\beta), J^m_-(\beta)] \subset H^{2m}_-(\beta) \subset H^m_-(\beta)$. Moreover, $J^m_-(\beta)$ normalizes θ so $\ker(\theta)$ is a normal subgroup of $J^m_-(\beta)$ and $H^m_-(\beta)/\ker(\theta)$ is central in $J^m_-(\beta)/\ker(\theta)$. So k_θ defines an alternating bilinear form on $J^m_-(\beta)/H^m_-(\beta)$. For nondegeneracy, we need

$$\theta[j,k] = 1 \qquad \text{for all } k \in J^m_-(\beta) \quad \text{if and only if} \quad h \in H^m_-(\beta).$$

The implication $\Leftarrow$ is immediate.

We deal first with the case $m = [\frac{r+1}{2}]$. Let $j, k \in J_-^{[\frac{r+1}{2}]}(\beta)$, so $j = C(x) = 1 + x'$, $k = C(y) = 1 + y'$ for some $x, y \in \mathfrak{J}_-^{[\frac{r+1}{2}]}(\beta)$ and $x', y' \in \mathfrak{J}^{[\frac{r+1}{2}]}(\beta)$. Then, by **[BK]** (3.2.12),

$$\theta[j,k] = \psi_{(1+x')^{-1}\beta(1+x')-\beta}(1+y').$$

We also have $(1+x')^{-1}\beta(1+x') - \beta = a_\beta(x') - (1+x')^{-1}x' a_\beta(x')$, and $x' a_\beta(x') \in \mathfrak{J}^{[\frac{r+1}{2}]}(\mathfrak{H}^{[\frac{r}{2}]+1})^*$ by **[BK]** (3.1.17) and this is contained in $(\mathfrak{J}^{[\frac{r+1}{2}]})^*$. Also, as in the proof of (6.3.6), $a_\beta(x') - a_\beta(x) \in \mathfrak{J}^{[\frac{r+1}{2}]}(\mathfrak{H}^{[\frac{r}{2}]+1})^* + (\mathfrak{H}^{[\frac{r}{2}]+1})^* \mathfrak{J}^{[\frac{r+1}{2}]} \subset (\mathfrak{J}^{[\frac{r+1}{2}]})^*$. Altogether, we have

$$(1+x')^{-1}\beta(1+x') - \beta \equiv a_\beta(x) \pmod{(\mathfrak{J}^{[\frac{r+1}{2}]})^*}$$

and, since both sides are skew, this is in fact $\pmod{(\mathfrak{J}_-^{[\frac{r+1}{2}]})^*}$. So $\theta[C(x), C(y)] = 1$, for all $y \in \mathfrak{J}^{[\frac{r+1}{2}]}$, if and only if $a_\beta(x) \in (\mathfrak{J}^{[\frac{r+1}{2}]})^*$, which, by **[BK]** (3.1.22), is if and only if $x \in (B_\beta + \mathfrak{H}^{[\frac{r}{2}]+1}) \cap \mathfrak{J}^{[\frac{r+1}{2}]} \cap A_- = \mathfrak{H}_-^{[\frac{r+1}{2}]}$.

Now suppose $m < [\frac{r+1}{2}]$ so $J^m_-(\beta) = P^m(\mathfrak{B}_\beta).J_-^{[\frac{r+1}{2}]}(\beta)$. Since $P^m(\mathfrak{B}_\beta)$ normalizes θ, the commutator subgroups $[P^m(\mathfrak{B}_\beta), P^m(\mathfrak{B}_\beta)]$ and $[P^m(\mathfrak{B}_\beta), J_-^{[\frac{r+1}{2}]}(\beta)]$ are both contained in $\ker(\theta)$. Take $j = uj' \in J^m_-(\beta)$ with $u \in P^m(\mathfrak{B}_\beta)$, $j' \in J_-^{[\frac{r+1}{2}]}$. Then $\theta[j, J^m_-] = 1$ if and only if $\theta[j', J^m_-] = 1$ which, by the first part, implies $j' \in H_-^{[\frac{r+1}{2}]}$. Then $j \in P^m(\mathfrak{B}_\beta).H_-^{[\frac{r+1}{2}]} = H^m_-(\beta)$.

Now suppose $m > [\frac{r+1}{2}]$. If β is minimal over F, this means $m \geq [\frac{r}{2}]+1$ and $J^m_- = H^m_- = P^m(\mathfrak{A})$ so the assertion is trivial. Otherwise, let $[\mathfrak{A}, n, r, \gamma]$ be a skew simple stratum equivalent to $[\mathfrak{A}, n, r, \beta]$. We have $[J^m_-(\beta), J^m_-(\beta)] \subset H^{2m}_-(\beta)$ by (6.1.5) and $H^{2m}_-(\beta) = H^{2m}_-(\gamma)$. Moreover, $2m \geq r+1$ so $\theta | H^{2m}_-(\beta) \in \mathcal{C}_-(\mathfrak{A}, 2m-1, \beta)$. The result now follows by induction along β, since $H^m_-(\gamma) = H^m_-(\gamma)$. ■

(6.4.2) Proposition (cf. [BK] (5.1.1)) *Let $[\mathfrak{A}, n, 0, \beta]$ be a skew simple stratum in A, $\theta \in \mathcal{C}_-(\mathfrak{A}, 0, \beta)$. There exists a unique irreducible representation $\eta(\theta)$ of $J^1_-(\beta, \mathfrak{A})$ such that $\eta(\theta)|H^1_-(\beta, \mathfrak{A})$ contains θ. Moreover, $\eta(\theta)|H^1_-(\beta, \mathfrak{A})$ is a multiple of θ and*

$$\dim(\eta(\theta)) = (J^1_-(\beta, \mathfrak{A}) : H^1_-(\beta, \mathfrak{A}))^{\frac{1}{2}}.$$

The G-intertwining of $\eta(\theta)$ is $J^1_-(\beta, \mathfrak{A}).B \cap G.J^1_-(\beta, \mathfrak{A})$.

Proof: Given (6.4.1), the proof is identical to that of **[BK]** (5.1.1). ■

7
A SPECIAL CASE

In this chapter we look at the case where the element β in our skew simple stratum $[\mathfrak{A}, n, 0, \beta]$ generates a maximal field extension $E = F[\beta]$ over F in A. In this case we are able to complete the construction of the type and construct supercuspidal representations of G. In the case where E/F is wildly ramified, these are new supercuspidals.

(7.1) Construction of types

Let $[\mathfrak{A}, n, 0, \beta]$ be a skew simple stratum in A such that $E = F[\beta]$ is a maximal field extension of F in A. Then the centralizer B of E in A is just the field E itself and $B \cap G = P(\mathfrak{o}_E) = N_1(E)$ is the group of norm-1 elements of E (for the norm N_{E/E_0}, where E_0 is the fixed field of the involution) and $\mathfrak{B} = \mathfrak{A} \cap B = \mathfrak{o}_E$.

Put $r = -k_0(\beta, \mathfrak{A})$ and let $[\mathfrak{A}, n, r, \gamma]$ be a skew simple stratum equivalent to $[\mathfrak{A}, n, r, \beta]$. Then we have

$$H^1_-(\beta) = P^1(\mathfrak{o}_E) H_-^{[\frac{r}{2}]+1}(\gamma);$$
$$J^1_-(\beta) = P^1(\mathfrak{o}_E) J_-^{[\frac{r+1}{2}]}(\gamma);$$
$$J_-(\beta) = P(\mathfrak{o}_E) J_-^{[\frac{r+1}{2}]}(\gamma).$$

In particular, $J_-(\beta) = P(\mathfrak{o}_E) J^1_-(\beta)$ and $J_-(\beta)/J^1_-(\beta) \simeq P(\mathfrak{o}_E)/P^1(\mathfrak{o}_E)$ is a finite cyclic group. (It is isomorphic to $P(k_E) = \{x \in k_E^\times : x\overline{x} = 1\}$, where $^-$ is the involution on k_E induced by the involution on $\mathfrak{o}_E$. If E/E_0 is ramified, this is just $\mathbb{Z}/2\mathbb{Z}$; if E/E_0 is unramified, this is $\mathbb{Z}/(q_0+1)\mathbb{Z}$, where $q_0 = \# k_{E_0}$.)

Let $\theta \in \mathcal{C}_-(\mathfrak{A}, 0, \beta)$ be a simple character and let η be the Heisenberg representation of $J^1_-(\beta)$ containing θ given by (6.4.2). Hence the intertwining of η is $\mathcal{I}_G(\eta) = J^1_-(\beta).B \cap G.J^1_-(\beta) = P(\mathfrak{o}_E) J^1_-(\beta) = J_-(\beta)$.

Now $J_-(\beta)/J^1_-(\beta)$ is cyclic so we can extend η to a representation κ of $J_-(\beta)$; indeed, all such extensions take the form $\kappa \otimes \chi$, for χ a character of $P(\mathfrak{o}_E)/P^1(\mathfrak{o}_E)$.

(7.1.1) Theorem *With notation as above, put $\pi = \mathrm{Ind}_{J_-}^G \kappa$,where $J_- = J_-(\beta)$. Then π is an irreducible supercuspidal representation of G. Moreover, (J_-, κ) is a $[G, \pi]_G$-type*

Proof: J_- is a compact open subgroup of G so certainly compact mod centre and we clearly have $\mathcal{I}_G(\kappa) \subset \mathcal{I}_G(\eta) = J_-$. Then, by **[Ca]** (1.5), π is irreducible and supercuspidal. Finally, (J_-, κ) is a $[G, \pi]_G$-type by **[BK2]** (5.4). ■

(7.2) Transfer

We recall from (6.2) that we have the involution $\sigma : A^\times \to A^\times$ given by $\sigma(x) = \overline{x}^{-1}$. We put $\Gamma = \langle\sigma\rangle$, a cyclic group of order 2. Then, for $0 \le m \le n-1$, we have a bijection $\mathcal{C}(\mathfrak{A}, m, \beta)^\Gamma \xrightarrow{\sim} \mathcal{C}_-(\mathfrak{A}, m, \beta)$, given by restriction of characters, as in (6.2.12).

Let $[\mathfrak{A}, n, 0, \beta]$ be a skew simple stratum such that $E = F[\beta]$ is a totally ramified maximal field extension of F in A. We fix a simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)^\Gamma$ and write $\theta_- = \theta|_{H^1_-(\beta,\mathfrak{A})}$.

By **[BK]** (5.1.1), there exists a unique irreducible representation η of $J^1 = J^1(\beta, \mathfrak{A})$ which contains θ. Further, there exist precisely two extensions of η to $J(\beta, \mathfrak{A})$ which are fixed by σ; they are κ^1_θ, given by

$$\kappa^1_\theta(\xi j) = \eta(j), \qquad \text{for } \xi \in \mu'_p(F), j \in J^1,$$

where $\mu'_p(F)$ is the group of roots of unity in F of order prime to p; and κ^2_θ, given by

$$\kappa^2_\theta(\xi j) = \chi(\xi)\eta(j), \qquad \text{for } \xi \in \mu'_p(F), j \in J^1,$$

where χ is the character of $\mu'_p(F)$ given by

$$\chi(\xi) = \begin{cases} 1 & \text{if } \xi \text{ is a square in } F; \\ -1 & \text{if } \xi \text{ is a non-square in } F. \end{cases}$$

The situation in the symplectic group is similar. By (6.4.2), there exists a unique irreducible representation η_- of $J^1_- = J^1_-(\beta, \mathfrak{A})$ which contains θ_-. (In fact, this is just the unique irreducible component of the restriction of η to J^1_-.) Further, there exist precisely two extensions of η_- to $J_-(\beta, \mathfrak{A})$; they are $\kappa^1_{\theta,-}$, given by

$$\kappa^1_{\theta,-}(\xi j) = \eta(j), \qquad \text{for } \xi = \pm 1, j \in J^1_-;$$

and $\kappa^2_{\theta,-}$, given by

$$\kappa^2_{\theta,-}(\xi j) = \xi\eta(j), \qquad \text{for } \xi = \pm 1, j \in J^1_-.$$

Now let π be an irreducible supercuspidal representation of $\mathrm{GL}_N(F) = A^\times$. Then π contains a simple type (J, λ) which is unique up to conjugacy in $\mathrm{GL}_N(F)$. This type is built from a simple character θ, which again is unique up to conjugacy in $\mathrm{GL}_N(F)$. Suppose that there is a skew simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A such that $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ and such that $E = F[\beta]$ is a totally ramified maximal extension of F in A. Then $\lambda = \kappa$, a β-extension of the unique irreducible representation η of $J^1 = J^1(\beta, \mathfrak{A})$ which contains θ.

Suppose that the (equivalence class of the) representation κ is fixed by the involution σ. In particular, this implies that $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)^\Gamma$. Then by the discussion above, $\kappa = \kappa^i_\theta$, for $i = 1$ or 2; this is determined by the restriction of the central character ω_π of π to $\mu'_p(F)$. We put $i(\pi) = 1$ if ω_π is trivial on $\mu'_p(F)$, $i(\pi) = 2$ otherwise; then $\kappa = \kappa_\theta^{i(\pi)}$.

Put $\theta_- = \theta|_{H^1_-(\beta,\mathfrak{A})}$. This character is not (in general) determined up to conjugacy in G, or even up to intertwining. Hence, in order to describe the representations of G obtained from π, we must consider all the simple types contained in π.

Let $\mathcal{K}$ be the set of κ such that (J, κ) is a simple type contained in π, κ is fixed by σ and κ is built from a skew simple stratum $[\mathfrak{A}, n, 0, \beta]$. Let Θ be the set of simple characters θ such that θ is contained in κ, for some $\kappa \in \mathcal{K}$. We put Θ_- to be the set of restrictions $\theta|_{H^1_-}$ for $\theta \in \Theta$. The group G acts on Θ_- by conjugation. Let ϑ be a set of representatives for this action.

We now put $\mathfrak{k} = \{\kappa_{\theta,-}^{i(\pi)} : \theta_- \in \vartheta\}$. Each $\kappa_- \in \mathfrak{k}$ induces to an irreducible supercuspidal representation π_{κ_-} of G, by (7.1.1). Note that the representations π_{κ_-} are not necessarily inequivalent, since we do not have an intertwining implies conjugacy theorem.

Finally, we put $\Pi = \Pi(\pi) = \{\pi_{\kappa_-} : \kappa_- \in \mathfrak{k}\}$. Hence we have associated to π a set of irreducible supercuspidal representations of $\mathbb{G}$.

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