

THE UNIFORM PRIMALITY CONJECTURE FOR THE TWISTED FERMAT CUBIC

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To Jupiter, bringer of joy.

ABSTRACT. On the twisted Fermat cubic, an elliptic divisibility sequence arises as the sequence of denominators of the multiples of a single rational point. We prove that the number of prime terms in the sequence is uniformly bounded. When the rational point is the image of another rational point under a certain 3-isogeny, all terms beyond the first fail to be primes.

1. INTRODUCTION

There abides a deep fascination with the subject of prime terms in naturally occurring integer sequences. The fanfare which accompanies the discovery of each new Mersenne prime [2] bears testimony, as does the search for factorizations of the Fermat and Fibonacci numbers [3, 21]. Elliptic curves [4, 32, 34] give rise naturally to rapidly growing sequences of integers, called *elliptic divisibility sequences*, generated from some nontorsion point via the geometric addition law. The first systematic study of elliptic divisibility sequences was given majestically by Morgan Ward in [40]. Recent times have witnessed a great deal of activity (the following represents a sample) because of their intrinsic interest [14, 16, 18, 19, 35, 36, 37, 38, 39] as well as in applications to Logic [6, 10, 27, 28, 31] and Cryptography [22, 30]. The Chudnovsky brothers [5] first proposed studying primality in elliptic divisibility sequences arising from elliptic curves in Weierstrass form. This was continued in [7, 11, 15] with the following outcome: a heuristic argument, together with lots of computational evidence and proofs in some special cases, suggests that any elliptic divisibility sequence contains only finitely many prime terms. This has become known as the *primality conjecture* for elliptic curves in Weierstrass form. In [15, Theorem 4.1] the same conjecture was proved for curves in homogeneous form. It is with such curves that our business lies in this paper.

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Let C denote an elliptic curve of the form

$$C : U^3 + V^3 = m, \quad (1)$$

where m is a nonzero integer. It is sometimes said that C is a twist of the Fermat cubic. The set of rational (or real) points form a group under the chord and tangent method: the (projective) point at infinity with homogeneous coordinates $(1 : -1 : 0)$ is the identity, and inversion is given by reflection in the line $U = V$. Given a nontorsion point $R \in C(\mathbb{Q})$, write, in lowest terms,

$$nR = \left(\frac{U_n}{W_n}, \frac{V_n}{W_n} \right).$$

The sequence (W_n) is a (strong) divisibility sequence, see Proposition 3.3. Our starting point is the result stated earlier, that only finitely many terms W_n are prime [15, Theorem 4.1]. A related result [13, Theorem 1.2] states that W_n has a primitive divisor for all $n > 1$: results from [13] will be employed in the proofs of our theorems.

To focus now upon the question we study: a stronger form of the primality conjecture has been proposed, assuming that the Weierstrass equation is in minimal form, predicting that the number of prime terms in an elliptic divisibility sequence is uniformly bounded (independently of both the curve and the point). Unsurprisingly, this conjecture is known as the *uniform primality conjecture* for curves in Weierstrass form. A conditional proof is known, under Lang's Conjecture, assuming the generating point is the image of a rational point under a nontrivial isogeny [12, Theorem 1.4]. This raises the natural question as to whether a uniform primality result holds for curves in homogeneous form and it is this question we address. The first of our two results is an unconditional result to that effect, and will be stated now. The proof will appear to be quite short although, in fact, it relies upon two substantial results from [13] and [17].

Theorem 1.1. *Assuming $m \in \mathbb{Z}$ is cube-free, the number of prime (power) terms W_n is uniformly bounded. Moreover, assuming the ABC conjecture holds in \mathbb{Z} , there is a uniform upper bound on the index n such that W_n is a prime power.*

The assumption about m being cube-free is necessary given the homogeneous nature of the curve C , since one could scale any given elliptic divisibility sequence to obtain as many prime terms as desired. This will be explained in detail in subsection 3.3. It is also probable that a uniform bound on the index n would follow from the Hall-Lang conjecture, as in recent work of Mahé [24].

Our second theorem relies upon the existence of a rational isogeny in line with work for curves in Weierstrass form [11, 15]. There is a 3-isogeny, detailed in [8], $\tau : C' \rightarrow C$ from the elliptic curve

$$C' : U'V'(U' + V') = m. \quad (2)$$

Theorem 1.2. *Suppose $m \in \mathbb{Z}$ is cube-free and $R \in C(\mathbb{Q})$ is the image of a point in $C'(\mathbb{Q})$ under τ . Then W_n is not a prime power, for all $n > 1$. Further, the number of terms W_n with at most two distinct prime divisors is uniformly bounded.*

The conclusion is clearly best possible and what is remarkable is the small size of the bound 1; this is certainly much stronger than any corresponding result for elliptic curves in Weierstrass form.

The proofs of the two theorems rely upon the same principle, which will be articulated in section 3 below. This is preceded by a section discussing possible improvements to our results and is followed by proofs of the theorems in order.

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2. BETTER-BEST?

2.1. With isogeny. Taking our two theorems in reverse over, Theorem 1.2 cannot be improved. But to how large a class of examples does it apply? If R is 3 times another rational point then it will be such an image [8]. A short analysis is included to shed light in greater generality. In truth, throughout the paper, we will not work explicitly with the isogeny mentioned in the statement of Theorem 1.2. Instead, our work relies upon a bi-rational map between C and the Mordell curve

$$E : Y^2 = X^3 - 432m^2, \quad (3)$$

where $P = (X, Y)$ corresponds to $R = (U, V)$ under the bi-rational transformation given by

$$\begin{aligned} X &= \frac{2^2 3m}{U+V}, & Y &= \frac{2^2 3^2 m(U-V)}{U+V}, \\ U &= \frac{2^2 3^2 m + Y}{6X}, & V &= \frac{2^2 3^2 m - Y}{6X}. \end{aligned} \quad (4)$$

There is a 3-isogeny $\sigma : E' \rightarrow E$ onto the curve E from the elliptic curve

$$E' : y^2 = x^3 + 16m^2,$$

given by

$$\sigma(x, y) = (X, Y) = \left(x + \frac{64m^2}{x^2}, \frac{y(y+12m)(y-12m)}{(y+4m)(y-4m)} \right). \quad (5)$$

It is with this isogeny that we work. The isogeny condition stated in Theorem 1.2 is equivalent to the point $P \in E(\mathbb{Q})$ being equal to $\sigma(P')$ for some point $P' \in E'(\mathbb{Q})$. Table 1 shows the 22 values $m < 100$ when this occurs

for a point P which is a generator for the group $E(\mathbb{Q})$. (Thus all points in $E(\mathbb{Q})$ are in the image of the isogeny.) In total, there are 42 rank one curves with $m < 100$. We conclude that, at least for small values of m , the hypothesis of Theorem 1.2 is not infrequently met.

TABLE 1. Rank 1 curves satisfying Theorem 1.2 with $m < 100$

m	P	P'
6	[28, 80]	[-8, 8]
12	[52, 280]	[-12, 24]
15	[49, 143]	[-15, 15]
20	[84, 648]	[-16, 48]
22	[553/9, 4085/27]	[33, 209]
33	[97, 665]	[-24, 60]
34	[273, 4455]	[-16, 120]
42	[172, 2080]	[-24, 120]
50	[2716/9, 138736/27]	[24, 232]
51	[5473/36, 333935/216]	[144, 1740]
58	[3148/9, 173600/27]	[-24, 200]
68	[55156/225, 12043304/3375]	[240, 3728]
69	[553, 12925]	[-23, 253]
70	[156, 1296]	[-40, 120]
75	[601, 14651]	[-24, 276]
78	[217, 2755]	[-39, 195]
84	[148, 440]	[-48, 48]
85	[18361/9, 2487509/27]	[-15, 335]
87	[1029841/1225, 1042214111/42875]	[840, 24348]
90	[364, 6688]	[-36, 288]
92	[2548/9, 117800/27]	[48, 496]
94	[62511752209/2480625, 15629405421521177/3906984375]	[25200, 4000376]

2.2. Without isogeny. Turning now to Theorem 1.1, we will draw upon a result of Ingram [17] bounding the number of integral multiples of a given integral point. In its most general form, this involves a non-explicit constant. In his paper, he gives a very strong (actually best possible) explicit constant in the case of congruent number curves. It seems a good bet that his analysis could be re-worked in our context to obtain an explicit (perhaps even best possible) form of Theorem 1.1. This analysis was done in certain cases by the second author in her thesis [26]: a special case of [26, Theorem 1.0.4] is recorded here.

Theorem 2.1. *Assume $m \equiv \pm 1, \pm 3, \pm 4 \pmod{9}$ is cube-free. Assuming $P \in E(\mathbb{Q})$ has $\gcd(x(P), m) = 1$ and $2P$ and $3P$ are not integral then at most one value of $n > 1$ yields a prime power term W_n .*

Of course the most desirable goal is to be able to compute all of the prime power terms in a given sequence, as permitted by Theorem 1.2. It is not clear whether Theorem 1.1 could be improved to allow that as an outcome. However, as part of the proof of Theorem 1.2, we do prove the following unconditional result:

Lemma 2.2. *Suppose $m \in \mathbb{Z}$ is cube-free. Then W_n is not a prime power, for $2 < n \leq 22$.*

2.3. Daylight. There will always be daylight between our theorems as our final remark demonstrates. It seems likely that without the isogeny hypothesis, there are infinitely many values of m and generating points R yielding a prime term W_2 . For, suppose $R = [u, v] \in C(\mathbb{Q})$ is an integral point of infinite order. The point $2R$ can be expressed in the form

$$2R = \left(\frac{-2vu^3 - v^4}{u^3 - v^3}, \frac{u^4 + 2v^3u}{u^3 - v^3} \right).$$

Suppose $u - v = 1$. Then

$$u^3 - v^3 = (u - v)(u^2 + uv + v^2) = 3u^2 - 3u + 1.$$

Applying the Bateman-Horn conjecture [1] to the polynomial

$$f(u) := 3u^2 - 3u + 1$$

implies that $f(u)$ is prime for infinitely many positive integers u . By a result of Erdős [9], the corresponding value of $m = u^3 + (u - 1)^3$ will be cube-free for infinitely many of these values. To show that the prime so constructed does not cancel with the numerator, compute the resultants of the two polynomials $u^2 + uv + v^2$ and $2u^3 + v^3$ in each variable: they are $9u^6$ and $9v^6$ so the only non-trivial common division is by a divisor of 9. But $3u^2 - 3u + 1$ is coprime to 3 so there can be no cancellation. This style of argument will be used repeatedly in the proof of Theorem 1.2.

3. PROVING THE PRIMALITY CONJECTURE

This section is elementary although quite technical because of the case-by-case nature of the proofs. It is the statements of the results which provide important input for sections 4 and 5, rather than their proofs. Readers whose taste is not excited by the details could take note of Lemma 3.1, Proposition 3.3 and subsection 3.3 then skip directly to sections 4 and 5.

3.1. Constraints to cancellation. From the bi-rational transformation (4), we have

$$\frac{U_n}{W_n} = \frac{2^2 3^2 m B_n^3 + C_n}{6 A_n B_n}, \quad (6)$$

and

$$\frac{V_n}{\bar{W}_n} = \frac{2^2 3^2 m B_n^3 - C_n}{6 A_n B_n}, \quad (7)$$

where

$${}_nR = \left(\frac{U_n}{W_n}, \frac{V_n}{W_n} \right) \text{ and } {}_nP = \left(\frac{A_n}{B_n^2}, \frac{C_n}{B_n^3} \right)$$

are written in lowest terms.

Lemma 3.1. *Any cancellation in the fractions (6) and (7) comes from the term $6A_n$ and divides $72m$.*

Since A_n and B_n are coprime, the following important principle arises as a consequence of Lemma 3.1:

We can be certain that W_n possesses at least two coprime factors if two conditions are met. The first is that $B_n > 1$ and the second is that the term $6A_n$ is not completely cancelled.

Proof of Lemma 3.1. Consider the fraction on the right-hand side of (6). Let $d = p^r$ be a common factor of $(2^23^2mB_n^3 + C_n)$ and $6A_nB_n$ with p a prime number and $r \in \mathbb{N}$ the highest order of p dividing both terms. If $d' := \gcd(d, B_n) \neq 1$, then $d' \mid d \mid (2^23^2mB_n^3 + C_n)$ implies $d' \mid C_n$, which contradicts the fact that B_n and C_n are coprime. Thus $\gcd(d, B_n) = 1$, so that d comes from the term $6A_n$.

Notice, moreover, that any cancellation in the right-hand side of (6) and (7) is the same in each term. This is because both terms have the same denominators as their cubes sum to an integer. Hence d has to divide both $(2^23^2mB_n^3 + C_n)$ and $(2^23^2mB_n^3 - C_n)$, so that $d \mid 72m$. Thus any cancellation of the fractions (6) and (7) divides $72m$. \square

We will need to be even more precise about the cancellation that occurs. Write d_n for the cancellation in the fraction (7).

Corollary 3.2. *We have*

$$\text{ord}_p(d_n) = \begin{cases} \text{ord}_p(m) & \text{if } p \mid A_n, p > 3; \\ \text{ord}_p(A_n) + 1 & \text{if } p \mid A_n, p \leq 3; \\ 0 & \text{if } p \nmid A_n. \end{cases}$$

Note in particular that if $p \leq 3$ and $p \mid A_n$ then p is completely cancelled. Thus for primes $p \leq 3$, we have $p \mid W_n$ if and only if $p \nmid A_n$. For primes greater than 3, the cancellation is exactly as large as allowed by Lemma 3.1; therefore, whenever $p > 3$ divides A_n , we have

$$\text{ord}_p(W_n) = \text{ord}_p(A_n) - \text{ord}_p(m). \quad (8)$$

Proof. Suppose first that $p \nmid A_n$. By Lemma 3.1, we have $\text{ord}_p(d_n) = 0$ except possibly if $p \leq 3$ so suppose we are in this situation. The defining equation (3) yields

$$C_n^2 = A_n^3 - 432m^2B_n^6, \quad (9)$$

whence $p \nmid C_n$ so p cannot cancel in (7).

Now suppose $p \mid A_n$ and $p \geq 3$. From (9) we find

$$\text{ord}_p(A_n) \geq \text{ord}_p(C_n) = \text{ord}_p(m),$$

since $p \mid A_n$, and it follows that $\text{ord}_p(d_n) = \text{ord}_p(m)$.

Finally, suppose $p \mid A_n$ and $p \leq 3$, and put $\delta = \text{ord}_p(\gcd(p, m))$. The arguments are fiddly but elementary so we only sketch them, beginning with the slightly simpler case $p = 3$.

If $\text{ord}_3(A_n) > 1 + \delta$ then, comparing 3-adic valuations, we see that equation (9) is impossible to solve. Hence $1 \leq \text{ord}_3(A_n) \leq 2$ and we also find that $\text{ord}_3(C_n) \geq \text{ord}_3(A_n) + 1$. Thus the numerator of (7) has 3-adic order at least $\text{ord}_3(A_n) + 1$ whereas the 3-adic order of the denominator is exactly $\text{ord}_3(A_n) + 1$, giving $\text{ord}_3(d_n) = \text{ord}_3(A_n) + 1$.

Now assume that $p = 2$. If $\text{ord}_2(A_n) > 2 + \delta$ then, dividing (9) through by $2^4 3^2 m^2$ yields a congruence $x^2 \equiv -1 \pmod{4}$, which is impossible. Moreover, $\text{ord}_2(A_n) = 1$ is impossible by comparing 2-adic valuations in (9).

If $\text{ord}_2(A_n) = 2 + \delta$ then $\text{ord}_2(C_n) = 2 + \text{ord}_2(m)$ so the numerator of (7) is $2^{2+\text{ord}_2(m)}$ times the sum of two odd numbers. Thus the 2-adic order of the numerator is at least $3 + \text{ord}_2(m)$ while that of the denominator is exactly $3 + \delta$ so $\text{ord}_2(d_n) = \text{ord}_2(A_n) + 1$.

This leaves the case $\text{ord}_2(A_n) = 2$ with $2 \mid m$. Here $\text{ord}_2(C_n) > 2$ so the 2-adic order of the numerator of (7) is at least 3, while that of the denominator is exactly 3, again giving $\text{ord}_2(d_n) = \text{ord}_2(A_n) + 1$. \square

3.2. W is a divisibility sequence. The aim of this subsection is a proof of the following:

Proposition 3.3. *The sequence (W_n) is a strong divisibility sequence: in other words, for all $r, n \in \mathbb{N}$,*

$$\gcd(W_r, W_n) = W_{\gcd(r, n)}.$$

In particular, $W_n \mid W_{nk}$, for all $n, k \in \mathbb{N}$.

The divisibility property will be used repeatedly in this paper but we cannot find it proved explicitly in the literature. The strong divisibility property will be used only in subsection 3.3. However, it is a natural property and is proved with little more effort.

We admit that the proof we give lacks elegance. This is due to the evil influence of the primes 2 and 3 and, to a lesser extent, those dividing m . However, the proof is completely self-contained and uses only elementary methods.

A more sophisticated proof of Proposition 3.3 uses formal groups [32, Chapter IV]. Writing $w = U/V$ and $z = 1/V$ in (1) yields the equation

$$w^3 = mz^3 - 1.$$

Now the binomial theorem yields a power series

$$w = \sum_{n=0}^{\infty} a_n z^{3n} \text{ with } a_n \in \mathbb{Q},$$

which converges p -adically for all primes p with $p \mid z$ (only the case $p = 3$ is at all tricky). Applying the geometric group law on the points (z_1, w_1) and (z_2, w_2) now yields a power series $F(z_1, z_2)$ – the *formal group* of our elliptic curve. The proof of Proposition 3.3 follows from the standard properties of formal groups [32, Chapters IV, VII]: in particular, the filtration into subgroups C_r , $r \in \mathbb{N}$, defined by $|U|_p \geq p^r$. However arguing as above requires quite a lot of explanation and checking and it would not shorten the paper. For example, the statement of Proposition 3.3 per se does not appear in [32]. Instead, the filtration referred to yields directly the relation (10) below.

Proof of Proposition 3.3. The divisibility property follows from the following relation: for any prime p and $k \in \mathbb{N}$, if $p \mid W_n$ then

$$\text{ord}_p(W_{nk}) = \text{ord}_p(W_n) + \text{ord}_p(k). \quad (10)$$

Much of the spade work here is supplied by the following two relations. Firstly, from [13, Lemma 3.2], for $p > 3$, if $\text{ord}_p(A_n) > 0$ and $3 \nmid k$ then

$$\text{ord}_p(A_{nk}) = \text{ord}_p(A_n) + \text{ord}_p(k). \quad (11)$$

Secondly, for all primes p , if $\text{ord}_p(B_n) > 0$ and $k \in \mathbb{Z}$ then

$$\text{ord}_p(B_{nk}) = \text{ord}_p(B_n) + \text{ord}_p(k). \quad (12)$$

The second relation arises from a local analysis using the formal group for the Weierstrass equation and follows from [32, Chapter VII]. A much more general proof, valid over a Dedekind domain, is given in [19, Proposition 1]. Note that (12) implies that $B = (B_n)$ is a strong divisibility sequence. Further work is needed to take us from (10) to the same conclusion for the sequence $W = (W_n)$: this will be supplied at the end.

Assume p denotes a prime with $p \mid W_n$. We will show (10) for all $k \in \mathbb{N}$. There are three cases in total:

- (a) $p \mid B_n$,
- (b) $p \nmid B_n$, $p > 3$,
- (c) $p \nmid B_n$, $p \leq 3$.

Case (a) No cancellation occurs in (7) so $\text{ord}_p(W_n) = \text{ord}_p(B_n)$ if $p \nmid 6$ or $1 + \text{ord}_p(B_n)$ if $p \mid 6$. Then (10) follows from (12), the corresponding result for the sequence $B = (B_n)$.

Case (b) Since $p \mid W_n$, the two conditions given imply $p \mid A_n$. When $3 \nmid k$, we have $p \mid A_{nk}$ by (11) so we can apply (8) with nk replacing n . Then

$$\begin{aligned} \text{ord}_p(W_{nk}) &= \text{ord}_p(A_{nk}) - \text{ord}_p(m) && \text{by (8),} \\ &= \text{ord}_p(A_n) + \text{ord}_p(k) - \text{ord}_p(m) && \text{by (11),} \\ &= \text{ord}_p(W_n) + \text{ord}_p(k) && \text{by (8).} \end{aligned}$$

When $3 \mid k$, we use the triplication law [13, (8)], repeated here for convenience:

$$x(3Q) = \frac{x^9(Q) + 2^9 3^4 x^6(Q) m^2 + 2^{12} 3^7 x^3(Q) m^4 - 2^{18} 3^9 m^6}{9x^2(Q)(x^3(Q) - 2^6 3^3 m^2)^2}. \quad (13)$$

It follows from (13) and (8) that

$$\text{ord}_p(B_{3n}) = \text{ord}_p(A_n) - \text{ord}_p(m) = \text{ord}_p(W_n). \quad (14)$$

Since $3n \mid nk$ we deduce from (12) that $p \mid B_{nk}$. Therefore

$$\begin{aligned} \text{ord}_p(W_{nk}) &= \text{ord}_p(B_{nk}) && \text{since } p > 3, \\ &= \text{ord}_p(B_{3n}) + \text{ord}_p(k/3) && \text{since } p \mid B_{3n}, \\ &= \text{ord}_p(B_{3n}) + \text{ord}_p(k) && \text{since } p > 3, \\ &= \text{ord}_p(W_n) + \text{ord}_p(k) && \text{by (14).} \end{aligned}$$

Case (c) Since $p \mid W_n$ and $p \leq 3$, we have $p \nmid A_n$ and, as there is no cancellation, $\text{ord}_p(W_n) = 1$. Also, since $p \nmid A_n$ it follows that $p \nmid A_{nk}$ for all $k \in \mathbb{N}$: this is because the point nP has good reduction at p so nkP has good reduction mod p as the set of points with good reduction form a group and it follows that nkP cannot reduce mod p to a point with zero x -coordinate. Thus there is no cancellation in (7) when n is replaced by nk . Moreover, as $p \nmid B_n$ also, the reduction of nP is a non-identity point and, since E has additive reduction modulo p , the point nkP reduces to the identity if and only if $p \mid k$. (Note that we are reducing the curve E , not its minimal form.)

There are now two possibilities: If $p \nmid k$ then $p \nmid B_{nk}$, in which case $\text{ord}_p(W_{nk}) = 1 = \text{ord}_p(W_n) + \text{ord}_p(k)$. Otherwise $p \mid k$ and $p \mid B_{nk}$; it is easily checked by an explicit calculation with the doubling and tripling formulae (see (13) for the latter) that $\text{ord}_p(B_{np}) = 1$ so we get

$$\begin{aligned} \text{ord}_p(W_{nk}) &= \text{ord}_p(B_{nk}) + 1 \\ &= \text{ord}_p(B_{np}) + \text{ord}_p(k/p) + 1 && \text{by (12),} \\ &= 1 + \text{ord}_p(k) \\ &= \text{ord}_p(W_n) + \text{ord}_p(k). \end{aligned}$$

Now we have proved (10), the strong divisibility property comes from the following claim:

Claim If a prime p divides $\text{gcd}(W_r, W_n)$ then $p \mid W_{\text{gcd}(r,n)}$.

To see that this suffices, let $d = \gcd(r, n)$ and suppose $r = dk, n = dl$. For any prime p , one of $\text{ord}_p(k)$ and $\text{ord}_p(l)$ is 0 because k and l are co-prime. From (10), the exact power of p dividing $\gcd(W_{dk}, W_{dl})$ is $\text{ord}_p(W_d)$ as desired.

It remains to prove the claim. First let p be a prime not dividing $6m$. Note that $p \mid W_n$ if and only if $p \mid A_n B_n$ in this case (since no cancellation can occur), which is equivalent to nP reducing mod p to an element of the subgroup G of $E(\overline{\mathbb{F}}_p)$ generated by a 3-torsion point T with $x(T) = 0$. If rP and nP both reduce to an element of G then so does any integer linear combination; in particular, so does $\gcd(r, n)P$ and we deduce that $p \mid W_{\gcd(r, n)}$.

Now suppose $p > 3$ is a prime dividing m such that $p \mid \gcd(W_r, W_n)$. There are three possibilities, and in each case we will show that $p \mid W_{\gcd(r, n)}$:

- (i) $p \mid B_r$ and $p \mid B_n$, in which case $p \mid B_{\gcd(r, n)}$, since B is a strong divisibility sequence. Thus $p \mid W_{\gcd(r, n)}$, since there is no cancellation.
- (ii) $p \mid B_r$ and $p \mid A_n$, in which case nP is a point of bad reduction, and P is also. Thus $p \mid A_1$ and, from the analysis in case (b) above, we see that $3 \mid r, 3 \nmid n$. Then, from (14), we have $p \mid B_{3n}$ so, from strong divisibility, $p \mid B_{\gcd(r, 3n)} = B_{3 \gcd(r, n)}$. Moreover, since $3 \nmid \gcd(r, n)$, we have $p \mid A_{\gcd(r, n)}$ so, from (14) again,

$$\text{ord}_p(W_{\gcd(r, n)}) = \text{ord}_p(B_{3 \gcd(r, n)}) > 0.$$

- (iii) $p \mid A_r$ and $p \mid A_n$, in which case P is again a point of bad reduction, $3 \nmid r, 3 \nmid n$, so $p \mid A_{\gcd(r, n)}$. From (14), we have $p \mid \gcd(B_{3r}, B_{3n}) = B_{3 \gcd(r, n)}$ so that $\text{ord}_p(W_{\gcd(r, n)}) = \text{ord}_p(B_{3 \gcd(r, n)}) > 0$.

Finally, suppose $p \leq 3$. Since $p \mid W_n$ we have $p \nmid A_n$ so nP is a point of good reduction modulo p . The same is true of rP and so also of $\gcd(r, n)P$. Thus $p \nmid A_{\gcd(r, n)}$ and we deduce from Corollary 3.2 that $p \mid W_{\gcd(r, n)}$.

This completes the proof of the claim, and of Proposition 3.3. \square

3.3. Cube-free m . We conclude with a remark justifying the assumption in both our theorems that m is cube-free.

The key remark needed follows from Proposition 3.3: if $\gcd(m, n) = 1$ then $\gcd(W_m, W_n) = 1$. From [13, Theorem 1.2], the terms W_n with $n > 1$ all see a primitive prime divisor: in other words, a prime $p \mid W_n$ which has not appeared in an earlier term. Let S denote a finite set of primes. For each $l \in S$, write $W_l = w_l p_l^{e_l}$ with $w_l, e_l \in \mathbb{N}$ and $p_l \nmid w_l$ equal to a primitive prime divisor. Now write

$$M = \prod_{l \in S} w_l$$

and rescale equation (1) by multiplication with M^3 . If $W'_l = W_l / \gcd(W_l, M)$ denotes the resulting denominator then W'_l is a prime power by the starting

remark. Therefore, by expanding S arbitrarily, we see that, at the cost only of multiplying m by a cube, we may produce elliptic divisibility sequences with arbitrarily large numbers of prime power terms.

4. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. We will prove firstly that, for $n > 12$, the term A_n in the denominator is not completely cancelled. From [13, Theorem 2.3], all terms A_n with $n > 12$ possess a primitive prime divisor p_n . Then $p_n \nmid A_1$, because p_n is a primitive prime divisor, so it follows that P is a point of good reduction for p_n . Thus all multiples of P , in particular nP , are points of good reduction. However, for primes dividing $6m$, the point on the reduced curve with x -coordinate zero is a point of bad reduction so it follows that $p_n \nmid 6m$. In particular, p_n is not cancelled because any cancellation divides $72m$.

Secondly, from [17, Theorem 1], with the notation used there, there is a uniform constant C such that $n > CM(P)^{16}$ forces $B_n > 1$, except for at most one value of n . The quantity $M(P)$ is related to the Tamagawa number. Since the Mordell curve E has integral j -invariant, along the same lines as in [17], E always has $M(P) \leq 6$. It follows that the number of prime power terms W_n is bounded by $C6^{16} + 1$, a uniform constant.

Finally, assume the *ABC* conjecture holds for \mathbb{Z} . As above we always get a prime factor from A_n , for $n > 12$, so we need only show that $B_n > 1$ for all n greater than some uniform bound. For this we will use the theory of heights, which will also be essential for the explicit bound in Theorem 1.2.

Recall that the (*naïve*) *Weil height* of the point P is

$$h(P) = h(x(P)) = \log \max \{|A_1|, B_1^2\}.$$

On the other hand the *canonical height* $\hat{h}(P)$ is given by

$$\hat{h}(P) = \lim_{n \rightarrow \infty} \frac{h(2^n P)}{4^n}.$$

Silverman gives explicit upper and lower bounds for the difference between the Weil height and the canonical height for curves in short Weierstrass form. Note that our heights are twice those in [33] so the inequalities are multiplied by 2.

Theorem 4.1 ([33, Remark 1.2]). *Given an elliptic curve in short Weierstrass form,*

$$E/\mathbb{Q} : y^2 = x^3 + ax + b,$$

and $Q \in E(\mathbb{Q})$, we have

$$-\frac{1}{6}h(j) - \frac{1}{6}h(\Delta) - 2.14 \leq h(Q) - \hat{h}(Q) \leq \frac{1}{4}h(j) + \frac{1}{6}h(\Delta) + 1.946. \quad (15)$$

where $\Delta = -16(4a^3 + 27b^2)$ and $j = -48a^3/\Delta$.

We return to the proof of the final assertion in Theorem 1.1. From (3), we have

$$C_n^2 = A_n^3 - 432m^2B_n^6.$$

Suppose first that $|A_n| > B_n^2$. Since $\gcd(A_n, C_n) \mid 432m^2$, for any $\varepsilon > 0$, the *ABC* conjecture gives

$$|A_n^3| \ll |mA_nB_nC_n|^{1+\varepsilon}$$

and, since $|C_n| < |A_n|^{3/2}$, we get

$$|A_n|^{1/2-\varepsilon} \ll mB_n.$$

Thus, taking $\varepsilon = 1/4$ and writing $|A_n| = h(nP)$, we have

$$\log B_n > \frac{1}{4}h(nP) - K_1 \log m, \quad (16)$$

for some constant $K_1 > 0$. On the other hand, if $|A_n| < B_n^2$ then (16) is trivially satisfied, since $h(nP) = B_n^2$.

Finally, from Theorem 4.1 we have

$$h(nP) > \hat{h}(nP) - K_2 \log m = n^2\hat{h}(P) - K_2 \log m,$$

for some constant $K_2 > 0$. Moreover, $\hat{h}(P) > K_3 \log m$, for some constant $K_3 > 0$, by [20, Proposition 1]. Putting these together with (16), we get

$$\log B_n > (K_4n^2 - K_5) \log m,$$

for constants $K_4, K_5 > 0$. In particular, this bounds, independently of m and P , the index n for which we may have $B_n = 1$. \square

5. PROOF OF THEOREM 1.2

The idea of the proof of Theorem 1.2 is to use heights on the Weierstrass elliptic curve (3) to show that, for $n > 22$, we are guaranteed to get a prime divisor of W_n coming from the term B_n ; that is, we use the isogeny to make explicit the constants in the argument above using the *ABC* conjecture. As we have seen, we also get a prime divisor coming from A_n in this case, so that W_n is not a prime power. The remaining values $2 \leq n \leq 22$ are treated case-by-case, using the explicit form of the point nR as a rational function in the coordinates (u, v) of $R \in C(\mathbb{Q})$.

Lemma 5.1. *Let E be defined as in (3) and suppose $P \in E(\mathbb{Q})$ is the image of a point in $E'(\mathbb{Q})$ under the isogeny σ as in (5). Then, with B_n as in (7),*

$$B_n > 1$$

for all $n > 22$.

Proof. Let $P \in E(\mathbb{Q})$ such that $\sigma(P') = P$, for some $P' \in E'(\mathbb{Q})$. Write

$$x_n := x(nP') = \frac{a_n}{b_n^2},$$

with $\gcd(a_n, b_n) = 1$. From (5),

$$\frac{A_n}{B_n^2} = X(nP) = x_n + \frac{64m^2}{x_n^2} = \frac{a_n^3 + 64m^2b_n^6}{a_n^2b_n^2}. \quad (17)$$

Claim: $B_n > 1$, provided $\max\{|a_n|, b_n^2\} > 8m$.

Before this claim can be settled, we must examine the fraction on the right-hand side of (17) for possible cancellation. Let $d = p^r$ be a common factor of $(a_n^3 + 64m^2b_n^6)$ and $a_n^2b_n^2$, where p is a prime and $r \in \mathbb{N}$ is the highest order of p dividing both terms. Since $\gcd(a_n, b_n) = 1$, either $d \mid a_n^2$ or $d \mid b_n^2$. If the latter occurs, then $d \mid (a_n^3 + 64m^2b_n^6)$ implies $d \mid a_n^3$, which is impossible as a_n and b_n are coprime. Thus d can only come from the term a_n^2 , so that $d \mid a_n^3$. We have now that

$$d \mid (a_n^3 + 64m^2b_n^6), \quad d \mid a_n^3, \quad \text{and} \quad d \nmid b_n^6,$$

so $p^r = d \mid 64m^2$. Hence the greatest common divisor of numerator and denominator of the fraction on the right-hand side of (17), say g , has to divide $64m^2$ as well.

To turn to the claim, if $|a_n| > 8m$, then

$$B_n^2 = \frac{a_n^2b_n^2}{g} \geq \frac{a_n^2}{g} \geq \frac{a_n^2}{64m^2} > \frac{64m^2}{64m^2} = 1.$$

On the other hand, if $b_n^2 > 8m$, then $g \mid a_n^2$ implies

$$B_n^2 = \frac{a_n^2b_n^2}{g} \geq b_n^2 > 8m > 1.$$

Thus we need to ensure that $\max\{|a_n|, b_n^2\} > 8m$, for our purposes and to that end we turn. Note that the logarithm of the expression on the left is the Weil height $h(nP')$ of nP' which we know, from Theorem 4.1, is close to the canonical height $\hat{h}(nP')$.

Write $h = \hat{h}(P)$ and $h' = \hat{h}(P')$; then

$$h = \hat{h}(P) = \hat{h}(\sigma(P')) = 3\hat{h}(P') = 3h'$$

as σ is a 3-isogeny. Applying the inequality (15) to the curve E' with $Q = nP'$, $j = 0$ and $\Delta = -16^3 3^3 m^4$, we obtain

$$\log \max\{|a_n|, b_n^2\} = h(nP') > h'n^2 - \frac{2}{3} \log m - \frac{1}{2} \log 48 - 2.14. \quad (18)$$

The height bound [13, (14)] gives

$$h' = \frac{h}{3} > \frac{1}{81} \log m - 0.039 \quad (19)$$

for all $m \geq 1$. Then (18) becomes

$$\log \max\{|a_n|, b_n^2\} > \left(\frac{1}{81} \log m - 0.039 \right) n^2 - \frac{2}{3} \log m - \frac{1}{2} \log 48 - 2.14.$$

Then we can ensure $\max\{|a_n|, b_n^2\} > 8m$ provided that

$$\left(\frac{1}{81} \log m - 0.039\right) n^2 - \frac{2}{3} \log m - \frac{1}{2} \log 48 - 2.14 > \log(8m). \quad (20)$$

With a manipulation, (20) will be guaranteed for $n > 12$ but only for all sufficiently large m . However the amount of checking for smaller values of m is infeasible. More realistically, if $m > 353$ then (20) is true provided $n > 22$. Thus it can be concluded that for all $m > 353$, we have $B_n > 1$ if $n > 22$. The proof for values $m \leq 353$ follows in an appendix, see section 6. \square

We are now in a position to prove Theorem 1.2 using Lemma 5.1.

Proof of Theorem 1.2. When $n > 12$, the term A_n is not completely cancelled from the denominator of (7), exactly as before. Also, it follows directly from Lemma 5.1 that $B_n > 1$ for all $n > 22$. Therefore, the term W_n possesses at least two coprime factors for all $n > 22$. We will go on to prove the same for every $2 \leq n \leq 22$ case by case.

Before this, we give a proof of the second claim in Theorem 1.2. Under the isogeny hypothesis, it follows from [12, Theorem 1.4] that the number of prime power terms B_n is uniformly bounded. Combining this with our knowledge that A_n has a primitive prime divisor for $n > 12$ means that the number of terms W_n which have at most two distinct prime divisors is uniformly bounded.

With sleeves rolled up, we will now show that every term W_n fails to be a prime power when $2 < n \leq 22$. We will not invoke the isogeny hypothesis so these results apply in complete generality; that is, we are giving a proof of Lemma 2.2. Finally we will deal with the case when $n = 2$ assuming the isogeny hypothesis.

Proof of Lemma 2.2. Since (W_n) is a divisibility sequence, it suffices to consider W_n when n is an odd prime less than 22 or $n = 4$. We begin with the case when $R = (u, v)$ is an integral point; at the end of the proof we will explain how the general case follows.

Case $n = 4$ We deal firstly with the case when $n = 4$. Write

$$4R = \left(\frac{U_4}{W_4}, \frac{V_4}{W_4} \right) = \left(\frac{f_4(u, v)}{g_4(u, v)}, \frac{f'_4(u, v)}{g_4(u, v)} \right),$$

where

$$\frac{f_4(u, v)}{g_4(u, v)} = \frac{-u^{16} + 8v^3u^{13} + 32v^6u^{10} + 28v^9u^7 + 10v^{12}u^4 + 4v^{15}u}{-u^{15} - 13v^3u^{12} - 10v^6u^9 + 10v^9u^6 + 13v^{12}u^3 + v^{15}},$$

and

$$\frac{f'_4(u, v)}{g_4(u, v)} = \frac{v^{16} - 8u^3v^{13} - 32u^6v^{10} - 28u^9v^7 - 10u^{12}v^4 - 4u^{15}v}{-u^{15} - 13v^3u^{12} - 10v^6u^9 + 10v^9u^6 + 13v^{12}u^3 + v^{15}}.$$

We may consider the second coordinate, and factorize $g_4(u, v)$ as a product of four terms:

$$\begin{aligned}
 g_{4,1}(u, v) &:= v - u \\
 g_{4,2}(u, v) &:= u^2 + uv + v^2 \equiv (v - u)^2 \pmod{3} \\
 g_{4,3}(u, v) &:= u^4 + 2u^3v + 2uv^3 + v^4 \equiv (v - u)^4 \pmod{3} \\
 g_{4,4}(u, v) &:= u^8 - 2u^7v + 4u^6v^2 + 4u^5v^3 - 5u^4v^4 \\
 &\quad + 4u^3v^5 + 4u^2v^6 - 2uv^7 + v^8 \\
 &\equiv (v - u)^8 \pmod{3}
 \end{aligned}$$

We claim that at least two of these factors can avoid being cancelled by the numerator $f'_4(u, v)$. Choosing to consider $g_{4,3}$ and $g_{4,4}$, we can see that the resultants between them and f'_4 with respect to u and v are

$$R_u(f'_4, g_{4,3}) = 3^{16}v^{64} \text{ and } R_v(f'_4, g_{4,3}) = 3^{16}u^{64},$$

respectively, and also

$$R_u(f'_4, g_{4,4}) = 3^{32}v^{128} \text{ and } R_v(f'_4, g_{4,4}) = 3^{32}u^{128}.$$

As u and v are coprime,

$$\gcd(f'_4(u, v), g_{4,3}(u, v)) \mid 3^{16} \text{ and } \gcd(f'_4(u, v), g_{4,4}(u, v)) \mid 3^{32}.$$

Next we will show that both $g_{4,3}(u, v)$ and $g_{4,4}(u, v)$ are not equal, up to a sign, to any power of 3. Suppose, for a contradiction, that $g_{4,3}(u, v) = \pm 3^k$, for some $k > 1$. Then

$$(v - u)^4 \equiv g_{4,3}(u, v) \equiv 0 \pmod{3}.$$

Hence $u \equiv v \pmod{3}$, so $u^3 \equiv v^3 \pmod{3^2}$. Replacing this in the expression of $g_{4,3}(u, v)$, we get

$$0 \equiv u^4 + 2u^3v + 2u^4 + u^3v \equiv 3u^3(u + v) \pmod{3^2}.$$

Then $3 \mid u$ or $3 \mid (u + v)$. Since $u \equiv v \pmod{3}$, the former implies $3 \mid v$, and the latter implies $3 \mid u$ and $3 \mid v$ which is a contradiction as $\gcd(u, v) = 1$. Thus the possibilities for k such that $g_{4,3}(u, v) = \pm 3^k$ are only 0 and 1. Calculating with PARI-GP [25], without assuming the GRH, shows that the only solutions (u, v) of the equation $g_{4,3}(u, v) = \pm 1$ are $[0, \pm 1], [\pm 1, 0]$ and there are no solutions to $g_{4,3}(u, v) = \pm 3$.

A similar argument will be applied for the second factor $g_{4,4}(u, v)$. Suppose $g_{4,4}(u, v) = \pm 3^k$ for some $k > 2$. As $(v - u)^8 \equiv g_{4,4}(u, v) \equiv 0 \pmod{3}$, we have $u \equiv v \pmod{3}$, so that

$$u^3 \equiv v^3 \pmod{3^3}, \quad 10u^3 \equiv v^3 \pmod{3^3}, \quad \text{or } 19u^3 \equiv v^3 \pmod{3^3}.$$

Replacing each of these in the expression of $g_{4,4}$, we find that there are no solutions to $g_{4,4}(u, v) = \pm 3^k$ when $k > 2$. Thus it remains to solve the equations $g_{4,4}(u, v) = \pm 3^k$ when $0 \leq k \leq 2$. The only solutions to

$$g_{4,4}(u, v) = \pm 1 \text{ are } [\pm 1, 0], [0, \pm 1], [-1, 1], [1, -1].$$

However these correspond to values of m (± 1 or 0) which yield rank zero (or singular) curves. Finally, GP says there are no solutions at all to $g_{4,4}(u, v) = \pm 3^k$ when $k = 1, 2$.

We will prove moreover that the multiple $g_{4,3}(u, v)g_{4,4}(u, v)$ cannot be a prime power. As above, $g_{4,3}$ and $g_{4,4}$ are not powers of 3, so write

$$g_{4,3}(u, v) = \pm 3^m p_1^{m_1} \cdots p_r^{m_r} \text{ and } g_{4,4}(u, v) = \pm 3^n q_1^{n_1} \cdots q_s^{n_s},$$

where the p_i and q_j are primes other than 3. Considering the resultant between $g_{4,3}$ and $g_{4,4}$, we get $\gcd(g_{4,3}(u, v), g_{4,4}(u, v)) \mid 3^{10}$. Thus there is at least one prime p_i which is not equal to any prime q_j . This implies W_4 is not a prime power.

Case $n = 3$ The expression for $3R$ can be written as

$$3R = \left(\frac{u^9 + 6u^6v^3 + 3u^3v^6 - v^9}{3uv(u^6 + u^3v^3 + v^6)}, \frac{-u^9 + 3u^6v^3 + 6u^3v^6 + v^9}{3uv(u^6 + u^3v^3 + v^6)} \right).$$

For convenience, let

$$f_3(u, v) = -u^9 + 3u^6v^3 + 6u^3v^6 + v^9 \text{ and } g_3(u, v) = u^6 + u^3v^3 + v^6.$$

By the theory of resultants, we obtain

$$\gcd(f_3(u, v), g_3(u, v)) \mid 3^9.$$

Since at least one of u, v is not a unit, to complete the proof in this case we have to prove that the denominator $g_3(u, v)$ is not (up to sign) a power of 3. Suppose $g_3(u, v) = \pm 3^k$ for some $k > 1$. Then $(u - v)^6 \equiv g_3(u, v) \equiv 0 \pmod{3}$. Thus $u^3 \equiv v^3 \pmod{3^2}$, and hence

$$0 \equiv u^6 + u^3v^3 + v^6 \equiv 3u^6 \pmod{3^2},$$

so $3 \mid u$. This implies $3 \mid v$ which is impossible. For the remaining values, when $k = 1$ the solutions are $[1, 1]$ and $[-1, -1]$ and when $k = 0$, the only solutions are given by

$$(u, v) = [\pm 1, 0], [0, \pm 1], [1, -1], [-1, 1].$$

As before, these all correspond to impossible values of m ($0, \pm 1, \pm 2$). Since $\gcd(u, v) = 1$ and u and v are coprime to both $f_3(u, v)$ and $g_3(u, v)$ but are not both units (as $m \neq 0, \pm 2$), W_3 possesses at least two coprime divisors.

Case $n \equiv 1 \pmod{3}$ The proof in this case proceeds exactly in the same way as in the case $n = 4$, by the following steps.

(i) Write

$$nR = \left(\frac{U_n}{W_n}, \frac{V_n}{W_n} \right) = \left(\frac{f_n(u, v)}{g_n(u, v)}, \frac{f'_n(u, v)}{g_n(u, v)} \right),$$

and factor the denominator $g_n(u, v)$ as $g_{n,1}(u, v), g_{n,2}(u, v), \dots, g_{n,k}(u, v)$, all of which are homogeneous in u and v . By the theory of resultants, we have found fortunately that for each n , $\gcd(f'_n(u, v), g_{n,i}(u, v))$ divides a power of 3 for every $i = 1, \dots, k$.

- (ii) Pick two factors of g_n , say $g_{n,i}(u, v)$ and $g_{n,j}(u, v)$. It can be proved that both of them cannot be a power of 3 (up to sign) by using the following facts:

$$\begin{aligned} g_{n,i}(u, v) &\equiv (u - v)^{\deg(g_{n,i})} \pmod{3}, \\ g_{n,j}(u, v) &\equiv (u - v)^{\deg(g_{n,j})} \pmod{3}. \end{aligned}$$

- (iii) Show that the multiple $g_{n,i}g_{n,j}$ is not a prime power, for which it is sufficient to prove that the resultant of $g_{n,i}$ and $g_{n,j}$ is a power of 3.

Case $n \equiv 2 \pmod{3}$ In this case, the situation is more complicated. For all n , $f'_n(u, v)$ and $g_n(u, v)$ also behave as in the previous case in steps (i) and (iii). However, matters are slightly different in step (ii). We need to employ some facts about Newton polygons over 3-adic fields to know about the 3-adic valuation of $g_{n,i}$. We will show how to do this for $n = 5$ (for other n , the proofs will proceed in the same way). We have

$$g_{5,1}(u, v) = u^8 - 2u^7v - 2u^6v^2 + u^5v^3 - 5u^4v^4 + u^3v^5 - 2u^2v^6 - 2uv^7 + v^8,$$

and

$$\begin{aligned} g_{5,2}(u, v) &= u^{16} + 2u^{15}v + 6u^{14}v^2 - 2u^{13}v^3 + 11u^{12}v^4 \\ &\quad + 21u^{11}v^5 - 11u^{10}v^6 - u^9v^7 + 27u^8v^8 - u^7v^9 - 11u^6v^{10} \\ &\quad + 21u^5v^{11} + 11u^4v^{12} - 2u^3v^{13} + 6u^2v^{14} + 2uv^{15} + v^{16}. \end{aligned}$$

We will explore their roots of the polynomials $h_{5,i}(X) = g_{5,i}(1+X, 1)$, where we find

$$h_{5,1}(X) = X^8 + 6X^7 + 12X^6 + 3X^5 - 30X^4 - 63X^3 - 63X^2 - 36X - 9,$$

and

$$\begin{aligned} h_{5,2}(X) &= X^{16} + 18X^{15} + 156X^{14} + 852X^{13} + 3261X^{12} + 9279X^{11} \\ &\quad + 20394X^{10} + 35496X^9 + 49617X^8 + 55971X^7 + 50814X^6 \\ &\quad + 36774X^5 + 20871X^4 + 9072X^3 + 2916X^2 + 648X + 81. \end{aligned}$$

The Newton polygons for $h_{5,1}$ and $h_{5,2}$ with $p = 3$, as shown in Figures 1 and 2, reveal that the slope of the only segment of each polygon is $-\frac{1}{4}$.

By the standard facts about Newton Polygons, all roots of $h_{5,i}(X)$ (and all roots of $h_{11,i}(X)$ and $h_{17,i}(X)$) have 3-adic absolute values $3^{-\frac{1}{4}}$. If α is a root of $h_{5,i}(X)$ then, since $|u/v - 1|_3 \neq |\alpha|_3$,

$$\left| \frac{u}{v} - (1 + \alpha) \right|_3 \geq |\alpha|_3 = 3^{-\frac{1}{4}}.$$

Thus, since $|u|_3 = |v|_3 = 1$,

$$|g_{5,i}(u, v)|_3 = \left| h_{5,i} \left(\frac{u}{v} - 1 \right) \right|_3 = \prod_{\alpha} \left| \frac{u}{v} - (1 + \alpha) \right|_3 \geq (3^{-\frac{1}{4}})^{\deg(g_{5,i})},$$

where α ranges over all roots of $h_{5,i}(X)$; that is, the 3-adic valuation of $g_{5,i}(u, v)$ is at most $\deg(g_{5,i})/4$. It remains only to solve the equations

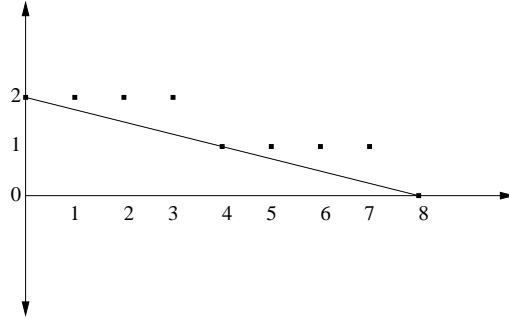


FIGURE 1. Newton polygon of $h_{5,1}(X)$

$g_{5,i}(u, v) = \pm 3^k$, with $0 \leq k \leq \deg(g_{5,i})/4$. We find that the only solutions to $g_{5,1}(u, v) = \pm 3^0$ are $[0, \pm 1], [\pm 1, 0]$ and there are no solutions to $g_{5,1}(u, v) = \pm 3^k$ for other k . Similarly, the only solutions for $g_{5,2}$ arise for $g_{5,2}(u, v) = 3^0$, with $[\pm 1, 0], [0, \pm 1], [-1, 1], [1, -1]$, and for $g_{5,2}(u, v) = 3^4$, with $[1, 1]$ and $[-1, -1]$. As before, these correspond to inadmissible values of m .

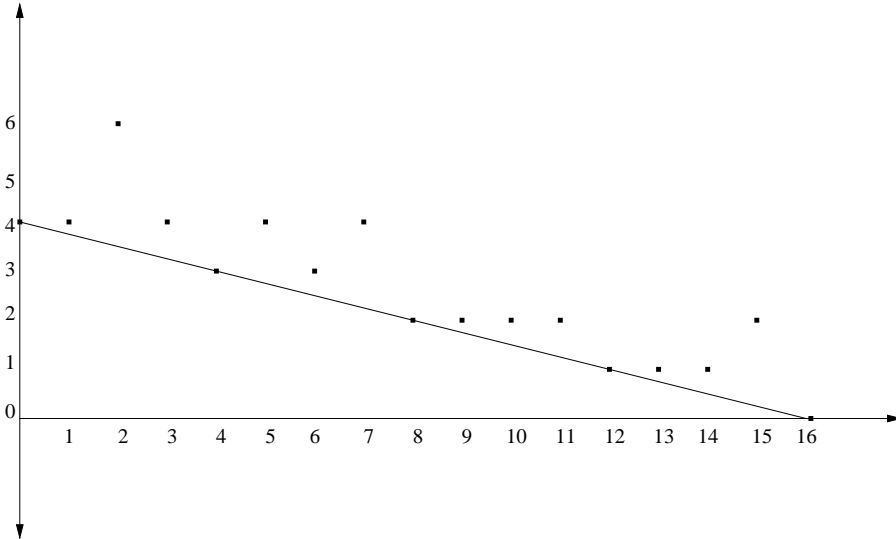


FIGURE 2. Newton polygon of $h_{5,2}(X)$

This concludes the proof of Lemma 2.2 when we consider integral points. In the case of rational points, write

$$R = \left(\frac{u_0}{w_0}, \frac{v_0}{w_0} \right) \in C(\mathbb{Q})$$

in lowest terms. The condition that m is cube-free implies u_0 and v_0 are coprime. Replacing u and v in the expressions of nR in previous cases by u_0/w_0 and v_0/w_0 respectively, we obtain

$$nR = \left(\frac{f_n(u_0, v_0)}{w_0 g_n(u_0, v_0)}, \frac{f'_n(u_0, v_0)}{w_0 g_n(u_0, v_0)} \right).$$

Now proceed with the proof working with $f_n(u_0, v_0)$ and $g_n(u_0, v_0)$: the conclusion follows as before. Thus we have proved Lemma 2.2. \square

Case $n = 2$ To complete the proof of Theorem 1.2 it remains only to treat the case $n = 2$; the salient details follow. Here we begin with the curve C' in (2). In a change of notation, we write $(u, v) \in C'(\mathbb{Q})$ for a point which maps to R under the isogeny τ . For now we assume it is integral, noting that this implies $\gcd(u, v) = 1$ since m is cube-free. We write

$$2R = \left(\frac{f_2(u, v)}{g_2(u, v)}, \frac{f'_2(u, v)}{g_2(u, v)} \right),$$

where

$$g_2(u, v) = -3(u-v)(u+2v)(2u+v)(u^2+uv+v^2) \\ (u^6 + 3u^5v + 60u^4v^2 + 115u^3v^3 + 60u^2v^4 + 3uv^5 + v^6).$$

Write $g_{2,1}(u, v)$ for the product of the linear factors and $g_{2,2}(u, v)$ for the degree 6 factor.

Firstly, we claim that the factor $g_{2,1}(u, v)$ does not completely cancel with the numerator $f_2(u, v)$. This is easy to check: compute the resultant of each of the linear factors in $g_{2,1}(u, v)$ with $f_2(u, v)$. Every time you obtain a power of 3. If $g_{2,1}(u, v)$ cancels then each of the linear factors is a power of 3 (possibly 3^0). This cannot happen since

$$(2u + v) - (u + 2v) = u - v$$

and the equation $3^a + 3^b = 3^c$ has no solutions.

Similarly, the factor $g_{2,2}(u, v)$ is not cancelled by $f_2(u, v)$. Checking resultants shows any common division is a power of 3. Now $g_{2,2}(u, v) \equiv 0 \pmod{3^6}$ forces $u \equiv v \equiv 0 \pmod{3}$ so we only need to consider

$$g_{2,2}(u, v) = \pm 3^r \text{ with } \gcd(u, v) = 1 \text{ and } r = 0, \dots, 5.$$

The only solutions occur when $g_{2,2}(u, v) = 1$ and $g_{2,2}(u, v) = 3^5$. In the first case, they are

$$[1, 0], [-1, 0], [1, -1], [0, -1], [-1, 1], [0, 1],$$

each giving the inadmissible value $m = 0$. In the second case, they are

$$[1, -2], [2, -1], [-1, -1], [-2, 1], [1, 1], [-1, 2],$$

giving the inadmissible values $m = \pm 2$.

Finally, the only common divisor of $g_{1,2}(u, v)$ and $g_{2,2}(u, v)$ is a power of 3 so we must obtain one non-trivial factor of W_2 from the first term and a coprime factor from the second, which proves that W_2 is not a prime power.

When (u, v) is *not* integral, things are slightly different from the situation for $2 < n \leq 22$ above, because u and v might have different denominators. Write $u = a/hb$ and $v = c/hd$ with $\gcd(a, b) = \gcd(c, d) = \gcd(b, d) = 1$. Then also $\gcd(a, c) = 1$ (as m is cube-free). Writing $u_0 = ad$ and $v_0 = bc$, which are coprime, we obtain

$$2R = \left(\frac{f_2(u_0, v_0)}{bdhg_2(u_0, v_0)}, \frac{f'_2(u_0, v_0)}{bdhg_2(u_0, v_0)} \right)$$

so we can proceed as before. \square

6. APPENDIX

In order to complete the proof of Lemma 5.1, it remains to check the statement (20) for all cube-free integers m up to 353, as mentioned at the end of the proof. In this part, we deal with the particular computations to find a uniform bound, N_0 , on the indices n for such m . Ranks and generators of $E : Y^2 = X^3 - 432m^2$ were computed using MAGMA [23]. Note that when $m = 337$, we were unable to find the generator and rank using MAGMA. This was found instead using SAGE [29]. For rank-1 curves, we tested the elliptic divisibility sequence (B_n) arising from the generator for $n = 1, \dots, 22$. A special argument is required for the curves of rank 2, with two parts needed to find the bound N_0 . Firstly find the finite set of pairs (i, j) , $i, j \in \mathbb{Z}$, such that the canonical height of each point $iP + jQ$ is less than 40, where P and Q represent the generators. Then compute the elliptic divisibility sequence (B_n) arising from each point $iP + jQ$, for $n = 1, \dots, 22$. Now we get a bound, say N'_0 , for the indices n from the points of canonical height less than 40. To treat all cases, when $h > 40$, we return to the proof of Lemma 5.1 again, replacing the estimate (19) by

$$h' > \frac{h}{3} > \frac{40}{3}.$$

This leads to

$$\frac{40}{3}n^2 - \frac{2}{3}\log m - \frac{1}{2}\log 48 - 2.14 > \log(8m).$$

Taking specific values for m such that E has rank-2 gives another bound, say N''_0 , for the indices n . Comparing N'_0 and N''_0 , let

$$N_0 = \min\{N'_0, N''_0\}.$$

In no case did N_0 exceed 1 for curves of rank 1 and 2. There are no curves of higher rank appearing in that range.

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