

The supercuspidal representations of p -adic classical groups

Shaun Stevens*

12 November 2007

Abstract

Let G be a unitary, symplectic or special orthogonal group over a locally compact non-archimedean local field of odd residual characteristic. We construct many new supercuspidal representations of G , and Bushnell-Kutzko types for these representations. Moreover, we prove that every irreducible supercuspidal representation of G arises from our constructions.

Introduction

Let G be a unitary, symplectic or special orthogonal group over a locally compact non-archimedean local field of odd residual characteristic. This article completes a series of papers (in particular [31, 32]) whose major goal was the construction of all irreducible supercuspidal representations of G .

More precisely, in this paper, we construct pairs (J, λ) , where J is a compact open subgroup of G and λ is an irreducible representation of J , such that the compactly induced representation $c\text{-Ind}_J^G \lambda$ is irreducible and supercuspidal. Moreover, any irreducible supercuspidal representation π of G contains some such pair, in the sense that λ is a component of the restriction of π to J , and hence $\pi \simeq c\text{-Ind}_J^G \lambda$. In particular, this shows that every irreducible supercuspidal representation of G can be obtained by irreducible compact induction from a compact open subgroup.

The study of the representation theory of p -adic groups by restriction to compact open subgroups goes back to the work of Howe [17] and, for the general linear group in arbitrary (residual) characteristic, was completed by Bushnell and Kutzko in [8]. For classical groups G in odd residual characteristic, we can go back to the work of Moy on $U(2, 1)$ [23] and GSp_4 [24, 25], and, for a general classical group, of Morris in a series of papers (see, for example, [19, 20]) which laid much of the necessary groundwork.

More recently, results have been obtained for arbitrary connected reductive groups. The level-zero representations – that is, those representations which contain the trivial character on restriction to the pro-unipotent radical of some parahoric subgroup – were classified independently by Morris [22] and Moy and Prasad [26]. For positive-level representations in the so-called *tame* case, the most general constructions of supercuspidal representations are due to Yu [33].

Recently in [18], Kim has proved that, under some restrictive conditions on the ground field (dependent on the group G), Yu's constructions do give all supercuspidal representations. Although our work here is not for such a general class of group, it has the notable advantage that the only restriction on the field is that the residual characteristic be odd.

*The research for this paper was supported by EPSRC grant GR/T21714/01.

Our work for classical groups G follows the spirit, and the methods, of Bushnell and Kutzko's work [8]. So we begin with a non-archimedean local field F of odd residual characteristic with a (possibly trivial) galois involution with fixed field F_0 , and a finite-dimensional F -vector space V equipped with a nondegenerate ε -hermitian form h . Then G is the group of F_0 -rational points of the connected component of the linear algebraic group preserving the form. It will also be useful to have the full group G^+ of automorphisms of V preserving the form.

The construction of supercuspidal representations begins with certain local data, namely a *skew semisimple stratum* $[\Lambda, n, 0, \beta]$ (see §2.1). Here we recall only that Λ is a lattice sequence in V and β is an endomorphism of V which is skew for the adjoint involution induced by h and such that $E = F[\beta]$ is a direct sum of fields each preserved by this involution – that is, β is an elliptic semisimple element of the Lie algebra of G . We write G_E for the centralizer of β in G ; this is a product of classical groups over extensions of F_0 . The lattice sequence Λ defines a compact open subgroup $P(\Lambda)$ in G , together with a filtration, and we write $P(\Lambda_{\mathfrak{o}_E})$ for the intersection $P(\Lambda) \cap G_E$.

In [32], we defined the set $\mathcal{C}_-(\Lambda, 0, \beta)$ of *skew semisimple characters* attached to a skew semisimple stratum, generalizing the notion of *simple character* from [8]. These are certain arithmetically defined abelian characters θ of a compact open subgroup $H^1(\beta, \Lambda)$ with nice intertwining properties. Moreover, for fixed β but varying lattice sequence Λ , these semisimple characters exhibit remarkable functorial properties, known as *transfer properties* – again, these generalize such properties for general linear groups.

Attached to our stratum, there are two other compact open subgroups $J(\beta, \Lambda) \supseteq J^1(\beta, \Lambda)$ containing $H^1(\beta, \Lambda)$ and normalizing each semisimple character θ . Moreover, it is shown in [32] that there is a unique irreducible representation η of $J^1(\beta, \Lambda)$ (a *Heisenberg extension*) containing a given semisimple character θ , that the intertwining of η is the same as that of θ and, moreover, that every intertwining space is 1-dimensional.

Our first task in this article is to extend the representation η to the group $J(\beta, \Lambda)$ – this is known as the β -*extension* problem. In fact, finding such an extension is not difficult – given the formalism and transfer properties of semisimple characters, the same arguments as in the case of general linear groups in [8] can be applied. The major difficulty is in knowing whether we have a *suitable* extension – in general, many of the extensions are not at all appropriate for our purposes.

For $GL_n(F)$ (or, more generally, general linear groups over a division algebra – see [27]) the crucial property of β -extensions is that their intertwining is the same as that of the representation η which they extend. In our situation for classical groups, we have not been able to prove that any extensions have this property. Instead, we characterize β -extensions by certain compatibility (or transfer) properties, beginning with the case where the group $P(\Lambda_{\mathfrak{o}_E})$ is a maximal compact open subgroup of G_E , analogously to the techniques in [8]. However, even these compatibilities have subtleties which do not appear for $GL_n(F)$.

In the case where the order $P(\Lambda_{\mathfrak{o}_E})$ is maximal, we actually extend η to the group $J^+(\beta, \Lambda)$, which is the group analogous to $J(\beta, \Lambda)$ in G^+ – the representation of $J(\beta, \Lambda)$ we seek will then be given by restriction. The methods of [8] generalize to our case and we can construct an extension κ of η whose restriction to a pro- p Sylow subgroup of $J^+(\beta, \Lambda)$ is determined by the transfer properties of semisimple characters. Indeed, in most cases any extension will do – the only exception is in the case where one of the simple components of the quotient $J^+(\beta, \Lambda)/J^1(\beta, \Lambda)$, which is a product of reductive groups over finite fields but may be disconnected, is isomorphic to $SL_2(\mathbb{F}_3)$. This is similar to the situation for $GL_n(F)$, where the only difficulty is with $GL_2(\mathbb{F}_2)$.

For general Λ , we choose a lattice sequence Λ^M such that $P(\Lambda_{\mathfrak{o}_E}^M)$ is a maximal compact subgroup of

G_E containing $P(\Lambda_{\mathfrak{o}_E})$ and let κ_M be a β -extension to $J^+(\beta, \Lambda^M)$ corresponding to the semisimple character in $\mathcal{C}_-(\Lambda^M, 0, \beta)$ which is the transfer of θ . We observe here that, unlike the case of $GL_n(F)$, this does *not* imply that the corresponding subgroups of G satisfy $P(\Lambda^M) \supseteq P(\Lambda)$. This complication has already been encountered for general linear groups over division algebras in [27]. In the case where we do have this inclusion, we may follow [8] and define a β -extension κ to $J^+(\beta, \Lambda)$ to be the (unique) one which satisfies an induction compatibility with respect to κ_M , namely

$$\mathrm{Ind}_{J^+(\beta, \Lambda)}^{\mathrm{P}^+(\Lambda^M)} \kappa \simeq \mathrm{Ind}_{\mathrm{P}^+(\Lambda_{\mathfrak{o}_E})J^1(\beta, \Lambda^M)}^{\mathrm{P}^+(\Lambda^M)} \left(\kappa_M|_{\mathrm{P}^+(\Lambda_{\mathfrak{o}_E})J^1(\beta, \Lambda^M)} \right),$$

where $\mathrm{P}^+(\Lambda)$ is the compact open subgroup of G^+ defined by Λ . When we do not have the containment of subgroups, we imitate [27] and proceed in steps from Λ to Λ^M by tracing a line through the Bruhat-Tits building of G and cutting it into sufficiently small pieces. Here we are using the description of the building in terms of *lattice functions* (see [6] or §2.2).

There is an important difference with previous work to note here, namely that the different possible choices for Λ^M are not conjugate in G^+ (and, in general, not in the group of similitudes either). This gives the added complication that the definition of β -extension depends *a priori* on the choice of Λ^M . Indeed, it is easy to see that different choices can give different *numbers* of β -extensions. It is not clear whether or not there is, in general, an extension of η to $J^+(\beta, \Lambda)$ which is a β -extension for every choice of Λ^M and this is at the heart of the problem of computing intertwining. On the other hand, we do show (Corollary 6.13) that β -extensions for different choices of Λ^M cannot differ by too much. It is essentially this same problem which prevents us from adopting the “bottom-up” approach to β -extensions of [27], beginning with the case when $P(\Lambda_{\mathfrak{o}_E})$ is an Iwahori subgroup.

There is, nonetheless, a “canonical” choice for Λ^M . (In fact, there are two choices which could be described as canonical, and we choose one of them – see (4.11).) We call β -extensions defined relative to this canonical choice *standard β -extensions*. The point here is that $P(\Lambda_{\mathfrak{o}_E}^M)$ is chosen so that the Weyl group of its reductive quotient is as large as possible – so it is “as close as possible” to being special. Using this large Weyl group, and an Iwahori decomposition with respect to a suitable Levi subgroup of G , we are able to compute a lower bound for the intertwining of standard β -extensions (see Proposition 6.14), which is good enough for our purposes.

Now we add a level zero piece. Let $J^\circ(\beta, \Lambda)$ denote the inverse image in $J(\beta, \Lambda)$ of the connected component of the reductive quotient $J(\beta, \Lambda)/J^1(\beta, \Lambda) \cong P(\Lambda_{\mathfrak{o}_E})/P^1(\Lambda_{\mathfrak{o}_E})$; likewise, we write $P^\circ(\Lambda_{\mathfrak{o}_E})$ for the inverse image in $P(\Lambda_{\mathfrak{o}_E})$, which is a parahoric subgroup of G_E . Let ρ be the inflation to $J^\circ(\beta, \Lambda)$ of an irreducible *cuspidal* representation of the quotient $P^\circ(\Lambda_{\mathfrak{o}_E})/P^1(\Lambda_{\mathfrak{o}_E})$ and set $\lambda = \kappa \otimes \rho$.

We can now bound above the G -intertwining of λ (see § 6.4). The key point is that the intertwining of the restriction of κ to a pro- p Sylow subgroup is the same as that of η so we can first bound the intertwining in terms of the intertwining of the restriction of ρ to a pro- p Iwahori subgroup of $P^\circ(\Lambda_{\mathfrak{o}_E})$. Now this intertwining can be bounded by a slight weakening of the hypotheses of a result of Morris in [21] (see Proposition 1.1). The idea that knowing the intertwining of the restriction to the pro- p Iwahori can be enough to bound intertwining has been used already in a special case by Blondel (see [3, page 553]). Of course, this only works because the level zero piece ρ is *cuspidal*.

In the special case where $P(\Lambda_{\mathfrak{o}_E})$ is a maximal compact subgroup of G_E , the intertwining of λ is found to be just $J(\beta, \Lambda)$. In particular, if τ is any irreducible representation of $J(\beta, \Lambda)$ whose restriction to $J^\circ(\beta, \Lambda)$ contains cuspidal ρ , then the compactly induced representation

$$\pi = \mathrm{c}\text{-Ind}_{J(\beta, \Lambda)}^G (\kappa \otimes \tau)$$

is irreducible and hence supercuspidal. Moreover, the pair $(J(\beta, \Lambda), \kappa \otimes \tau)$ is a $[G, \pi]_G$ -type, in the sense of Bushnell-Kutzko [9].

Finally, it remains to show that we have constructed every supercuspidal representation of G in this way. The ideas are similar to those in [8] (see also [28] for the case of general linear groups over division algebras) though the details are somewhat more complicated. From [32], we know already that any irreducible supercuspidal representation π contains a semisimple character θ . Then, possibly after changing the lattice sequence Λ , it is not hard to see that π must contain a representation $\vartheta = \kappa \otimes \rho$ of $J^\circ(\beta, \Lambda)$, for κ a standard β -extension and ρ a *cuspidal* representation. The idea then is to show that, if the compact subgroup $P(\Lambda_{\mathfrak{o}_E})$ is not maximal then π has a non-zero Jacquet module, contradicting the cuspidality of π .

In order to construct such a Jacquet module we use Bushnell-Kutzko's method of *covers*. There is a (choice of) Levi subgroup M associated to the stratum such that, for any parabolic subgroup $P = MU$ with Levi factor M , the group $J^\circ(\beta, \Lambda)$ has an Iwahori decomposition with respect to (M, P) . We form the group

$$J_P^\circ = H^1(\beta, \Lambda)(J^\circ(\beta, \Lambda) \cap P)$$

and let ϑ_P be the natural representation of J_P° on the $(J^\circ(\beta, \Lambda) \cap U)$ -fixed vectors of ϑ . Then π certainly also contains ϑ_P . Moreover, the pair (J_P°, ϑ_P) is a *decomposed pair* – that is, ϑ_P restricts trivially to the unipotent radical of any parabolic subgroup with Levi component M , and the restriction $\vartheta_P|_{J_P^\circ \cap M}$ is irreducible.

In order to show that (J_P°, ϑ_P) is a cover, it remains to show that there is an invertible element of the Hecke algebra $\mathcal{H}(G, \vartheta_P)$ supported on $J_P^\circ \zeta J_P^\circ$, for some *strongly* (P, J_P°) -positive element ζ of the centre of M . Then the theory of covers shows that π has a non-zero Jacquet module.

We note that, since the compact open subgroup $P(\Lambda_{\mathfrak{o}_E})$ is non-maximal, the reductive quotient $P^\circ(\Lambda_{\mathfrak{o}_E})/P^1(\Lambda_{\mathfrak{o}_E})$ has a simple component isomorphic to some general linear group (over an extension of the residue field of F_0). There is then an involution on these simple components, induced by the Weyl group elements introduced earlier. We will need to divide into two cases:

(i) If the component of ρ on some such general linear group is *not* fixed by the involution, then the bound on intertwining from above gives the bound

$$I_G(\vartheta_P) \subseteq J_P^\circ M' J_P^\circ,$$

for some proper Levi subgroup M' of G containing M . Then it is straightforward that (J_P°, ϑ_P) is a cover of $(J_P^\circ \cap M', \vartheta_P|_{J_P^\circ \cap M'})$ (see §7.2.1).

(ii) Otherwise every component of ρ is fixed by the involution and we use an idea of Kutzko (which was also used in [16]). If Λ^M is a lattice sequence such that $P(\Lambda_{\mathfrak{o}_E}^M)$ is a maximal compact subgroup of G_E containing $P(\Lambda_{\mathfrak{o}_E})$ then, thinking of ρ as a representation of $P(\Lambda_{\mathfrak{o}_E})$, we get a support-preserving injection of Hecke algebras (see (7.3))

$$\mathcal{H}(P(\Lambda_{\mathfrak{o}_E}^M), \rho \otimes \chi) \hookrightarrow \mathcal{H}(G, \vartheta_P),$$

for some self-dual character χ (which comes from the problem of incompatibility of β -extensions relative to different maximal compact subgroups of G_E). Moreover, the Hecke algebra on the left, which is just in level zero, is understood via the very general results of Morris in [21]. In particular, it contains an invertible function supported on a certain Weyl group element (see §7.2.2 for details). If we do this for two choices of Λ^M , we obtain two invertible elements of $\mathcal{H}(G, \vartheta_P)$ whose convolution

is supported on a single double coset $J_P^\circ \xi J_P^\circ$, with $\xi \in M$. Raising this to a suitable power, we obtain the invertible function we were seeking.

We end with some remarks on questions which are still unanswered. Firstly, although we present here a construction which gives all irreducible supercuspidal representations of G , we do not know when two apparently distinct constructions give rise to equivalent representations. More precisely, we do not have an “intertwining implies conjugacy” result for the types we construct – indeed, we do not even have such a result for the strata underlying the types.

Secondly, there still remains the problem of constructing types for the non-supercuspidal components of the Bernstein spectrum. Types have been constructed in certain special cases: see [3] and [16], where types are constructed for self-dual supercuspidal representations of the Siegel Levi subgroup; and [4], where (sometimes putative) covers in smaller symplectic groups are “propagated” to larger ones – note that our work here shows that the hypothesis (H4) of [4] can always be satisfied. There are also some results in this direction in [14], where it is shown that any smooth irreducible representation of G contains some (suitably generalized) semisimple character. We remark that, throughout this work and its predecessor [32], numerous covers are constructed. It seems likely that a more careful examination, especially of the Hecke algebras in level zero used in proving exhaustion, should give types in all cases.

Acknowledgements I would like to thank: David Goldberg, and Purdue University, for their hospitality at an early stage of this work; the University of East Anglia, for granting me study leave, which gave me the chance finally to write up all the details of the arguments; and, in no particular order, Colin Bushnell, Guy Henniart, Corinne Blondel, Phil Kutzko and Vincent Sécherre, for their support, advice, and interest in this work at various times. I would also like to thank Corinne Blondel, Michitaka Miyauchi and the referee for their careful reading, and for pointing out some errors and missing explanations.

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1 A variant of a result of L. Morris

In this section, we give a mild weakening of the hypotheses in a result of Lawrence Morris ([21, Proposition 4.13]). Since this result is valid in great generality, we retain all the notation of *op. cit.*; in particular, the notation of this section is independent of that in the rest of the paper. We also refer the reader to *op. cit.* (or to [11], [12]) for details of the situation, which we recall briefly below.

Let \mathbf{G} be a connected reductive group over a non-archimedean local field F and let $G = \mathbf{G}(F)$ be its group of F -points. Let \mathbf{T} be a maximal F -split torus in \mathbf{G} and let N be its normalizer; we also write $T = \mathbf{T}(F)$ and $N = \mathbf{N}(F)$. Let B be an Iwahori subgroup of G , which is the stabilizer of a chamber in the apartment of the enlarged Bruhat-Tits building of G associated to T . Then (G, B, N) is a generalized affine BN-pair.

Associated to this affine BN-pair, we have the generalized affine Weyl group $W = N/B \cap N$; it is the semi-direct product of a subgroup Ω and a normal subgroup W' which is the affine Weyl group for some split affine root system Σ . We also have a set S of fundamental reflections in W' corresponding to a simple system of affine roots.

Given any proper subset $J \subseteq S$, we let W_J be the subgroup of W generated by the reflections in J and let N_J be the corresponding subgroup of N . Then $P_J = BN_JB$ is a subgroup of G , which we call a *standard parahoric subgroup*. For J, K subsets of S , Morris defines in [21, §3.10] (see also *loc. cit.* Variant 3.22) a (non-unique, in general) set of *distinguished double coset representatives* for $W_J \backslash W / W_K$, which we denote $D_{J,K}$. We can then choose $D'_{J,K}$ to be a set of inverse images in N of $D_{J,K}$, and this is a set of *distinguished double coset representatives* for $P_J \backslash G / P_K$.

Given P_J a standard parahoric subgroup of G , we denote by U_J its pro-unipotent radical, and by M_J the quotient P_J / U_J , (the points of) a connected reductive group over a finite field. We write U_B for the pro-radical U_\emptyset of $B = P_\emptyset$.

Finally, for ρ a representation of a compact subgroup K of G , we write $\mathcal{H}(\rho|K) = \text{End}_G(\text{c-Ind}_K^G \rho)$ for the associated Hecke algebra and $I_G(\rho|K)$ for the intertwining of ρ , which is the support of the Hecke algebra.

Proposition 1.1 (cf. [21, Proposition 4.13]). *Let J be a proper subset of S , let $D'_{J,J}$ be a set of distinguished double coset representatives for $P_J \backslash G / P_J$ and let $n \in D'_{J,J}$. Let ρ be the inflation to P_J of a cuspidal representation of M_J . If n lies in the support of $\mathcal{H}(\rho|U_B)$, then $wJ = J$, where $w \in W$ is the projection of n .*

Remark The only difference between this and [21, Proposition 4.13] is that the hypothesis there is that n lies in the support of $\mathcal{H}(\rho|P_J)$.

Proof Write $P = P_J$ and let P' denote nPn^{-1} , with radical U' , and $U'_B = nU_Bn^{-1}$. We note that we have $P \cap U' \subseteq U_{J \cap wJ} \subseteq U_B$, by [21, Lemma 3.21], and $U' \subseteq U'_B$ so we have $P \cap U' \subseteq U_B \cap U'_B$. Then, if we replace every $P \cap P'$ in the proof of *op. cit.* Proposition 4.13 by $U_B \cap U'_B$, the proof is identical. \blacksquare

The interest in weakening the hypothesis to one which says only that the restriction of ρ to U_B is intertwined by n is suggested by the work of Blondel in [3], in which she is able to compute sufficient information on intertwining without calculating it completely (*op. cit.* Proposition IV.1 and Lemma IV.2). We will wish to use it in two different cases:

- (i) The case when $P = P_J$ is a non-maximal parahoric subgroup of G . In this case, let ${}^c\Phi_J$ denote the closure of the root system generated by the gradients of the affine roots in J and let $\mathfrak{M}_J = \mathfrak{M}_J(F)$ be the group of points of the corresponding reductive subgroup of \mathbf{G} (see [21, §3.15]). Then $\mathfrak{M}_J \cap P_J$ is a parahoric subgroup of \mathfrak{M}_J with reductive quotient M_J again. We also put $N_J = N \cap \mathfrak{M}_J \cap P_J$. Then a set of representatives for $\{n \in N : wJ = J\} / N_J$ can be taken in the normalizer $N_N(\mathfrak{M}_J \cap P_J)$ of $\mathfrak{M}_J \cap P_J$ in N (see [21, §4.16]).
- (ii) The case where $P = P_J$ is a maximal parahoric subgroup of G . In this case, the conclusion from the previous lemma is that n normalizes P (see [21, Appendix A.1]). In the classical groups we will be looking at, the G -normalizer of P is compact.

We will also need to use the following stronger result from [21]:

Theorem 1.2 ([21, Theorem 4.15]). *Let J be a proper subset of S and let ρ be the inflation to P_J of a cuspidal representation of M_J . Then $I_G(\rho) \subseteq P_J N_N(\rho) P_J$, where $N_N(\rho) = \{n \in N_N(\mathfrak{M}_J \cap P_J) : {}^n\rho \simeq \rho\}$.*

2 Notations and preliminaries

After setting up the notation for the remainder of this paper, we will recall some of the basic objects needed in the theory. We give quite a rapid overview, and refer the reader to [8] and [32] for more details.

2.1 Strata

Let F be a non-archimedean local field equipped with a galois involution $\bar{}$ with fixed field F_0 ; we allow the possibility $F = F_0$. Let \mathfrak{o}_F be the ring of integers of F , \mathfrak{p}_F its maximal ideal and

$k_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue field; we assume throughout that the residual characteristic $p := \text{char } k_F$ is not 2. We denote by $\mathfrak{o}_0, \mathfrak{p}_0, k_0$ the same objects in F_0 , and will use similar notation for any non-archimedean local field. We fix a uniformizer ϖ_F of F such that $\overline{\varpi_F} = -\varpi_F$ if F/F_0 is ramified, $\overline{\varpi_F} = \varpi_F$ otherwise. We put $\varpi_0 = \varpi_F^2$ if F/F_0 is ramified, $\varpi_0 = \varpi_F$ otherwise; so ϖ_0 is a uniformizer of F_0 . We will use analogous notation for any field extension E of F to which the involution extends.

Let V be an N -dimensional vector space over F , equipped with a nondegenerate ϵ -hermitian form, with $\epsilon = \pm 1$. We put $A = \text{End}_F V$ and denote by $\bar{}$ the adjoint (anti-)involution on A induced by h . Set also $\tilde{G} = \text{Aut}_F V$ and let σ be the involution given by $g \mapsto \bar{g}^{-1}$, for $g \in \tilde{G}$. We also have an action of σ on the Lie algebra A given by $a \mapsto -\bar{a}$, for $a \in A$. We put $\Sigma = \{1, \sigma\}$, where 1 acts as the identity on both \tilde{G} and A .

We put $G^+ = \tilde{G}^\Sigma = \{g \in \tilde{G} : h(gv, gw) = h(v, w) \text{ for all } v, w \in V\}$, the F_0 -points of a unitary, symplectic or orthogonal group \mathbf{G}^+ over F_0 . Let G be the F_0 -points of the connected component \mathbf{G} of \mathbf{G}^+ , so that $G = G^+$ except in the orthogonal case.

We put $A_- = A^\Sigma \simeq \text{Lie } G$. In general, for S a subset of A , we will write S_- or S^- for $S \cap A_-$, and, for \tilde{H} a subgroup of \tilde{G} , we will write H for $\tilde{H} \cap G$.

Let ψ_0 be a character of the additive group of F_0 , with conductor \mathfrak{p}_0 . Then we put $\psi_F = \psi_0 \circ \text{tr}_{F/F_0}$, a character of the additive group of F with conductor \mathfrak{p}_F , and $\psi_A = \psi_F \circ \text{tr}_{A/F}$.

An \mathfrak{o}_F -lattice sequence in V is a function Λ from \mathbb{Z} to the set of \mathfrak{o}_F -lattices in V such that

- (i) $\Lambda(k) \subseteq \Lambda(j)$, for $k \geq j$;
- (ii) there exists a positive integer $e = e(\Lambda|\mathfrak{o}_F)$, called the \mathfrak{o}_F -period of Λ , such that $\varpi_F \Lambda(k) = \Lambda(k + e)$, for all $k \in \mathbb{Z}$.

For L an \mathfrak{o}_F -lattice in V , we put $L^\# = \{v \in V : h(v, L) \subseteq \mathfrak{p}_F\}$. Then we call an \mathfrak{o}_F -lattice sequence Λ *self-dual* if there exists $d \in \mathbb{Z}$ such that $\Lambda(k)^\# = \Lambda(d - k)$, for all $k \in \mathbb{Z}$. Without changing any of the objects associated to a self-dual \mathfrak{o}_F -lattice sequence Λ (except for a scale of the indices), **we may (and do) normalize all self-dual lattice sequences Λ so that $d = 1$ and so that all periods are even.** There is also a well-defined notion of the direct sum of lattice sequences (see [10, §2]). The orthogonal direct sum of self-dual lattice sequences is itself self-dual, by the assumption $d = 1$.

Associated to an \mathfrak{o}_F -lattice sequence Λ in V , we have a decreasing filtration $\{\mathfrak{a}_n(\Lambda) : n \in \mathbb{Z}\}$ of A by \mathfrak{o}_F -lattices; \mathfrak{a}_0 is a hereditary \mathfrak{o}_F -order in A and \mathfrak{a}_1 is its Jacobson radical. The filtration on A also gives rise to a valuation ν_Λ on A , with $\nu_\Lambda(0) = +\infty$. If Λ is self-dual, then each $\mathfrak{a}_n(\Lambda)$ is fixed by σ and $\mathfrak{a}_n^- = \mathfrak{a}_n^-(\Lambda) = \mathfrak{a}_n(\Lambda) \cap A_-$ gives a filtration of A_- by \mathfrak{o}_F -lattices. Moreover, ν_Λ is fixed by σ .

Given an \mathfrak{o}_F -lattice sequence Λ , we also put $\tilde{P} = \tilde{P}(\Lambda) = \mathfrak{a}_0(\Lambda)^\times$, a compact open subgroup of \tilde{G} , and $\tilde{P}_n = \tilde{P}_n(\Lambda) = 1 + \mathfrak{a}_n(\Lambda)$, for $n > 0$, a filtration of $\tilde{P}(\Lambda)$ by normal subgroups. If Λ is self-dual, then \tilde{P}, \tilde{P}_n are fixed by σ and we put $P^+ = P^+(\Lambda) = \tilde{P} \cap G^+$, a compact open subgroup of G^+ , and $P = P(\Lambda) = P^+ \cap G$. We have a filtration of $P(\Lambda)$ by normal subgroups $P_n = P_n(\Lambda) = \tilde{P}_n^\Sigma \cap G$, for $n > 0$. We also have, for $n > 0$, a bijection $\mathfrak{a}_n^- \rightarrow P_n$ given by the Cayley map $x \mapsto C(x) = (1 + \frac{x}{2})(1 - \frac{x}{2})^{-1}$, which is equivariant under conjugation by P .

The quotient group $\mathcal{G} = P/P_1$ is (the group of rational points of) a reductive group over the finite field k_{F_0} . However, it is not, in general, connected. We denote by $P^\circ = P^\circ(\Lambda)$ the inverse image

in P of (the group of rational points of) the connected component \mathcal{G}° of \mathcal{G} ; then P° is a parahoric subgroup of G .

Definition 2.1 ([8, §1.5], [10, §3.1]). (i) A *stratum* in A is a 4-tuple $[\Lambda, n, r, b]$, where Λ is an \mathfrak{o}_F -lattice sequence, $n \in \mathbb{Z}$ and $r \in \mathbb{Z}$ with $n \geq r \geq 0$ and $b \in \mathfrak{a}_{-n}(\Lambda)$.

(ii) Two strata $[\Lambda, n, r, b_i]$, $i = 1, 2$, are called *equivalent* if $b_1 - b_2 \in \mathfrak{a}_{-r}(\Lambda)$.

(iii) A stratum $[\Lambda, n, r, b]$ is called *skew* if Λ is self-dual and $b \in A_-$.

(iv) A stratum $[\Lambda, n, r, b]$ is called *null* if $n = r$ and $b = 0$.

For $n \geq r \geq \frac{n}{2} > 0$, an equivalence class of strata corresponds to a character of $\tilde{P}_{r+1}(\Lambda)$, by

$$[\Lambda, n, r, b] \mapsto (\psi_b : x \mapsto \psi_A(b(x-1))), \text{ for } x \in \tilde{P}_{r+1},$$

while an equivalence class of skew strata corresponds to a character of $P_{r+1}(\Lambda)$, by

$$[\Lambda, n, r, b] \mapsto \psi_b^- = \psi_b|_{P_{r+1}(\Lambda)}.$$

Definition 2.2 ([8, Definition 1.5.5], [10, Definition 5.1]). A stratum $[\Lambda, n, r, \beta]$ in A is called *simple* if either it is null or the following four conditions are satisfied:

(i) the algebra $E = F[\beta]$ is a field;

(ii) Λ is an \mathfrak{o}_E -lattice chain (we usually write $\Lambda_{\mathfrak{o}_E}$ when we are thinking of it as such);

(iii) $\nu_\Lambda(\beta) = -n$;

(iv) $k_0(\beta, \Lambda) < -r$.

Here, $k_0(\beta, \Lambda)$ is an integer related to the action of the adjoint map $a_\beta(x) = \beta x - x\beta$ on the filtration associated to Λ . We refer the reader to [8, §1.5] or [29, §5] for the formal definition and a discussion of it.

Now let $[\Lambda, n, r, \beta]$ be a stratum in A and suppose we have a decomposition $V = \bigoplus_{i=1}^l V^i$ into F -subspaces. Let Λ^i be the lattice sequence in V^i given by $\Lambda^i(k) = \Lambda(k) \cap V^i$ and put $\beta_i = \mathbf{1}^i \beta \mathbf{1}^i$, where $\mathbf{1}^i$ is the projection onto V^i with kernel $\bigoplus_{j \neq i} V^j$.

Definition 2.3. (i) We say that $V = \bigoplus_{i=1}^l V^i$ is a *splitting* for the stratum $[\Lambda, n, r, \beta]$ if we have $\Lambda(k) = \bigoplus_{i=1}^l \Lambda^i(k)$, for all $k \in \mathbb{Z}$, and $\beta = \sum_{i=1}^l \beta_i$.

(ii) If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_N\}$ is a basis for V then we say that \mathcal{B} *splits* the stratum $[\Lambda, n, r, \beta]$ if $V = \bigoplus_{i=1}^N F\mathbf{v}_i$ is a splitting for $[\Lambda, n, r, \beta]$.

Definition 2.4 ([32, Definition 3.2]). A stratum $[\Lambda, n, r, \beta]$ in A is called *semisimple* if either it is null or $\nu_\Lambda(\beta) = -n$ and there exists a splitting $V = \bigoplus_{i=1}^l V^i$ for the stratum such that

(i) for $1 \leq i \leq l$, $[\Lambda^i, q_i, r, \beta_i]$ is a simple or null stratum in A^{ii} , where $q_i = r$ if $\beta_i = 0$, $q_i = -\nu_{\Lambda^i}(\beta_i)$ otherwise; and

(ii) for $1 \leq i, j \leq l$, $i \neq j$, the stratum $[\Lambda^i \oplus \Lambda^j, q, r, \beta_i + \beta_j]$ is not equivalent to a simple or null stratum, with $q = \max\{q_i, q_j\}$.

In this case, the splitting is uniquely determined (upto ordering) by the stratum. We will use the block notation $A^{ij} = \text{Hom}_F(V^j, V^i)$ and put $\mathcal{M}_\beta = \bigoplus_{i=1}^l A^{ii}$.

Let $[\Lambda, n, r, \beta]$ be a semisimple stratum as above and put $E = F[\beta] = \bigoplus_{i=1}^l E_i$, where $E_i = F[\beta_i]$. We will sometimes say that Λ is an \mathfrak{o}_E -lattice sequence to mean that $\Lambda = \bigoplus_{i=1}^l \Lambda^i$ and each Λ^i is an \mathfrak{o}_{E_i} -lattice sequence in V^i .

We let $B = B_\beta$ denote the A -centralizer of β , so that $B = \bigoplus_{i=1}^l B_i$, where B_i is the centralizer of β_i in A^{ii} . We write $\tilde{G}_E = B^\times$, $\tilde{G}_i = \text{Aut}_F(V^i)$ and $\tilde{G}_{E_i} = B_i^\times = \tilde{G}_i \cap \tilde{G}_E$, so that $\tilde{M}_\beta = \mathcal{M}_\beta^\times$ is a Levi subgroup of \tilde{G} and $\tilde{G}_E = \prod_{i=1}^l \tilde{G}_{E_i} \subseteq \tilde{M}_\beta$. Each \tilde{G}_{E_i} is (the group of F_0 -points of) the restriction of scalars to F_0 of a general linear group over E_i . We also write $\mathfrak{b}_n(\Lambda) = \mathfrak{a}_n(\Lambda) \cap B$, for $n \in \mathbb{Z}$, which gives the filtration induced on B by thinking of Λ as an \mathfrak{o}_E -lattice sequence.

Definition 2.5. We say that a semisimple stratum $[\Lambda, n, r, \beta]$ is *skew* if Λ is self-dual, $\beta \in A_-$ and the associated splitting $V = \bigoplus_{i=1}^l V^i$ is orthogonal with respect to the form h .

If $[\Lambda, n, r, \beta]$ is skew semisimple with splitting $V = \bigoplus_{i=1}^l V^i$, then the strata $[\Lambda^i, q_i, r, \beta_i]$ are all skew simple strata, for the form h_i induced by h on V^i . The involution on F extends to each E_i , as the adjoint involution, and we write $E_{i,0}$ for the subfield of fixed points; it is a subfield of index 2 except in the case $E_i = F = F_0$ (so that $\beta_i = 0$).

For $[\Lambda, n, r, \beta]$ a skew semisimple stratum, we put $G_E^+ = B \cap G^+$ and $G_E = B \cap G$; then G_E^+ is (the group of F_0 -points of) a reductive group over F_0 and G_E is (the group of F_0 -points of) its connected component. We write $G_E^+ = \prod_{i=1}^l G_{E_i}^+$ and $G_E = \prod_{i=1}^l G_{E_i}$; then each $G_{E_i}^+$ is (the group of F_0 -points of) the restriction of scalars to F_0 of a unitary, symplectic or orthogonal group over $E_{i,0}$; moreover, at most one of the $G_{E_i}^+$ is symplectic or orthogonal – this happens in the case $\beta_i = 0$ and $F = F_0$. If $G_{E_i}^+$ is not orthogonal, then we have $G_{E_i} = G_{E_i}^+$; if it is orthogonal, then G_{E_i} is special orthogonal.

Note that, by [6, Lemmas 5.2–5.5], for each i there is a nondegenerate $E_i/E_{i,0}$ ϵ -hermitian form f_i on V^i such that the two notions of lattice duality for \mathfrak{o}_{E_i} -lattices in V^i given by h_i and by f_i coincide. Then $G_{E_i}^+$ is the group of fixed points in \tilde{G}_{E_i} of the involution determined by f_i .

We write $\tilde{P}_n(\Lambda_{\mathfrak{o}_E}) = \tilde{P}_n(\Lambda) \cap \tilde{G}_E = 1 + \mathfrak{b}_n(\Lambda)$ and $P_n(\Lambda_{\mathfrak{o}_E}) = P_n(\Lambda) \cap G_E$, for $n > 0$. Similarly, we write $\tilde{P}(\Lambda_{\mathfrak{o}_E}) = \tilde{P}(\Lambda) \cap \tilde{G}_E$ and $P^+(\Lambda_{\mathfrak{o}_E}) = P^+(\Lambda) \cap G_E^+$, $P(\Lambda_{\mathfrak{o}_E}) = P(\Lambda) \cap G_E$. Let $P^\circ(\Lambda_{\mathfrak{o}_E})$ denote the inverse image in $P(\Lambda_{\mathfrak{o}_E})$ of the connected component of the quotient $P(\Lambda_{\mathfrak{o}_E})/P_1(\Lambda_{\mathfrak{o}_E})$; note that this can be smaller than $P^\circ(\Lambda) \cap G_E$.

Lemma 2.6 (cf. [8, Theorem 1.6.1]). *Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum. Then*

$$\begin{aligned} P_1(\Lambda)xP_1(\Lambda) \cap G_E^+ &= P_1(\Lambda_{\mathfrak{o}_E})xP_1(\Lambda_{\mathfrak{o}_E}), \quad \text{for } x \in G_E^+; \\ P_1(\Lambda)xP_1(\Lambda) \cap G_E &= P_1(\Lambda_{\mathfrak{o}_E})xP_1(\Lambda_{\mathfrak{o}_E}), \quad \text{for } x \in G_E. \end{aligned}$$

Proof We begin by proving the corresponding statement for \tilde{G} : for $x \in \tilde{G}_E$,

$$\tilde{P}_1(\Lambda)x\tilde{P}_1(\Lambda) \cap \tilde{G}_E = \tilde{P}_1(\Lambda_{\mathfrak{o}_E})x\tilde{P}_1(\Lambda_{\mathfrak{o}_E}). \quad (2.7)$$

The simple case for this is given by [8, Theorem 1.6.1]. For the general case, we have $\tilde{G}_E \subseteq \tilde{M}_\beta$ and \tilde{M}_β is the fixed-point subgroup in \tilde{G} of a 2-group of automorphisms (namely the group of

conjugations by $\sum_{i=1}^l \varepsilon_i \mathbf{1}^i$, for $\varepsilon_i = \pm 1$). Since $p \neq 2$, $x \in \widetilde{M}_\beta$ and $\widetilde{P}_1(\Lambda)$ is a pro- p group, [29, Lemma 2.1] implies that $\widetilde{P}_1(\Lambda)x\widetilde{P}_1(\Lambda) \cap \widetilde{M}_\beta = (\widetilde{P}_1(\Lambda) \cap \widetilde{M}_\beta)x(\widetilde{P}_1(\Lambda) \cap \widetilde{M}_\beta)$. The result now follows from the simple case, by intersecting with \widetilde{G}_E .

Now to prove the first statement of the Lemma, we take the σ -fixed points of (2.7) and apply [29, Lemma 2.1] again. The second statement follows by intersection with G , since $P_1(\Lambda) \subseteq G$. \blacksquare

2.2 Buildings

We will need a few results on inclusions of the hereditary orders given by lattice sequences, which are most easily proved using the Bruhat-Tits affine building. We recall briefly the description of the building in terms of lattice sequences – see [5] and [6] for more details and proofs.

There is a (unique upto translation) \widetilde{G} -set isomorphism between the (extended) affine building $\mathcal{I}(\widetilde{G}, F_0)$ and the set of lattice *functions* on V , of which the lattice sequences are the subset of “rational points” (i.e. barycentres of vertices with rational weights). The involution σ on \widetilde{G} induces an involution on $\mathcal{I}(\widetilde{G}, F_0)$ which, for a unique choice of the isomorphism above, coincides with the map $\Lambda \mapsto \Lambda^\#$, where $\Lambda^\#(k) = (\Lambda(1-k))^\#$, on lattice sequences. Moreover, we can identify the set of fixed points with the affine building $\mathcal{I}(G^+, F_0)$ of G^+ . Thus the affine building is identified with the set of self-dual lattice functions on V . In particular, we will identify a self-dual lattice function with the corresponding points in $\mathcal{I}(G^+, F_0)$ and $\mathcal{I}(\widetilde{G}, F_0)$; thus we can, for example, talk about the line segment joining two lattice sequences.

Lemma 2.8 (cf. [27, Lemme 1.7]). *Given (self-dual) \mathfrak{o}_E -lattice sequences Λ, Λ' with $\mathfrak{b}_0(\Lambda) \supseteq \mathfrak{b}_0(\Lambda')$, there exists a (self-dual) \mathfrak{o}_E -lattice sequence Λ'' such that*

$$\mathfrak{b}_0(\Lambda'') = \mathfrak{b}_0(\Lambda') \quad \text{and} \quad \mathfrak{a}_0(\Lambda) \supseteq \mathfrak{a}_0(\Lambda'').$$

Note that we are identifying an \mathfrak{o}_E -lattice sequence both with a point of $\mathcal{I}(\widetilde{G}_E, F_0)$ and a point of $\mathcal{I}(\widetilde{G}, F_0)$, which gives a continuous affine injection of buildings $\mathcal{I}(\widetilde{G}_E, F_0) \hookrightarrow \mathcal{I}(\widetilde{G}, F_0)$ (see [5]).

Proof We use the following facts, translated into the language of lattice sequences from [27, Lemme 1.5]):

- (i) There is a neighbourhood of Λ in $\mathcal{I}(\widetilde{G}, F_0)$ such that, for all Λ'' in this neighbourhood, $\mathfrak{a}_0(\Lambda) \supseteq \mathfrak{a}_0(\Lambda'')$.
- (ii) There is a sequence of \mathfrak{o}_E -lattice sequences Λ_j such that $\mathfrak{b}_0(\Lambda_j) = \mathfrak{b}_0(\Lambda')$ and the limit of (Λ_j) is Λ .

Note that (ii) is given by [27, Lemme 1.5(ii)] only in the simple case; in the general case, we use the facts that $\mathcal{I}(\widetilde{G}_E, F_0) \simeq \prod_{i=1}^l \mathcal{I}(\widetilde{G}_{E_i}, F_0)$ and $\mathcal{I}(\widetilde{G}_{E_i}, F_0) \simeq \mathcal{I}(B_i^\times, E_i)$, so we can apply the simple case in each $\mathcal{I}(B_i^\times, E_i)$.

The result is now immediate: for large enough j , the sequence Λ_j of (ii) is inside the neighbourhood given by (i) so $\Lambda'' = \Lambda_j$ has the required properties.

For the self-dual case, note that we may assume that the sequence in (ii) consists of self-dual lattice sequences, by replacing Λ_j by the barycentre of Λ_j and $\Lambda_j^\#$. The result now follows in the same way. \blacksquare

Corollary 2.9. *Given any (self-dual) \mathfrak{o}_E -lattice sequence Λ , there exists a (self-dual) \mathfrak{o}_E -lattice sequence Λ^m such that $\mathfrak{b}_0(\Lambda^m)$ is a minimal (self-dual) \mathfrak{o}_E -order in B and $\mathfrak{a}_0(\Lambda^m) \subseteq \mathfrak{a}_0(\Lambda)$.*

Remark The same result is not true for maximal (self-dual) orders: given a (self-dual) \mathfrak{o}_E -lattice sequence Λ , there is *not* necessarily a (self-dual) \mathfrak{o}_E -lattice sequence Λ^M such that $\mathfrak{b}_0(\Lambda^M)$ is a maximal (self-dual) \mathfrak{o}_E -order in B and $\mathfrak{a}_0(\Lambda^M) \supseteq \mathfrak{a}_0(\Lambda)$.

Lemma 2.10. *Let Λ, Λ' be (self-dual) \mathfrak{o}_E -lattice sequences in A with $\mathfrak{b}_0(\Lambda) = \mathfrak{b}_0(\Lambda')$. Then there is a sequence $\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_t = \Lambda'$ of (self-dual) \mathfrak{o}_E -lattice sequences on the line segment $[\Lambda, \Lambda']$ such that, for $1 \leq i \leq t$:*

- (i) $\mathfrak{b}_0(\Lambda_i) = \mathfrak{b}_0(\Lambda)$;
- (ii) either $\mathfrak{a}_0(\Lambda_i) \subseteq \mathfrak{a}_0(\Lambda_{i-1})$ or $\mathfrak{a}_0(\Lambda_i) \supseteq \mathfrak{a}_0(\Lambda_{i-1})$.

Proof Note that any lattice sequence Λ'' on the line segment $[\Lambda, \Lambda']$ is certainly a (self-dual) \mathfrak{o}_E -lattice sequence and we have $\mathfrak{b}_0(\Lambda'') = \mathfrak{b}_0(\Lambda)$. On the other hand, the map $\Lambda'' \mapsto \mathfrak{a}_0(\Lambda'')$ is locally constant on this line segment, except at a discrete (so, in particular, finite) set of points – namely, the points where the line segment meets some wall in the building. This divides the line segment into a finite set of subsegments, on the interior of which $\Lambda'' \mapsto \mathfrak{a}_0(\Lambda'')$ is constant. Taking one point from the interior of each subsegment, together with the endpoints of the subsegments, gives the required sequence. \blacksquare

Remark 2.11. Suppose $\Lambda_1, \Lambda_2, \Lambda$ are \mathfrak{o}_E -lattice sequences such that $\mathfrak{a}_0(\Lambda_i) \supseteq \mathfrak{a}_0(\Lambda)$, for $i = 1, 2$. Then, for any Λ' on the line segment $[\Lambda_1, \Lambda_2]$, we also have $\mathfrak{a}_0(\Lambda') \supseteq \mathfrak{a}_0(\Lambda)$ – this is clear since $\mathfrak{a}_0(\Lambda') \supseteq \mathfrak{a}_0(\Lambda)$ if and only if Λ' is contained in the closure of the facet containing Λ .

Likewise, if $\mathfrak{a}_0(\Lambda_i) \subseteq \mathfrak{a}_0(\Lambda)$, for $i = 1, 2$, then, for any Λ' on the line segment $[\Lambda_1, \Lambda_2]$, we also have $\mathfrak{a}_0(\Lambda') \subseteq \mathfrak{a}_0(\Lambda)$. One can check this using the explicit description of the building in terms of lattice sequences as follows. We choose an apartment containing both Λ_1 and Λ_2 ; then also Λ is in the same apartment, since it is in the closure of each Λ_i . One can then use the explicit description of $\mathfrak{a}_0(\Lambda_i)$ from [5, Corollary 4.5]. The details are left to the reader.

If $\Lambda_1, \dots, \Lambda_t$ are (self-dual) \mathfrak{o}_E -lattice sequences then we write $[\Lambda_1, \dots, \Lambda_t]$ for their (closed) convex hull. If this is actually a polygon then we assume that the Λ_i are distinct and that we have ordered the lattice sequences so that the boundary is $\cup_{i=1}^t [\Lambda_i, \Lambda_{i+1}]$, with the understanding that $\Lambda_{t+1} = \Lambda_0$.

Lemma 2.12. *Let $\Lambda, \Lambda', \Lambda''$ be (self-dual) \mathfrak{o}_E -lattice sequences with $\mathfrak{b}_0(\Lambda)$ and $\mathfrak{b}_0(\Lambda')$ both contained in $\mathfrak{b}_0(\Lambda'')$. Then there is a decomposition of the triangle $[\Lambda, \Lambda', \Lambda'']$ into a union of triangles $[\Lambda_i, \Lambda'_i, \Lambda''_i]$, where $\Lambda_i, \Lambda'_i, \Lambda''_i$ are (self-dual) \mathfrak{o}_E -lattice sequences with the property that $\mathfrak{a}_0(\Lambda_i) \subseteq \mathfrak{a}_0(\Lambda'_i) \subseteq \mathfrak{a}_0(\Lambda''_i)$.*

Note that the hypotheses imply that $\Lambda, \Lambda', \Lambda''$ lie in a common apartment of $\mathcal{I}(\tilde{G}_E, F_0)$; since the embedding of buildings $\mathcal{I}(\tilde{G}_E, F_0) \hookrightarrow \mathcal{I}(\tilde{G}, F_0)$ respects apartments, they also lie in a common apartment of $\mathcal{I}(\tilde{G}, F_0)$, so that the convex hull $[\Lambda, \Lambda', \Lambda'']$ is genuinely a triangle. The same is true in the self-dual case.

Proof The idea is essentially the same as that of Lemma 2.10; indeed, it already gives the result if $\Lambda, \Lambda', \Lambda''$ are collinear, so we assume this is not the case.

Removing all walls of the building from $[\Lambda, \Lambda', \Lambda'']$ splits it into a disjoint union of convex polygons. We split the closure $[\Gamma_1, \dots, \Gamma_t]$ of each such polygon into triangles as follows: choose a point Γ_0 in the interior of the polygon and points Γ'_i on each (open) edge (Γ_i, Γ_{i+1}) , for $1 \leq i \leq t$; then the triangles are

$$[\Gamma_0, \Gamma'_i, \Gamma_i] \quad \text{and} \quad [\Gamma_0, \Gamma'_i, \Gamma_{i+1}], \quad \text{for } 1 \leq i \leq t.$$

Taking all these triangles for all the polygons gives a decomposition as required. \blacksquare

We note that, in the situation of Lemma 2.12, if we actually have $\mathfrak{b}_0(\Lambda) = \mathfrak{b}_0(\Lambda') = \mathfrak{b}_0(\Lambda'')$, then also $\mathfrak{b}_0(\Lambda_i) = \mathfrak{b}_0(\Lambda'_i) = \mathfrak{b}_0(\Lambda''_i) = \mathfrak{b}_0(\Lambda)$.

3 Semisimple characters and Heisenberg extensions

We will now recall results from [32] on the definition and properties of semisimple characters, in particular their “transfer property”. Then we look at the same transfer properties for the Heisenberg extensions which are induced by these characters, as in [8, §5.1] and [27, §2.2], and deduce the existence of certain extensions to some larger pro- p groups. Finally, we characterize these extensions in terms of their intertwining.

3.1 Semisimple characters

We will not repeat in detail the definitions of the objects used here but refer the reader to [8, §3] and [32, §3] for more details, in the simple and semisimple cases respectively.

Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum with associated splitting $V = \bigoplus_{i=1}^l V^i$. We use the same notation as in §2.1, so $\Lambda = \bigoplus_{i=1}^l \Lambda^i$, $\beta = \sum_{i=1}^l \beta_i$, $B = C_A(E) = \bigoplus_{i=1}^l B_i$, $\tilde{G}_E = B^\times$, $\tilde{P}(\Lambda_{\mathfrak{o}_E}) = \tilde{P}(\Lambda) \cap \tilde{G}_E$, etc.

In [32, §3.2] we define, inductively along $k_0(\beta, \Lambda)$, a pair of \mathfrak{o}_F -orders

$$\mathfrak{H}(\beta, \Lambda) \subseteq \mathfrak{J}(\beta, \Lambda) \subseteq \mathfrak{a}_0(\Lambda).$$

This gives us two compact open subgroups $\tilde{H}(\beta, \Lambda) = \mathfrak{H}(\beta, \Lambda)^\times$ and $\tilde{J}(\beta, \Lambda) = \mathfrak{J}(\beta, \Lambda)^\times$ of \tilde{G} , filtered by $\tilde{H}^{m+1}(\beta, \Lambda) = \tilde{H}(\beta, \Lambda) \cap \tilde{P}_{m+1}(\Lambda)$ and $\tilde{J}^{m+1}(\beta, \Lambda) = \tilde{J}(\beta, \Lambda) \cap \tilde{P}_{m+1}(\Lambda)$, for $m \geq 0$. Moreover, we have $\tilde{J}(\beta, \Lambda) = \tilde{P}(\Lambda_{\mathfrak{o}_E})\tilde{J}^1(\beta, \Lambda)$.

When the stratum $[\Lambda, n, 0, \beta]$ is skew, \mathfrak{H} and \mathfrak{J} are stable under the involution so, as usual, we put $J^+(\beta, \Lambda) = \tilde{J}(\beta, \Lambda) \cap G^+$ and $J(\beta, \Lambda) = \tilde{J}(\beta, \Lambda) \cap G$ etc. We also put $J^\circ(\beta, \Lambda) = P^\circ(\Lambda_{\mathfrak{o}_E})J^1(\beta, \Lambda)$, the inverse image in $J(\beta, \Lambda)$ of the connected component of $J(\beta, \Lambda)/J^1(\beta, \Lambda)$.

Returning to the case of a general semisimple stratum, in [32, Definition 3.13] we define, again inductively along $k_0(\beta, \Lambda)$, a set $\mathcal{C}(\Lambda, 0, \beta)$ of abelian characters $\tilde{\theta}$ of $\tilde{H}^1(\beta, \Lambda)$, called *semisimple characters*, by the following two properties:

- (i) we have $\tilde{\theta}|_{\tilde{H}^1 \cap \tilde{M}_\beta} = \bigotimes_{i=1}^l \tilde{\theta}_i$, for some *simple* characters $\tilde{\theta}_i \in \mathcal{C}(\Lambda^i, 0, \beta_i)$ (in the sense of [8, §3.3]);
- (ii) writing $r = -k_0(\beta, \Lambda)$ there is, by [32, Proposition 3.4], a semisimple stratum $[\Lambda, n, r, \gamma]$ equivalent to $[\Lambda, n, r, \beta]$, with $\gamma \in \mathcal{M}_\beta$; then $\tilde{\theta}|_{\tilde{H}^{[\frac{r}{2}]+1}(\beta, \Lambda)} = \tilde{\theta}_0|_{\tilde{H}^{[\frac{r}{2}]+1}(\beta, \Lambda)} \psi_{\beta-\gamma}$, for some semisimple character $\tilde{\theta}_0 \in \mathcal{C}(\Lambda, 0, \gamma)$.

Moreover, by [32, Lemma 3.15], $\tilde{\theta}$ is trivial on the unipotent radical \tilde{U} of any parabolic subgroup of \tilde{G} containing \tilde{M}_β .

When the stratum is skew, this set $\mathcal{C}(\Lambda, 0, \beta)$ is stable under the involution and the set $\mathcal{C}_-(\Lambda, 0, \beta)$ of restrictions to $H^1(\beta, \Lambda)$ of semisimple characters is called the set of *skew semisimple characters*. This can also be described in slightly different terms via the Glauberman correspondence (see [32, §3.6] and also §5.3).

Recall that, given a representation ρ of a subgroup K of G and $g \in G^+$, the g -intertwining space of ρ is

$$I_g(\rho) = I_g(\rho|_K) = \text{Hom}_{K \cap {}^g K}(\rho, {}^g \rho),$$

where ${}^g \rho$ is the representation of ${}^g K = gKg^{-1}$ given by ${}^g \rho(gkg^{-1}) = \rho(k)$. The G -intertwining of ρ is

$$I_G(\rho) = I_G(\rho|_K) = \{g \in G : I_g(\rho) \neq \{0\}\},$$

and, similarly, we have the G^+ -intertwining $I_{G^+}(\rho)$. We recall the following intertwining result:

Proposition 3.1 ([32, Proposition 3.27]). *Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum and $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ a skew semisimple character. Then θ is normalized by $J(\beta, \Lambda)$ and*

$$I_{G^+}(\theta) = J^1(\beta, \Lambda)G_E^+J^1(\beta, \Lambda), \quad I_G(\theta) = J^1(\beta, \Lambda)G_EJ^1(\beta, \Lambda).$$

Of equally great importance is the so-called ‘‘transfer property’’ of semisimple characters:

Proposition 3.2 ([32, Propositions 3.26, 3.32]). *Let $[\Lambda, n, 0, \beta]$ and $[\Lambda', n', 0, \beta']$ be semisimple strata in A . Then there exists a canonical bijection*

$$\tau_{\Lambda, \Lambda', \beta} : \mathcal{C}(\Lambda, 0, \beta) \rightarrow \mathcal{C}(\Lambda', 0, \beta')$$

such that, for $\tilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$, $\tilde{\theta}' := \tau_{\Lambda, \Lambda', \beta}(\tilde{\theta})$ is the unique semisimple character in $\mathcal{C}(\Lambda', 0, \beta')$ such that $\tilde{G}_E \cap I_{\tilde{G}}(\tilde{\theta}, \tilde{\theta}') \neq \emptyset$. Moreover $\tilde{G}_E \subseteq I_{\tilde{G}}(\tilde{\theta}, \tilde{\theta}')$.

If the strata are skew then the bijection $\tau_{\Lambda, \Lambda', \beta}$ commutes with the involution σ ; in particular, $\tau_{\Lambda, \Lambda', \beta}$ restricts to a bijection $\tau_{\Lambda, \Lambda', \beta} : \mathcal{C}_-(\Lambda, 0, \beta) \rightarrow \mathcal{C}_-(\Lambda', 0, \beta')$.

Remark 3.3. In [32, §3.5], the transfer is given in a more general situation between $\mathcal{C}(\Lambda, m, \beta)$ and $\mathcal{C}(\Lambda', m', \beta')$, where $\mathcal{C}(\Lambda, m, \beta)$ denotes the set of restrictions to $\tilde{H}^{m+1}(\beta, \Lambda)$ and m, m' are related by $\lfloor m/e(\Lambda|\mathfrak{o}_F) \rfloor = \lfloor m'/e(\Lambda'|\mathfrak{o}_F) \rfloor$. This is not quite the correct condition on the integers m, m' . It should be replaced as follows: put

$$e = \frac{e(\Lambda|\mathfrak{o}_F)}{\text{lcm}_{1 \leq i \leq l} \{e(E_i/F)\}},$$

and likewise e' ; then we require that

$$\lfloor m/e \rfloor = \lfloor m'/e' \rfloor. \tag{3.4}$$

In particular, this ensures that $\lfloor m/e(\Lambda^i|\mathfrak{o}_{E_i}) \rfloor = \lfloor m'/e(\Lambda'^i|\mathfrak{o}_{E_i}) \rfloor$, for each i , so the transfer from the simple case can be used in each block. Notice that the integers called m_0, m'_0 in [32, §3.5] also satisfy this condition since, by [32, §3.1] we have $k_0(\beta, \Lambda)/e(\Lambda|\mathfrak{o}_F) = k_0(\beta, \Lambda')/e(\Lambda'|\mathfrak{o}_F)$ so surely $k_0(\beta, \Lambda)/e = k_0(\beta, \Lambda')/e'$. In particular, the proof of [32, Proposition 3.26] is valid under the condition (3.4).

3.2 Heisenberg extensions

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum and $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ a skew semisimple character.

Proposition 3.5 ([32, Corollary 3.29, Proposition 3.31]). *There is a unique irreducible representation η of J^1 such that η contains θ . Moreover, $\dim \eta = (J^1 : \mathbb{H}^1)^{\frac{1}{2}}$ and the intertwining of η is given by*

$$\dim I_g(\eta) = \begin{cases} 1 & \text{if } g \in J^1 G_E^+ J^1; \\ 0 & \text{otherwise.} \end{cases}$$

We also have a relationship between the representations η for different \mathfrak{o}_E -lattice sequences Λ .

Lemma 3.6 (cf. [8, Proposition 5.1.2]). *Let $[\Lambda^i, n_i, 0, \beta]$ be skew semisimple strata, for $i = 1, 2$, and let $\theta_i \in \mathcal{C}(\Lambda^i, 0, \beta)$; let η_i be the representation of $J_i^1 = J^1(\beta, \Lambda^i)$ given by Proposition 3.5. Then*

$$\frac{\dim \eta_1}{\dim \eta_2} = \frac{(J_1^1 : J_2^1)}{(P_1(\Lambda_{\mathfrak{o}_E}^1) : P_1(\Lambda_{\mathfrak{o}_E}^2))}.$$

Proof From [32, Lemma 3.17], we have exact sequences in A_-

$$0 \longrightarrow \mathfrak{b}_1(\Lambda^i)_- \longrightarrow \mathfrak{J}^1(\beta, \Lambda^i)_- \xrightarrow{a_\beta} (\mathfrak{J}^1(\beta, \Lambda^i))_-^* \xrightarrow{s} \mathfrak{b}_0(\Lambda^i)_- \longrightarrow 0,$$

where $s : A \rightarrow B$ is a tame corestriction relative to E/F which commutes with the involution (see [8, §1.3] and [30, Lemma 4.4] for the definition). Since the Cayley map is a bijection $\mathfrak{J}^1(\beta, \Lambda^i)_- \rightarrow J_i^1$ (and similarly for H), the result follows from Proposition 3.5 exactly as in the proof of [8, Proposition 5.1.2]. \blacksquare

Now suppose we are given:

- skew semisimple strata $[\Lambda, n, 0, \beta]$, $[\Lambda^m, n_m, 0, \beta]$ and $[\Lambda^M, n_M, 0, \beta]$ with (necessarily) the same splitting, such that $\mathfrak{a}_0(\Lambda^m) \subseteq \mathfrak{a}_0(\Lambda) \subseteq \mathfrak{a}_0(\Lambda^M)$; by duality we then also have that $\mathfrak{a}_1(\Lambda^m) \supseteq \mathfrak{a}_1(\Lambda) \supseteq \mathfrak{a}_1(\Lambda^M)$.
- semisimple characters $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$, $\theta_m \in \mathcal{C}_-(\Lambda^m, 0, \beta)$ and $\theta_M \in \mathcal{C}_-(\Lambda^M, 0, \beta)$, related by the correspondences:
$$\theta_m = \tau_{\Lambda, \Lambda^m, \beta}(\theta), \quad \theta_M = \tau_{\Lambda, \Lambda^M, \beta}(\theta)$$
- η (respectively η_m, η_M) the unique irreducible representation of $J^1 = J^1(\beta, \Lambda)$ (respectively $J_m^1 = J^1(\beta, \Lambda^m)$, $J_M^1 = J^1(\beta, \Lambda^M)$) containing θ (respectively θ_m, θ_M), given by Proposition 3.5.

We form the group $J_{m,M}^1 = P_1(\Lambda_{\mathfrak{o}_E}^m)J_M^1$; note that this is indeed a group, since $P(\Lambda_{\mathfrak{o}_E}^M)$ normalizes J_m^1 and contains $P_1(\Lambda_{\mathfrak{o}_E}^m)$.

Proposition 3.7 (cf. [8, Propositions 5.1.4, 5.1.19], [27, Proposition 2.12]). *There exists a unique irreducible representation $\eta_{m,M}$ of $J_{m,M}^1$ such that*

- $\eta_{m,M}|_{J_M^1} = \eta_M$;
- $\eta_{m,M}$ and η_m induce equivalent irreducible representations of $P_1(\Lambda^m)$.

Moreover, the intertwining of $\eta_{m,M}$ is given by

$$\dim I_g(\eta_{m,M}) = \begin{cases} 1 & \text{if } g \in J_{m,M}^1 G_E^+ J_{m,M}^1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof Given Proposition 3.5, Lemma 3.6 and Lemma 2.6, the proof of the existence and uniqueness of $\eta_{m,M}$ is the same as that of [8, Proposition 5.1.14] while the proof of the intertwining is the same as that of *op. cit.* Proposition 5.1.19. \blacksquare

Remark The representation $\eta_{m,M}$ depends only on $\mathfrak{b}_0(\Lambda^m)$, not on the lattice sequence Λ^m . For suppose $\Lambda^{m'}$ is another \mathfrak{o}_E -lattice sequence with $\mathfrak{b}_0(\Lambda^m) = \mathfrak{b}_0(\Lambda^{m'})$ and $\mathfrak{a}_0(\Lambda^{m'}) \subseteq \mathfrak{a}_0(\Lambda^m)$, and let $\eta_{m',M}$ be the representation of $J_{m,M}^1$ given by Proposition 3.7 applied with $\Lambda^{m'}$ in place of Λ^m .

By Lemma 2.10, there is a sequence $\Lambda^m = \Lambda_0, \Lambda_1, \dots, \Lambda_t = \Lambda^{m'}$ of \mathfrak{o}_E -lattice sequences on the line segment $[\Lambda^m, \Lambda^{m'}]$ such that, for $1 \leq i \leq t$:

- (i) $\mathfrak{b}_0(\Lambda_i) = \mathfrak{b}_0(\Lambda^m)$;
- (ii) either $\mathfrak{a}_0(\Lambda_i) \subseteq \mathfrak{a}_0(\Lambda_{i-1})$ or $\mathfrak{a}_0(\Lambda_i) \supseteq \mathfrak{a}_0(\Lambda_{i-1})$.

Let $\theta_i = \tau_{\Lambda, \Lambda_i, \beta}(\theta)$ and let η_i be the unique irreducible representation of $J_i^1 = J^1(\beta, \Lambda_i)$ containing θ_i . Since each Λ_i is on the line segment $[\Lambda^m, \Lambda^{m'}]$, we have $\mathfrak{a}_0(\Lambda_i) \subseteq \mathfrak{a}_0(\Lambda^m)$ (see Remark 2.11). Let $\eta_{i,M}$ denote the representation of $J_{m,M}^1$ given by Proposition 3.7 applied with Λ_i in place of Λ^m .

Fix i and suppose $\mathfrak{a}_0(\Lambda_i) \subseteq \mathfrak{a}_0(\Lambda_{i-1})$ (the opposite case is symmetrically similar). Applying Proposition 3.7 with the pair $(\Lambda_i, \Lambda_{i-1})$ in place of (Λ^m, Λ^m) , we see that η_i, η_{i-1} induce equivalent irreducible representations of $P_1(\Lambda_i)$. But then

$$\text{Ind}_{J_{m,M}^1}^{P_1(\Lambda_i)} \eta_{i-1,M} \simeq \text{Ind}_{P_1(\Lambda_{i-1})}^{P_1(\Lambda_i)} \text{Ind}_{J_{m,M}^1}^{P_1(\Lambda_{i-1})} \eta_{i-1,M} \simeq \text{Ind}_{P_1(\Lambda_{i-1})}^{P_1(\Lambda_i)} \text{Ind}_{J_{i-1}^1}^{P_1(\Lambda_{i-1})} \eta_{i-1} \simeq \text{Ind}_{J_i^1}^{P_1(\Lambda_i)} \eta_i.$$

Hence, by the uniqueness in Proposition 3.7, we have $\eta_{i,M} = \eta_{i-1,M}$. Applying this with all pairs $(i, i-1)$, we see that $\eta_{m',M} = \eta_{m,M}$, as required.

Let $\eta_{\Lambda,M}$ denote the representation of $J_{\Lambda,M}^1 = P_1(\Lambda_{\mathfrak{o}_E}) J_M^1$ given by applying Proposition 3.7 with Λ in place of Λ^m . As in [8, Proposition 5.1.18] (and with the same proof), we have the following compatibility property:

Proposition 3.8. *With notation as above, we have*

$$\eta_{m,M} |_{J_{\Lambda,M}^1} = \eta_{\Lambda,M}.$$

3.3 Characters of Sylow subgroups

We continue with the notation of the previous section but suppose, in addition, that $\mathfrak{b}_0(\Lambda^m)$ is a minimal self-dual \mathfrak{o}_E -order in B . It will be useful, in this situation, to have another characterization of the extension $\eta_{m,M}$ of η_M . For this we examine the characters of the quotient group $J_{m,M}^1/J_M^1 \cong P_1(\Lambda_{\mathfrak{o}_E}^m)/P_1(\Lambda_{\mathfrak{o}_E}^M)$. We will think of this as a subgroup of $J_M^+/J_M^1 \cong P^+(\Lambda_{\mathfrak{o}_E}^M)/P_1(\Lambda_{\mathfrak{o}_E}^M)$, where $J_M^+ = J^+(\beta, \Lambda^M)$. Since $\mathfrak{b}_0(\Lambda^m)$ is a minimal self-dual \mathfrak{o}_E -order, the group $J_{m,M}^1$ is in fact a Sylow pro- p subgroup of J_M^+ .

We identify $P^+(\Lambda_{\mathfrak{o}_E}^M)/P_1(\Lambda_{\mathfrak{o}_E}^M)$ with the product of (the points of) reductive groups \mathcal{G}_i over $k_{E_i,0}$, which are not necessarily connected; the possibilities here are as follows:

- if $E_i/E_{i,0}$ is quadratic unramified then \mathcal{G}_i is the product of (the restriction of scalars to $k_{E_i,0}$ of) some general linear groups over k_{E_i} and at most two unitary groups over $k_{E_i,0}$;
- if $E_i/E_{i,0}$ is quadratic ramified then \mathcal{G}_i is the product of some general linear groups, at most one symplectic group, and at most one orthogonal group, over $k_{E_i} = k_{E_i,0}$.
- if $E_i = F = F_0$ then \mathcal{G}_i is a product of some general linear groups and at most two symplectic (if $\epsilon = -1$) or orthogonal (if $\epsilon = +1$) groups over k_F .

The image of $P_1(\Lambda_{\mathfrak{o}_E}^m)$ identifies with \mathcal{U} , the product of the unipotent radicals \mathcal{U}_i of Borel subgroups of \mathcal{G}_i .

We also remark, for future reference, that $P(\Lambda_{\mathfrak{o}_E}^M)$ is a *maximal* compact subgroup of G_E if and only if there are no factors isomorphic to a general linear group in any of the three cases above.

Lemma 3.9. *Let ϕ be a character of $J_{m,M}^1/J_M^1 \cong \mathcal{U}$ which is intertwined by all of J_M^+ . Then ϕ extends to a character of J_M^+ .*

Proof We think of $\phi = \otimes \phi_i$ as a character of $\mathcal{U} = \prod_{i=1}^l \mathcal{U}_i$, where ϕ_i is a character of \mathcal{U}_i intertwined by all of \mathcal{G}_i .

We claim that the characters ϕ_i extend to characters, also denoted ϕ_i , of \mathcal{G}_i ; indeed, in all but one case, ϕ_i will be trivial. Inflating $\phi = \otimes \phi_i$ to J_M gives the required extension.

We fix i and now omit it from our notation. Let \mathcal{T} be a maximal torus in \mathcal{G} , and \mathcal{B} a Borel subgroup with unipotent radical \mathcal{U} containing \mathcal{T} . Let Σ be the root system of \mathcal{G} relative to \mathcal{T} , and let Σ^+ be the positive roots, and Δ the simple roots, determined by \mathcal{B} . For $\alpha \in \Sigma$, write \mathcal{U}_α for the corresponding root subgroup, so $\mathcal{U} = \prod_{\alpha \in \Sigma^+} \mathcal{U}_\alpha$. By [15, page 129], the derived subgroup of \mathcal{U} is $\prod_{\alpha \in \Sigma^+ \setminus \Delta} \mathcal{U}_\alpha$ (note that the only (quasi-simple) exceptions to this are groups of type B_2 or F_4 over the field of 2 elements and G_2 over the field of 3 elements, none of which can occur in our situation). Hence ϕ must take the form

$$\phi = \prod_{\alpha \in \Delta} \phi_\alpha, \quad \text{for } \phi_\alpha \text{ a character of } \mathcal{U}_\alpha.$$

Now write $\Delta = \cup \Delta_j$, with each Δ_j irreducible. If $\text{card } \Delta_j \geq 2$ then, for all $\alpha \in \Delta_j$, there is a Weyl-group element w such that $w(\alpha) \in \Sigma^+ \setminus \Delta$. If n in the \mathcal{G} -normalizer of \mathcal{T} reduces to w then, since n intertwines ϕ , we have

$$\phi(u_\alpha) = \phi(nu_\alpha n^{-1}) = 1, \quad \text{for } u_\alpha \in \mathcal{U}_\alpha,$$

since $nu_\alpha n^{-1} \in \mathcal{U}_{w(\alpha)} \subseteq \ker \phi$. Hence ϕ_α is trivial. In particular, if $\text{card } \Delta_j \geq 2$ for all j , then ϕ is trivial and extends to the trivial representation of \mathcal{G} .

This leaves the cases when $\text{card } \Delta_j = 1$, for some j ; in this case the corresponding factor of \mathcal{G} is one of:

- (i) a general linear group of dimension 2, that is $GL_2(k_E)$;
- (ii) a symplectic group of dimension 2, that is $SL_2(k_E)$;
- (iii) a unitary group of dimension 2 or 3;
- (iv) an orthogonal group of dimension 3 or 4.

These cases need to be treated individually. We omit the subscript j .

(i) The proof in this case is contained in the proof of [8, Proposition 5.2.4] (see bottom of page 168). Indeed, since $p \neq 2$, we have $|k_E| > 2$ so the proof there shows that the character ϕ is in fact trivial and extends to the trivial character of \mathcal{G} .

(ii) We write the elements of $\mathcal{G} \simeq SL_2(k_E)$ as matrices and \mathcal{U} as the group of upper triangular unipotent matrices. The character ϕ is given by

$$\phi \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \chi(n), \quad \text{for } n \in k_E,$$

where χ is an additive character of k_E . Since ϕ is intertwined by all of \mathcal{G} , it is normalized by $\text{diag}(a, a^{-1})$, for $a \in k_E^\times$. Hence

$$\chi((a^2 - 1)n) = 1, \quad \text{for all } a \in k_E^\times, n \in k_E.$$

Providing $|k_E| > 3$, there is an $a \in k_E^\times$ such that $a^2 \neq 1$ and hence χ is trivial and ϕ extends to the trivial representation of \mathcal{G} .

If $|k_E| = 3$ then the two non-trivial characters of \mathcal{U} are also intertwined by all of $\mathcal{G} = SL_2(\mathbb{F}_3)$. However, in this case \mathcal{U} is a quotient of \mathcal{G} (by its 2-Sylow subgroup) so ϕ extends in any case. (In the case of the non-trivial characters of \mathcal{U} , it extends to a 1-dimensional cuspidal representation of \mathcal{G} .)

(iii) The proof here is almost identical, though there is no exceptional case as in (ii). In the 2-dimensional case we conjugate by $\text{diag}(a, \bar{a}^{-1})$ and use the facts that the norm map $k_E \rightarrow k_{E,0}$ is surjective and $|k_{E,0}| > 2$; in the 3-dimensional case, we conjugate by $\text{diag}(a, 1, \bar{a}^{-1})$.

(iv) Again, the proof here is the same, with no exceptional case. In the 3-dimensional case we conjugate by $\text{diag}(a, 1, a^{-1})$ and in the 4-dimensional case by $\text{diag}(a, 1, 1, a^{-1})$. \blacksquare

Lemma 3.10. *Let ϕ be a character of $J_{m,M}^1/J_M^1 \cong \mathcal{U}$ which is intertwined by all of G_E . Then ϕ is trivial.*

Proof Using the notation of the proof of Lemma 3.9, we have $\phi = \otimes \phi_i$, and each character ϕ_i is intertwined by G_{E_i} . It is enough to show that each ϕ_i is trivial so it is enough to check the simple case. Moreover, we have seen in the proof of Lemma 3.9 that ϕ is trivial except in the case where \mathcal{G} has a factor isomorphic to $SL(2, \mathbb{F}_3)$. Hence we may restrict to the case (in the notation of the proof of Lemma 3.9) when $|\Delta| = 1$ and $\mathcal{G} \simeq SL_2(\mathbb{F}_3)$.

We choose a self-dual E -basis which splits the lattice sequence $\Lambda_{\mathfrak{o}_E}^m$ (so also $\Lambda_{\mathfrak{o}_E}^M$). Then, after scaling and permuting the basis if necessary, this identifies

$$\begin{aligned} \mathfrak{b}_0(\Lambda_{\mathfrak{o}_E}^m) & \text{ with } \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & \mathfrak{o}_E \end{pmatrix}; & \text{ and} \\ \mathfrak{b}_0(\Lambda_{\mathfrak{o}_E}^M) & \text{ with } \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{o}_E & \mathfrak{o}_E \end{pmatrix} \text{ or } \begin{pmatrix} \mathfrak{o}_E & \mathfrak{p}_E^{-1} \\ \mathfrak{p}_E & \mathfrak{o}_E \end{pmatrix}. \end{aligned}$$

(The second case occurs when E/E_0 is ramified and the form h is hermitian.) We consider

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ \varpi_E & 1 \end{pmatrix}$$

in the two cases, so that $xP_1(\Lambda_{\mathfrak{o}_E}^M)$ generates \mathcal{U} . We put $g = \text{diag}(\varpi_E, \varpi_E^{-1})$ or $\text{diag}(\varpi_E^{-1}, \varpi_E)$ in the two cases. Then $gxg^{-1} \in P_1(\Lambda_{\mathfrak{o}_E}^M)$. Since $g \in G_E$ intertwines ϕ and ϕ is trivial on $P_1(\Lambda_{\mathfrak{o}_E}^M)$, we see that

$$\phi(x) = \phi(gxg^{-1}) = 1.$$

Since $xP_1(\Lambda_{\mathfrak{o}_E}^M)$ generates \mathcal{U} , the character ϕ is indeed trivial. \blacksquare

Corollary 3.11. *If $\mathfrak{b}_0(\Lambda^m)$ is a minimal self-dual \mathfrak{o}_E -order contained in $\mathfrak{b}_0(\Lambda^M)$, then the representation $\eta_{m,M}$ is the unique extension of η_M to $J_{m,M}^1$ which is intertwined by all of G_E .*

Proof Suppose η' is another such extension. Then $\eta' = \eta_{m,M} \otimes \phi$, for some character ϕ of $J_{m,M}^1/J_M^1$ which is intertwined by all of G_E . By Lemma 3.10, such a character is trivial so $\eta' = \eta_{m,M}$. \blacksquare

3.4 Analogous results for \tilde{G}

The results of the previous two sections all have analogues for \tilde{G} . The proofs are almost identical, though a little simpler. Indeed, since so far we have only been looking at representations of pro- p subgroups, we could have obtained the results above by first proving the analogous results for \tilde{G} and then using the Glauberman transfer, as in previous work (see [30], [32]). We state those results we will need for \tilde{G} here, leaving the modifications of the proofs of the previous sections as an exercise for the reader.

Proposition 3.12. *Let $[\Lambda^m, n_m, 0, \beta]$ and $[\Lambda^M, n_M, 0, \beta]$ be semisimple strata with $\mathfrak{a}_0(\Lambda^m) \subseteq \mathfrak{a}_0(\Lambda^M)$. Let $\theta_m \in \mathcal{C}(\Lambda^m, 0, \beta)$ be a semisimple character and $\theta_M = \tau_{\Lambda^m, \Lambda^M, \beta}(\theta_m)$. Then there is a unique irreducible representation $\tilde{\eta}_m$ (respectively $\tilde{\eta}_M$) of $\tilde{J}^1(\beta, \Lambda^m)$ (respectively $\tilde{J}^1(\beta, \Lambda^M)$) which contains θ_m (respectively θ_M).*

Put $\tilde{J}_{m,M}^1 = \tilde{P}_1(\Lambda_{\mathfrak{o}_E}^m) \tilde{J}^1(\beta, \Lambda^M)$. There is a unique irreducible representation $\tilde{\eta}_{m,M}$ of $\tilde{J}_{m,M}^1$ such that

$$(i) \quad \tilde{\eta}_{m,M}|_{\tilde{J}^1(\beta, \Lambda^M)} = \tilde{\eta}_M;$$

$$(ii) \quad \tilde{\eta}_{m,M} \text{ and } \tilde{\eta}_m \text{ induce equivalent irreducible representations of } \tilde{P}_1(\Lambda^m).$$

Moreover, if $\mathfrak{b}_0(\Lambda^m)$ is a minimal \mathfrak{o}_E -order in B then $\tilde{\eta}_{m,M}$ is the unique extension of $\tilde{\eta}_M$ to $\tilde{J}_{m,M}^1$ which is intertwined by \tilde{G}_E . Finally, the intertwining of $\tilde{\eta}_{m,M}$ is given by

$$\dim I_g(\tilde{\eta}_{m,M}) = \begin{cases} 1 & \text{if } g \in \tilde{J}_{m,M}^1 \tilde{G}_E \tilde{J}_{m,M}^1; \\ 0 & \text{otherwise.} \end{cases}$$

4 Beta extensions

This section is devoted to the study of β -extensions, which are certain rather special extensions of the Heisenberg representations η to $J^+(\beta, \Lambda)$. In the case of $GL_N(F)$ (or $GL_m(D)$ for D an F -central division algebra) these extensions are characterized by their intertwining, which should contain the whole centralizer of β (see [8, §5.2], [27, §2.4] respectively). In our situation we have been unable to do this, but instead give a different characterization (which, from the results of [8, §5.2], is also valid for $GL_N(F)$) in terms of the representation $\eta_{m,M}$ of the previous section and by transfer properties.

4.1 Beta-extensions in the maximal case

Let $[\Lambda^M, n, 0, \beta]$ be a skew semisimple stratum such that $\mathfrak{b}_0(\Lambda^M)$ is a maximal self-dual \mathfrak{o}_E -order in B . Let $\theta_M \in \mathcal{C}_-(\Lambda^M, 0, \beta)$ be a skew semisimple character and let η_M be the unique irreducible representation of $J_M^1 = J^1(\beta, \Lambda^M)$ containing θ_M . By Corollary 2.9, there exists a self-dual \mathfrak{o}_E -lattice

sequence Λ^m in V such that $\mathfrak{b}_0(\Lambda^m)$ is a minimal self-dual \mathfrak{o}_E -order in B and $\mathfrak{a}_0(\Lambda^m) \subseteq \mathfrak{a}_0(\Lambda^M)$. Let $\eta_{m,M}$ be the representation of $J_{m,M}^1 = P_1(\Lambda_{\mathfrak{o}_E}^m)J_M^1$ given by Proposition 3.7. Recall that this depends only on the group $J_{m,M}^1$, not on the choice of Λ^m .

Following the ideas of [8, §5.2], we show that there is a representation of $J_M^+ = J_M^+(\beta, \Lambda)$ which extends $\eta_{m,M}$.

Theorem 4.1 (cf. [8, Proposition 5.2.4]). *With notation as above, there exists a representation κ_M of J_M^+ which extends $\eta_{m,M}$. Moreover, if κ'_M is another such representation then $\kappa'_M = \kappa_M \otimes \chi$, for some character χ of $P^+(\Lambda_{\mathfrak{o}_E}^M)/P_1(\Lambda_{\mathfrak{o}_E}^M)$ which is trivial on the subgroup generated by all its unipotent subgroups.*

We call such a representation κ_M a β -extension of η_M . Note that the definition of κ_M is independent of the choice of Λ^m , since $P^+(\Lambda_{\mathfrak{o}_E}^M)$ acts transitively by conjugation on the minimal self-dual \mathfrak{o}_E -orders contained in $\mathfrak{b}_0(\Lambda^M)$ and the representation $\eta_{m,M}$ depends only on $\mathfrak{b}_0(\Lambda^m)$, not on the lattice sequence Λ^m .

Proof The proof is essentially the same as that of [8, Proposition 5.2.4] (see also [2, Lemme 6.4]) so we will omit some of the details and refer the reader to *loc. cit.*

Since J_M^+ normalizes η_M , we can extend η_M to a projective representation of J_M^+ . This gives rise to a 2-cocycle α of J_M in \mathbb{C}^\times . Since the dimension of η_M is a power of p , the order of the cohomology class of this cocycle is a power of p . But η_M admits an extension to $J_{m,M}^1$ so the restriction of the cocycle to $J_{m,M}^1$ is cohomologous to zero. However, $J_{m,M}^1$ is a Sylow pro- p subgroup of J_M^+ so the original cohomology class of α is trivial and hence η_M extends to a linear representation λ of J_M^+ .

Now we compare $\lambda|_{J_{m,M}^1}$ to $\eta_{m,M}$; it is of the form $\eta_{m,M} \otimes \phi$, for some abelian character ϕ of $P_1(\Lambda_{\mathfrak{o}_E}^m)/P_1(\Lambda_{\mathfrak{o}_E}^M)$ which is intertwined by all of $P^+(\Lambda_{\mathfrak{o}_E}^M)$. By Lemma 3.9, any such character extends to a character, also denoted ϕ , of $P^+(\Lambda_{\mathfrak{o}_E}^M)$. Then $\kappa_M := \lambda \otimes \phi^{-1}$ extends $\eta_{m,M}$, as required. Note also that the final assertion of the Theorem is clear. \blacksquare

Remark 4.2. The conclusions of Theorem 4.1 are also true without the assumption that $\mathfrak{b}_0(\Lambda^M)$ is *maximal*, as is the case for \tilde{G} (see [8, Remark, p.170]). However, as for \tilde{G} , this gives many more extensions than are useful for our purposes – instead, we will define β -extensions in the non-maximal case by a compatibility property with the maximal case.

4.2 Beta-extensions in the general case

Let $[\Lambda, n, 0, \beta]$ be any skew semisimple stratum, let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ be a skew semisimple character and let η be the representation of $J^1 = J^1(\beta, \Lambda)$ given by Proposition 3.5. We are going to define β -extensions of η in the case where $\mathfrak{b}_0(\Lambda)$ is not necessarily a maximal self-dual \mathfrak{o}_E -order, by transfer from the maximal case of Theorem 4.1. In order to obtain compatibility bijections, we will see that we do not want to extend our representations all the way to the group $J^+(\beta, \Lambda)$, but rather to the group $J^\circ = J^\circ(\beta, \Lambda) = P^\circ(\Lambda_{\mathfrak{o}_E})J^1$, the inverse image in $J(\beta, \Lambda)$ of the connected component of $J(\beta, \Lambda)/J^1(\beta, \Lambda) \cong P(\Lambda_{\mathfrak{o}_E})/P_1(\Lambda_{\mathfrak{o}_E})$. Nevertheless, we initially define β -extensions to the group $J^+(\beta, \Lambda)$ and then restrict back to $J^\circ(\beta, \Lambda)$; this is because we need to be sure that our β -extensions to J° extend to $J^+(\beta, \Lambda)$.

Lemma 4.3 (cf. [8, Proposition 5.2.5], [27, Lemme 2.23, Proposition 2.9]). *Using the notation above, let $[\Lambda^m, n_m, 0, \beta]$ be a skew semisimple stratum such that $\mathfrak{b}_0(\Lambda) \supseteq \mathfrak{b}_0(\Lambda^m)$. Let $\theta_m = \tau_{\Lambda, \Lambda^m, \beta}(\theta)$*

and let η_m be the unique irreducible representation of $J_m^1 = J^1(\beta, \Lambda^m)$ which contains θ_m . Then there is a canonical bijection between the set of extensions κ_m of η_m to $J_m^+ = J^+(\beta, \Lambda^m)$ and the set of extensions κ of η to $J_{m,\Lambda}^+ = P^+(\Lambda_{\mathfrak{o}_E}^m)J^1$. If $\mathfrak{a}_0(\Lambda) \supseteq \mathfrak{a}_0(\Lambda^m)$ then it is given as follows: given κ_m (respectively κ) there is a unique κ (respectively κ_m) such that $\kappa_m|_{J_m^+}$ and $\kappa|_{J_{m,\Lambda}^+}$ induce equivalent irreducible representations of $P^+(\Lambda_{\mathfrak{o}_E}^m)P_1(\Lambda^m)$.

Moreover, if $g \in G_E^+$ then g intertwines $\kappa_m|_{J_m^+}$ if and only if it intertwines the corresponding $\kappa|_{J_{m,\Lambda}^+}$.

Note that, in the special case of Lemma 4.3 when $\mathfrak{b}_0(\Lambda) = \mathfrak{b}_0(\Lambda^m)$, we have $J_{m,\Lambda}^+ = J^+(\beta, \Lambda)$.

Proof *Case 1* We assume first that $\mathfrak{a}_0(\Lambda) \supseteq \mathfrak{a}_0(\Lambda^m)$. The proof of the first statement is identical to that of [8, Proposition 5.2.5]. Given the simple intersection property Lemma 2.6, the proof of the second statement is identical to that of [27, Proposition 2.9].

For the general case, we first need a compatibility condition:

Lemma 4.4 (cf. [8, (5.2.14)]). *Suppose $[\Lambda, n, 0, \beta]$, $[\Lambda', n', 0, \beta]$ and $[\Lambda^M, n_M, 0, \beta]$ are semisimple strata such that $\mathfrak{a}_0(\Lambda) \subseteq \mathfrak{a}_0(\Lambda') \subseteq \mathfrak{a}_0(\Lambda^M)$. Let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ and $\theta' = \tau_{\Lambda, \Lambda', \beta}$, $\theta_M = \tau_{\Lambda, \Lambda^M, \beta}$, and let η (respectively η' , η_M) be the irreducible representation of $J^1(\beta, \Lambda)$ (respectively, $J^1(\beta, \Lambda')$, $J^1(\beta, \Lambda^M)$) given by Proposition 3.5. Put $J_{\Lambda, M}^+ = P^+(\Lambda_{\mathfrak{o}_E})J^1(\beta, \Lambda^M)$, $J_{\Lambda', M}^+ = P^+(\Lambda'_{\mathfrak{o}_E})J^1(\beta, \Lambda^M)$ and $J_{\Lambda, \Lambda'}^+ = P^+(\Lambda_{\mathfrak{o}_E})J^1(\beta, \Lambda')$ so $J_{\Lambda, M}^+ \subseteq J_{\Lambda', M}^+$. Then the following diagram commutes, where the maps are given by (Case 1 of) Lemma 4.3, or by restriction:*

$$\begin{array}{ccc} \left\{ \text{extensions of } \eta_M \text{ to } J_{\Lambda', M}^+ \right\} & \xrightarrow{\text{res}} & \left\{ \text{extensions of } \eta_M \text{ to } J_{\Lambda, M}^+ \right\} \\ \downarrow & & \searrow \\ \left\{ \text{extensions of } \eta' \text{ to } J^+(\beta, \Lambda') \right\} & \xrightarrow{\text{res}} & \left\{ \text{extensions of } \eta' \text{ to } J_{\Lambda, \Lambda'}^+ \right\} \longrightarrow \left\{ \text{extensions of } \eta \text{ to } J^+(\beta, \Lambda) \right\} \end{array}$$

Proof Suppose κ_M is an extension of η_M to $J_{\Lambda', M}^+$, let κ' be the corresponding extension of η' to $J^+(\beta, \Lambda')$, and let κ be the extension of η to $J^+(\beta, \Lambda)$ which corresponds to $\kappa'|_{J_{\Lambda, \Lambda'}^+}$. We form the group $P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda')$ and note that we have

$$\text{Ind}_{J_{\Lambda, \Lambda'}^+}^{P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda')} \text{Res}_{J_{\Lambda, \Lambda'}^+}^{J^+(\beta, \Lambda')} \kappa' \simeq \text{Res}_{P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda')}^{P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda)} \text{Ind}_{J^+(\beta, \Lambda')}^{P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda)} \kappa',$$

since there is only one double coset in the Mackey restriction formula here. Similarly, we have

$$\begin{aligned} \text{Ind}_{J_{\Lambda, M}^+}^{P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda')} \text{Res}_{J_{\Lambda, M}^+}^{J_{\Lambda', M}^+} \kappa_M &\simeq \text{Res}_{P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda')}^{P^+(\Lambda'_{\mathfrak{o}_E})P_1(\Lambda')} \text{Ind}_{J_{\Lambda', M}^+}^{P^+(\Lambda'_{\mathfrak{o}_E})P_1(\Lambda')} \kappa_M \\ &\simeq \text{Res}_{P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda')}^{P^+(\Lambda'_{\mathfrak{o}_E})P_1(\Lambda')} \text{Ind}_{J^+(\beta, \Lambda')}^{P^+(\Lambda'_{\mathfrak{o}_E})P_1(\Lambda')} \kappa', \end{aligned}$$

where the second equivalence is from the compatibility of κ' and κ_M . Hence

$$\begin{aligned} \text{Ind}_{J_{\Lambda, M}^+}^{P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda')} \text{Res}_{J_{\Lambda, M}^+}^{J_{\Lambda', M}^+} \kappa_M &\simeq \text{Ind}_{P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda')}^{P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda)} \text{Res}_{P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda')}^{P^+(\Lambda'_{\mathfrak{o}_E})P_1(\Lambda')} \text{Ind}_{J^+(\beta, \Lambda')}^{P^+(\Lambda'_{\mathfrak{o}_E})P_1(\Lambda')} \kappa' \\ &\simeq \text{Ind}_{J_{\Lambda, \Lambda'}^+}^{P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda')} \text{Res}_{J_{\Lambda, \Lambda'}^+}^{J^+(\beta, \Lambda')} \kappa' \simeq \text{Ind}_{J^+(\beta, \Lambda)}^{P^+(\Lambda_{\mathfrak{o}_E})P_1(\Lambda)} \kappa. \end{aligned}$$

Hence κ does indeed correspond to $\kappa_M|_{J_{\Lambda, M}^+}$ via (Case 1 of) Lemma 4.3. ■

Case 2 We return to the proof of Lemma 4.3 and consider now the case where $\mathfrak{b}_0(\Lambda) = \mathfrak{b}_0(\Lambda^m)$. Let $\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_t = \Lambda^m$ be a sequence of self-dual \mathfrak{o}_E -lattice sequences given by Lemma 2.10, so that $\mathfrak{b}_0(\Lambda_i) = \mathfrak{b}_0(\Lambda)$ and either $\mathfrak{a}_0(\Lambda_i) \subseteq \mathfrak{a}_0(\Lambda_{i-1})$ or $\mathfrak{a}_0(\Lambda_i) \supseteq \mathfrak{a}_0(\Lambda_{i-1})$, for $1 \leq i \leq t$. For each i , let $\theta_i = \tau_{\Lambda, \Lambda_i, \beta}(\theta)$ and let η_i be the representation of $J^1(\beta, \Lambda_i)$ given by Proposition 3.5. Then, using the first case repeatedly (with $\{\Lambda_i, \Lambda_{i+1}\}$ in place of $\{\Lambda, \Lambda^m\}$) and composing the bijections it gives, we get the required bijection between the set of extensions κ_m of η_m to J_m^+ and the set of extensions κ of η to $J_{m, \Lambda}^+ = J^+(\beta, \Lambda)$. The assertion concerning intertwining also follows by repeated use of the first case.

To prove that this bijection is independent of the choices of the Λ_i , we must use Lemma 4.4. So suppose that $\Lambda = \Lambda'_0, \Lambda'_1, \dots, \Lambda'_{t'} = \Lambda^m$ is another sequence of self-dual \mathfrak{o}_E -lattice sequences such that $\mathfrak{b}_0(\Lambda_i) = \mathfrak{b}_0(\Lambda)$ and either $\mathfrak{a}_0(\Lambda'_i) \subseteq \mathfrak{a}_0(\Lambda'_{i-1})$ or $\mathfrak{a}_0(\Lambda'_i) \supseteq \mathfrak{a}_0(\Lambda'_{i-1})$, for $1 \leq i \leq t'$.

We consider the triangles

$$[\Lambda, \Lambda_i, \Lambda_{i+1}], \text{ for } 1 \leq i \leq t-1, \quad \text{and} \quad [\Lambda'_{i-1}, \Lambda'_i, \Lambda^m], \text{ for } 1 \leq i \leq t'-1.$$

Using Lemma 2.12, each can be decomposed into smaller triangles $[\Gamma_j, \Gamma'_j, \Gamma''_j]$ such that $\mathfrak{a}_0(\Gamma_j) \subseteq \mathfrak{a}_0(\Gamma'_j) \subseteq \mathfrak{a}_0(\Gamma''_j)$. We write $\theta_{\Gamma_j}, \theta_{\Gamma'_j}, \theta_{\Gamma''_j}$ for the transfers of θ , and $\eta_{\Gamma_j}, \eta_{\Gamma'_j}, \eta_{\Gamma''_j}$ for the representations given by Proposition 3.5. Then Lemma 4.4 gives us commutative diagrams

$$\begin{array}{ccc} \left\{ \text{extensions of } \eta_{\Gamma''_j} \text{ to } J^+(\beta, \Gamma''_j) \right\} & & \\ \downarrow & \searrow & \\ \left\{ \text{extensions of } \eta_{\Gamma'_j} \text{ to } J^+(\beta, \Gamma'_j) \right\} & \longrightarrow & \left\{ \text{extensions of } \eta_{\Gamma_j} \text{ to } J^+(\beta, \Gamma_j) \right\} \end{array}$$

Putting together all these commutative diagrams, we see that the bijection

$$\left\{ \text{extensions of } \eta \text{ to } J^+(\beta, \Lambda) \right\} \longleftrightarrow \left\{ \text{extensions of } \eta_m \text{ to } J_m^+ \right\}$$

is indeed independent of the choices.

Case 3 Finally, consider the general case $\mathfrak{b}_0(\Lambda) \supseteq \mathfrak{b}_0(\Lambda^m)$. By Lemma 2.8, there is a self-dual \mathfrak{o}_E -lattice sequence Λ' such that $\mathfrak{b}_0(\Lambda') = \mathfrak{b}_0(\Lambda^m)$ and $\mathfrak{a}_0(\Lambda) \supseteq \mathfrak{a}_0(\Lambda')$. Put $\theta' = \tau_{\Lambda, \Lambda', \beta}$ and let η' be the irreducible representation of $J^1(\beta, \Lambda')$ given by Proposition 3.5.

Applying Case 2 (with Λ' in place of Λ), we get a canonical bijection between the set of extensions κ_m of η_m to J_m^+ and the set of extensions κ' of η' to $J^+(\beta, \Lambda')$. Now applying Case 1 (with Λ' in place of Λ^m), we get a canonical bijection between the set of extensions κ' of η' to $J^+(\beta, \Lambda')$ and the set of extensions κ of η to $J_{m, \Lambda}^+ = P^+(\Lambda'_{\mathfrak{o}_E})J^1$. The required bijection is the composition of these two and the assertion on intertwining is immediate.

It remains to check that the bijection is independent of the choice of Λ' . Suppose Λ'' is another \mathfrak{o}_E -lattice sequence such that $\mathfrak{b}_0(\Lambda'') = \mathfrak{b}_0(\Lambda^m)$ and $\mathfrak{a}_0(\Lambda) \supseteq \mathfrak{a}_0(\Lambda'')$, put $\theta'' = \tau_{\Lambda, \Lambda'', \beta}$ and let η'' be the irreducible representation of $J^1(\beta, \Lambda'')$ given by Proposition 3.5. Then we have a diagram in which the arrows are given by Cases 1 and 2 of Lemma 4.3

$$\begin{array}{ccc} \left\{ \text{extensions of } \eta \text{ to } J_{m, \Lambda}^+ \right\} & \longrightarrow & \left\{ \text{extensions of } \eta'' \text{ to } J^+(\beta, \Lambda'') \right\} \\ \downarrow & \swarrow & \downarrow \\ \left\{ \text{extensions of } \eta' \text{ to } J^+(\beta, \Lambda') \right\} & \longrightarrow & \left\{ \text{extensions of } \eta_m \text{ to } J_m^+ \right\} \end{array}$$

The lower right triangle commutes by Case 2, while we can see that the upper left triangle commutes by splitting $[\Lambda', \Lambda'', \Lambda]$ into triangles (using Lemma 2.12) and arguing as in Case 2. Hence the whole diagram commutes, as required. \blacksquare

We can now define β -extensions for an arbitrary skew semisimple stratum $[\Lambda, n, 0, \beta]$ and character $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$.

Definition 4.5. Let $[\Lambda^M, n_M, 0, \beta]$ be a skew semisimple stratum with $\mathfrak{b}_0(\Lambda^M)$ a maximal self-dual \mathfrak{o}_E -order in B and $\mathfrak{b}_0(\Lambda^M) \supseteq \mathfrak{b}_0(\Lambda)$. Let $\theta_M = \tau_{\Lambda, \Lambda^M, \beta}$, let η_M be the unique irreducible representation of $J_M^1 = J^1(\beta, \Lambda^M)$ which contains θ_M , and let κ_M be a β -extension of η_M to $J_M^+ = J^+(\beta, \Lambda^M)$, given by Theorem 4.1. Then the β -extension of η to J^+ relative to Λ^M , compatible with κ_M is the unique representation κ of J^+ which corresponds to $\kappa_M|_{P^+(\Lambda_{\mathfrak{o}_E})J_M^1}$ under the canonical bijection of Lemma 4.3.

The definition of β -extension, together with Lemma 4.3, immediately gives us a lower bound for their intertwining

Corollary 4.6. *Let κ be a β -extension of η relative to Λ^M . Then $P^+(\Lambda_{\mathfrak{o}_E}^M) \subseteq I_{G^+}(\kappa)$.*

Proof Note first that certainly $P^+(\Lambda_{\mathfrak{o}_E}^M) \subseteq I_{G^+}(\kappa_M)$, since $P^+(\Lambda_{\mathfrak{o}_E}^M) \subseteq J_M^+$. In particular, $P^+(\Lambda_{\mathfrak{o}_E}^M)$ intertwines $\kappa_M|_{P^+(\Lambda_{\mathfrak{o}_E})J_M^1}$ so, by Lemma 4.3, it also intertwines κ . \blacksquare

This is, of course, much less than the situation for simple characters for \tilde{G} , where β -extensions are intertwined by the whole \tilde{G} -centralizer of β . When $P(\Lambda_{\mathfrak{o}_E}^M)$ is a good maximal compact subgroup of G_E it should also be possible to prove such a result here, using the Cartan decomposition and imitating the proofs in [8, §5.2]. Alternatively, if G is an unramified unitary group then the techniques of §5 (see also [27]) may provide a proof of this for any Λ^M . In general, the situation is much less clear. Although, in §6.3 we do improve our lower bound for β -extensions, we do not pursue these strong intertwining results here since we will not need them.

It is useful also to note that we can describe the restriction of a β -extension to the pro- p Sylow subgroup of $J^+(\beta, \Lambda)$ in terms of the representation given by Proposition 3.7. The proof is identical to that of [8, Proposition 5.2.6].

Proposition 4.7 (cf. [8, Proposition 5.2.6]). *Let $[\Lambda, n, 0, \beta]$, $[\Lambda^m, n_m, 0, \beta]$ be skew semisimple strata with $\mathfrak{b}_0(\Lambda^m)$ a minimal self-dual \mathfrak{o}_E -order in B and $\mathfrak{a}_0(\Lambda^m) \subseteq \mathfrak{a}_0(\Lambda)$. Let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$, let $\theta_m = \tau_{\Lambda^m, \Lambda, \beta}(\theta)$, and let η (respectively η_m) be the irreducible representation of $J^1(\beta, \Lambda)$ (respectively, $J^1(\beta, \Lambda^m)$) given by Proposition 3.5. Put $J_{m, \Lambda}^1 = P_1(\Lambda_{\mathfrak{o}_E}^m)J^1(\beta, \Lambda)$, a pro- p Sylow subgroup of $J^+(\beta, \Lambda)$, and let $\eta_{m, \Lambda}$ be the irreducible representation of $J_{m, \Lambda}^1$ given by Proposition 3.7 applied with Λ in place of Λ^M . Then, for any β -extension κ of η to $J^+(\beta, \Lambda)$,*

$$\kappa|_{J_{m, \Lambda}^1} = \eta_{m, \Lambda}.$$

If κ is a β -extension of η to $J^+(\beta, \Lambda)$ then we say that the restriction $\kappa|_{J^0(\beta, \Lambda)}$ is a β -extension of η to $J^0(\beta, \Lambda)$. We have a compatibility property analogous to Lemma 4.3 for extensions to $J^0(\beta, \Lambda)$. Indeed, the following Proposition shows that this compatibility is a bijection for β -extensions to $J^0(\beta, \Lambda)$, as in [8, (5.2.14)].

Proposition 4.8 (cf. [8, (5.2.14)]). *Let $[\Lambda, n, 0, \beta]$ and $[\Lambda', n', 0, \beta]$ be skew semisimple strata such that there exists $[\Lambda^M, n_M, 0, \beta]$ a skew semisimple stratum with $\mathfrak{b}_0(\Lambda^M)$ a maximal self-dual \mathfrak{o}_E -order containing both $\mathfrak{b}_0(\Lambda)$ and $\mathfrak{b}_0(\Lambda')$. Let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ be a skew semisimple character and let $\theta' = \tau_{\Lambda, \Lambda', \beta}(\theta)$. Let η, η' be the representations of $J^1(\beta, \Lambda), J^1(\beta, \Lambda')$ respectively, given by Proposition 3.5. Then there is a canonical bijection between the set of β -extensions κ' of η' to $J^\circ(\beta, \Lambda')$ relative to Λ^M and the set of β -extensions κ of η to $J^\circ(\beta, \Lambda)$ relative to Λ^M . If $\mathfrak{a}_0(\Lambda) \subseteq \mathfrak{a}_0(\Lambda')$ then it is given as follows: given κ (respectively κ') there is a unique κ' (respectively κ) such that $\kappa'|_{P^\circ(\Lambda_{\mathfrak{o}_E})J^1(\beta, \Lambda')}$ and κ induce equivalent irreducible representations of $P^\circ(\Lambda_{\mathfrak{o}_E})P_1(\Lambda)$.*

We say that κ' is *compatible with κ* if they correspond under the bijection of Proposition 4.8.

Proof We observe first that, using Lemma 4.3 (or rather, its analogue for J°), we can reduce to the case where $\mathfrak{a}_0(\Lambda^M)$ contains both $\mathfrak{a}_0(\Lambda)$ and $\mathfrak{a}_0(\Lambda')$: Lemma 2.8 shows that there is a self-dual \mathfrak{o}_E -lattice sequence Λ'' with $\mathfrak{b}_0(\Lambda'') = \mathfrak{b}_0(\Lambda)$ and $\mathfrak{a}_0(\Lambda'') \subseteq \mathfrak{a}_0(\Lambda^M)$ and, by Lemma 4.3, there is a canonical bijection between the extensions of η to $J^\circ(\beta, \Lambda)$ and the extensions of η'' to $J^\circ(\beta, \Lambda'')$. Then we replace Λ by Λ'' (and similarly for Λ').

When $\mathfrak{a}_0(\Lambda) \subseteq \mathfrak{a}_0(\Lambda')$, from the commutativity of the diagram in Lemma 4.4, and by restriction, we get a well-defined *surjection*

$$\left\{ \beta\text{-extensions of } \eta' \text{ to } J^\circ(\beta, \Lambda') \text{ relative to } \Lambda^M \right\} \rightarrow \left\{ \beta\text{-extensions of } \eta \text{ to } J^\circ(\beta, \Lambda) \text{ relative to } \Lambda^M \right\}. \quad (4.9)$$

To see that we have a bijection, we need the following Lemma:

Lemma 4.10. *Let $[\Lambda^M, n_M, 0, \beta]$ be a skew semisimple stratum with $\mathfrak{b}_0(\Lambda^M)$ a maximal self-dual \mathfrak{o}_E -order in B containing $\mathfrak{b}_0(\Lambda)$. Let $\theta_M \in \mathcal{C}_-(\Lambda^M, 0, \beta)$, let η_M be the representation of $J_M^1 = J^1(\beta, \Lambda^M)$ given by Proposition 3.5, and let κ_M, κ'_M be β -extensions of η_M to $J_M^+ = J^+(\beta, \Lambda^M)$, given by Theorem 4.1. Put $J_M^\circ = J^\circ(\beta, \Lambda^M)$ and $J_{\Lambda, M}^\circ = P^\circ(\Lambda_{\mathfrak{o}_E})J_M^1$. Then*

$$\kappa_M|_{J_{\Lambda, M}^\circ} \simeq \kappa'_M|_{J_{\Lambda, M}^\circ} \iff \kappa_M|_{J_M^\circ} \simeq \kappa'_M|_{J_M^\circ}.$$

Proof By Theorem 4.1, we have $\kappa'_M = \kappa_M \otimes \chi$, for some character χ of $\mathcal{G}^+ = P^+(\Lambda_{\mathfrak{o}_E}^M)/P_1(\Lambda_{\mathfrak{o}_E}^M)$ which is trivial on the subgroup generated by all its unipotent subgroups. If $\kappa_M|_{J_{\Lambda, M}^\circ} \simeq \kappa'_M|_{J_{\Lambda, M}^\circ}$ then χ is also trivial on $\mathcal{P}^\circ = P^\circ(\Lambda_{\mathfrak{o}_E})/P_1(\Lambda_{\mathfrak{o}_E}^M)$, which is a parabolic subgroup of \mathcal{G}° . Hence χ is trivial on all of \mathcal{G}° and $\kappa_M|_{J_M^\circ} \simeq \kappa'_M|_{J_M^\circ}$. The converse is trivial. ■

Returning to Proposition 4.8, we see that Lemmas 4.10 and 4.3 (together with the analogous Lemma for $J^\circ(\beta, \Lambda)$) imply that both sides of (4.9) have the same number of elements, which completes the proof when $\mathfrak{a}_0(\Lambda) \subseteq \mathfrak{a}_0(\Lambda')$. The general case follows immediately. ■

Remark In the simple case for $\tilde{G} = GL_N(F)$, all the maximal \mathfrak{o}_E -orders in B containing $\mathfrak{b}_0(\Lambda)$ are conjugate by \tilde{G}_E so the set of β -extensions relative to Λ^M is independent of the choice of Λ^M . This is no longer true in our situation. However, we will see in Corollary 6.13 that there is some sort of compatibility between the β -extensions defined relative to different maximal self-dual \mathfrak{o}_E -orders in B . On the other hand, it will be useful in many situations to have a ‘‘canonical’’ choice for Λ^M , as follows:

Given a skew semisimple stratum $[\Lambda, n, 0, \beta]$, write $\Lambda = \oplus_i \Lambda^i$, with Λ^i an \mathfrak{o}_{E_i} -lattice sequence. Define the \mathfrak{o}_{E_i} lattice sequence \mathfrak{M}_Λ^i by

$$\mathfrak{M}_\Lambda^i(2r + s) = \mathfrak{p}_{E_i}^r \Lambda^i(s), \quad \text{for } r \in \mathbb{Z}, s = 0, 1, \quad (4.11)$$

and put $\mathfrak{M}_\Lambda = \bigoplus_i \mathfrak{M}_\Lambda^i$. Then $[\mathfrak{M}_\Lambda, n_{\mathfrak{M}}, 0, \beta]$ is a skew semisimple stratum, for some integer $n_{\mathfrak{M}}$, and $\mathfrak{b}_0(\mathfrak{M}_\Lambda)$ is a maximal self-dual \mathfrak{o}_E -order containing $\mathfrak{b}_0(\Lambda)$.

Let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ and let η be the unique irreducible representation of $J^1(\beta, \Lambda)$ containing θ . We call a β -extension of η relative to \mathfrak{M}_Λ a *standard β -extension*.

In particular, if $[\Lambda, n, 0, \beta]$ and $[\Lambda', n', 0, \beta]$ are skew semisimple strata with $\mathfrak{M}_\Lambda = \mathfrak{M}_{\Lambda'}$ then, by Proposition 4.8 there is a bijection between the standard β -extensions of η to $J^\circ(\beta, \Lambda)$ and those of η' to $J^\circ(\beta, \Lambda')$, where η' is the unique irreducible representation of $J^1(\beta, \Lambda')$ containing $\theta' = \tau_{\Lambda, \Lambda', \beta}(\theta)$.

4.3 Analogous results for \tilde{G}

If $[\Lambda, n, 0, \beta]$ is a semisimple stratum and $\tilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$ is a semisimple character then the results of the previous sections can be imitated to construct β -extensions $\tilde{\kappa}$ of $\tilde{\eta}$ to $\tilde{J}(\beta, \Lambda)$. Given Proposition 3.12, the proofs are identical to those for G and are left as an exercise for the reader. In this situation the notion of β -extension is independent of the choice of Λ^M such that $\mathfrak{b}_0(\Lambda^M)$ is a maximal \mathfrak{o}_E -order containing $\mathfrak{b}_0(\Lambda)$, the reductive quotient $\tilde{J}(\beta, \Lambda)/\tilde{J}^1(\beta, \Lambda)$ is always connected and there is a canonical bijection between β -extensions, as in Proposition 4.8.

We will not, in fact, use these results for semisimple characters, but only for simple characters, when they all come from [8, §5.2].

5 Iwahori factorizations

In this section we imitate the constructions of [8, §7] to get Iwahori factorizations of semisimple characters. We do this first for \tilde{G} and then, by Glauberman transfer, for G^+ and G . We end with some similar results for β -extensions.

5.1 Iwahori decompositions

Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum with associated splitting $V = \bigoplus_{i=1}^l V^i$ and all the other usual notation. Let $V = \bigoplus_{j=1}^m W^{(j)}$ be a decomposition of V into subspaces such that

- (i) $W^{(j)} = \bigoplus_{i=1}^l (W^{(j)} \cap V^i)$, for $1 \leq j \leq m$, so also $V^i = \bigoplus_{j=1}^m (W^{(j)} \cap V^i)$, for $1 \leq i \leq l$;
- (ii) $W^{(j)} \cap V^i$ is an E_i -subspace of V^i , for $1 \leq j \leq m$ and $1 \leq i \leq l$.

Definition 5.1. We say that the decomposition $V = \bigoplus_{j=1}^m W^{(j)}$ is:

- (i) *subordinate to* $[\Lambda, n, 0, \beta]$ if, for all $r \in \mathbb{Z}$, $\Lambda(r) = \bigoplus_{j=1}^m (\Lambda(r) \cap W^{(j)})$;
- (ii) *properly subordinate to* $[\Lambda, n, 0, \beta]$ if it is subordinate to $[\Lambda, n, 0, \beta]$ and, for all $r \in \mathbb{Z}$ and for $1 \leq i \leq l$, there is at most one j , $1 \leq j \leq m$, such that

$$\left(\Lambda(r) \cap W^{(j)} \cap V^i \right) \supsetneq \left(\Lambda(r+1) \cap W^{(j)} \cap V^i \right).$$

An explanation of how these decompositions arise is in order at this point. We consider first the case when the stratum $[\Lambda, n, 0, \beta]$ is simple. Let \mathcal{B} be an E -basis for V which splits the lattice sequence Λ . If we decompose \mathcal{B} as a disjoint union of subsets \mathcal{B}_j , $1 \leq j \leq m$, and define $W^{(j)}$

to be the E -linear span of \mathcal{B}_j , then we obtain a decomposition of V subordinate to the stratum. Moreover, any decomposition subordinate to the stratum arises in this way.

For a properly subordinate decomposition, we need to be a little more careful. Writing $e = e(\Lambda|\mathfrak{o}_E)$ for the \mathfrak{o}_E -period of Λ , we decompose \mathcal{B} into a disjoint union of subsets \mathcal{B}_j , $1 \leq j \leq e$, by saying that a vector $\mathbf{w} \in \mathcal{B}$ lies in \mathcal{B}_j if and only if there exists $r \in \mathbb{Z}$ such that $\mathbf{w} \in \Lambda(j+re) \setminus \Lambda(j+re+1)$. Let $W^{(j)}$ be the E -linear space of \mathcal{B}_j . Then any decomposition of V which coarsens the decomposition $V = \bigoplus_{j=1}^e W^{(j)}$ (that is, each subspace is a sum of some $W^{(j)}$) is properly subordinate to the stratum. Again, any decomposition properly subordinate to the stratum arises in this way.

For the semisimple case we take, for $1 \leq i \leq l$, a decomposition $V^i = \bigoplus_{j=1}^m W^{(i,j)}$ which is (properly) subordinate to the simple stratum $[\Lambda^i, n_i, 0, \beta_i]$ (and where we allow some of the subspaces $W^{(i,j)}$ to be trivial). Then we get a decomposition of V which is (properly) subordinate to $[\Lambda, n, 0, \beta]$ by setting $W^{(j)} = \bigoplus_{i=1}^l W^{(i,j)}$.

Proposition 5.2 (cf. [8, Theorem 7.1.14]). *Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum and let $V = \bigoplus_{j=1}^m W^{(j)}$ be a decomposition as above. Let \widetilde{M} be the Levi subgroup of \widetilde{G} which is the stabilizer of this decomposition and let \widetilde{P} be any parabolic subgroup with Levi component \widetilde{M} and unipotent radical \widetilde{U} .*

- (i) *If the decomposition is subordinate to $[\Lambda, n, 0, \beta]$, then the groups $\widetilde{H}^1(\beta, \Lambda)$ and $\widetilde{J}^1(\beta, \Lambda)$ have Iwahori decompositions with respect to $(\widetilde{M}, \widetilde{P})$.*
- (ii) *If the decomposition is properly subordinate to $[\Lambda, n, 0, \beta]$, then the group $\widetilde{J}(\beta, \Lambda)$ also has an Iwahori decomposition with respect to $(\widetilde{M}, \widetilde{P})$ and, moreover, $\widetilde{J}(\beta, \Lambda) \cap \widetilde{U} = \widetilde{J}^1(\beta, \Lambda) \cap \widetilde{U}$.*

Proof (i) For $1 \leq j \leq m$, let $\mathbf{e}^{(j)}$ denote the projection onto $W^{(j)}$ with kernel $\bigoplus_{k \neq j} W^{(k)}$. The definition of subordinate (Definition 5.1(i)) ensures that $\mathbf{e}^{(j)} \in \mathfrak{a}_0(\Lambda)$. Moreover, the property that each $W^{(j)} \cap V^i$ is an E_i -subspace of V^i , for $1 \leq i \leq l$, ensures that $\mathbf{e}^{(j)} \in B$. Hence $\mathbf{e}^{(j)} \in \mathfrak{b}_0(\Lambda)$.

Now $\widetilde{J}^1(\beta, \Lambda) = 1 + \mathfrak{J}^1(\beta, \Lambda)$ and, by [32, Lemma 3.10], $\mathfrak{J}^1(\beta, \Lambda)$ is a $\mathfrak{b}_0(\Lambda)$ -bimodule. In particular, $\mathbf{e}^{(j)} \mathfrak{J}^1(\beta, \Lambda) \mathbf{e}^{(k)} \subseteq \mathfrak{J}^1(\beta, \Lambda)$ so, by [7, Proposition 10.4], $\widetilde{J}^1(\beta, \Lambda)$ has an Iwahori decomposition with respect to $(\widetilde{M}, \widetilde{P})$. The same proof works for $\widetilde{H}^1(\beta, \Lambda)$.

(ii) Suppose now that the decomposition is properly subordinate to $[\Lambda, n, 0, \beta]$. We start with a lemma.

Lemma 5.3 (cf. [8, Lemma 7.1.15]). *In this situation we have $\widetilde{P}(\Lambda_{\mathfrak{o}_E}) \cap \widetilde{U} = \widetilde{P}_1(\Lambda_{\mathfrak{o}_E}) \cap \widetilde{U}$.*

Proof We prove the corresponding additive statement, which is $\mathbf{e}^{(j)} \mathfrak{b}_0(\Lambda) \mathbf{e}^{(k)} \subseteq \mathfrak{b}_1(\Lambda)$, for $j \neq k$. So suppose $x \in \mathbf{e}^{(j)} \mathfrak{b}_0(\Lambda) \mathbf{e}^{(k)}$. For any $r \in \mathbb{Z}$ and for $s \neq k$ we have $x(\Lambda(r) \cap W^{(s)}) = 0$. On the other hand, for any $r \in \mathbb{Z}$ and $1 \leq i \leq l$, either $\Lambda(r) \cap W^{(k)} \cap V^i = \Lambda(r+1) \cap W^{(k)} \cap V^i$ or $\Lambda(r) \cap W^{(j)} \cap V^i = \Lambda(r+1) \cap W^{(j)} \cap V^i$, since the decomposition is properly subordinate. In either case, we get $x(\Lambda(r) \cap W^{(k)} \cap V^i) \subseteq \Lambda(r+1) \cap W^{(j)} \cap V^i$. Hence

$$x\Lambda(r) = x \left(\Lambda(r) \cap W^{(k)} \right) = \bigoplus_{i=1}^l x \left(\Lambda(r) \cap W^{(k)} \cap V^i \right) \subseteq \Lambda(r+1),$$

and $x \in \mathfrak{a}_1(\Lambda)$ as required. ■

Now the same proof as that of [8, Theorem 7.1.14] shows that $\tilde{\mathbb{P}}(\Lambda_{\mathfrak{o}_E})$ has an Iwahori decomposition with respect to (\tilde{M}, \tilde{P}) and that $\tilde{\mathbb{P}}(\Lambda_{\mathfrak{o}_E}) = \left(\tilde{\mathbb{P}}(\Lambda_{\mathfrak{o}_E}) \cap \tilde{M}\right) \tilde{\mathbb{P}}_1(\Lambda_{\mathfrak{o}_E})$. Hence $J(\beta, \Lambda) = \left(\tilde{\mathbb{P}}(\Lambda_{\mathfrak{o}_E}) \cap \tilde{M}\right) J^1(\beta, \Lambda)$ has an Iwahori decomposition with respect to (\tilde{M}, \tilde{P}) , and the final assertion follows also. \blacksquare

Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum and let $V = \bigoplus_{j=1}^m W^{(j)}$ be a decomposition subordinate to the stratum. We write $\Lambda^{(j)}$ for the lattice sequence in $W^{(j)}$ given by

$$\Lambda^{(j)}(r) = \Lambda(r) \cap W^{(j)}, \quad \text{for } r \in \mathbb{Z},$$

and put $\beta^{(j)} = \mathbf{e}^{(j)} \beta \mathbf{e}^{(j)}$, where $\mathbf{e}^{(j)}$ denotes the projection onto $W^{(j)}$ with kernel $\bigoplus_{k \neq j} W^{(k)}$. Then, since each $V^i \cap W^{(j)}$ is an E_i -subspace of V^i , the stratum $[\Lambda^{(j)}, n^{(j)}, 0, \beta^{(j)}]$ is a semisimple stratum in $A^{(j)} = \text{End}_F(W^{(j)})$, for some integer $n^{(j)}$, with splitting $W^{(j)} = \bigoplus_{i=1}^l (V^i \cap W^{(j)})$. Note that some of the pieces of this splitting may be trivial so, strictly speaking, the splitting consists only of the non-zero pieces. We also write $B^{(j)}$ for the centralizer of $\beta^{(j)}$ in $A^{(j)}$.

Proposition 5.4. *Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum and let $V = \bigoplus_{j=1}^m W^{(j)}$ be a decomposition subordinate to the stratum. Let \tilde{M} be the Levi subgroup of \tilde{G} which is the stabilizer of this decomposition. Then*

$$\tilde{\mathbb{H}}^1(\beta, \Lambda) \cap \tilde{M} = \prod_{j=1}^m \tilde{\mathbb{H}}^1(\beta^{(j)}, \Lambda^{(j)}),$$

and likewise for $\tilde{\mathbb{J}}^1(\beta, \Lambda)$. If the decomposition is properly subordinate to the stratum then we also have

$$\tilde{\mathbb{J}}(\beta, \Lambda) \cap \tilde{M} = \prod_{j=1}^m \tilde{\mathbb{J}}(\beta^{(j)}, \Lambda^{(j)}).$$

Proof We will prove the additive statement

$$\mathbf{e}^{(j)} \mathfrak{H}(\beta, \Lambda) \mathbf{e}^{(j)} = \mathfrak{H}(\beta^{(j)}, \Lambda^{(j)}).$$

The same proof will give the corresponding statement for $\mathfrak{J}(\beta, \Lambda)$ and the result follows.

Let \mathcal{M} denote the stabilizer in A of the decomposition, so that $\tilde{M} = \mathcal{M}^\times$. We proceed by induction on $k_0(\beta, \Lambda)$, noting that the base case is a null stratum, where the result is clear since $\mathfrak{H}(\beta, \Lambda) = \mathfrak{a}_0(\Lambda)$. So we turn to the inductive step and put $r = -k_0(\beta, \Lambda)$.

Put $W^{(i,j)} = V^i \cap W^{(j)}$ so that $V^i = \bigoplus_{j=1}^m W^{(i,j)}$ is (properly) subordinate to the simple stratum $[\Lambda^i, n_i, 0, \beta_i]$. We choose an E_i -basis $\mathcal{B}_{i,j}$ for each $W^{(i,j)}$ which splits the lattice sequence $\Lambda^{(i,j)}$ given by $\Lambda^{(i,j)}(r) = \Lambda(r) \cap W^{(i,j)}$, and let $Y^{(i,j)}$ be the F -linear span of $\mathcal{B}_{i,j}$. Putting $Y^i = \bigoplus_{j=1}^m Y^{(i,j)}$, we get a ‘‘generalized (W, E) -decomposition’’ $V^i = Y^i \otimes_F E_i$ (see [10, §5.3]), and hence an embedding $\iota_{Y^i} : \text{End}_F(E_i) \hookrightarrow A^{ii} = \text{End}_F(V^i)$ with image in $A^{ii} \cap \mathcal{M}$.

According to [32, Proposition 3.4], we can choose a semisimple stratum $[\Lambda, n, r, \gamma]$ equivalent to $[\Lambda, n, r, \beta]$ with γ in the image of $\prod_i \iota_{Y^i}$. In particular, we have $\gamma \in \mathcal{M}$ so the decomposition is subordinate to $[\Lambda, n, r, \gamma]$ also. Hence, by the inductive hypothesis, we have

$$\mathbf{e}^{(j)} \mathfrak{H}(\gamma, \Lambda) \mathbf{e}^{(j)} = \mathfrak{H}(\gamma^{(j)}, \Lambda^{(j)}),$$

where $\gamma^{(j)} = \mathbf{e}^{(j)}\gamma\mathbf{e}^{(j)}$. The semisimple stratum $[\Lambda^{(j)}, n^{(j)}, r, \gamma^{(j)}]$ is equivalent to $[\Lambda^{(j)}, n^{(j)}, r, \beta^{(j)}]$ so, since $\mathbf{e}^{(j)} \in \mathfrak{b}_0(\Lambda)$, we get

$$\begin{aligned} \mathbf{e}^{(j)}\mathfrak{H}(\beta, \Lambda)\mathbf{e}^{(j)} &= \mathbf{e}^{(j)}\mathfrak{b}_0(\Lambda)\mathbf{e}^{(j)} + \mathbf{e}^{(j)}\mathfrak{H}[\frac{r}{2}]^{+1}(\gamma, \Lambda)\mathbf{e}^{(j)} \\ &= \mathfrak{b}_0(\Lambda^{(j)}) + \mathfrak{H}[\frac{r}{2}]^{+1}(\gamma^{(j)}, \Lambda^{(j)}) \\ &= \mathfrak{H}(\beta^{(j)}, \Lambda^{(j)}), \end{aligned}$$

as required. ■

Continuing in the same situation, let \tilde{P} be any parabolic subgroup of \tilde{G} with Levi subgroup \tilde{M} , and let \tilde{U} denote its unipotent radical. We define the groups

$$\tilde{\mathfrak{H}}_{\tilde{P}}^1 = \tilde{\mathfrak{H}}^1(\beta, \Lambda) \left(\tilde{\mathfrak{J}}^1(\beta, \Lambda) \cap \tilde{U} \right), \quad \text{and} \quad \tilde{\mathfrak{J}}_{\tilde{P}}^1 = \tilde{\mathfrak{H}}^1(\beta, \Lambda) \left(\tilde{\mathfrak{J}}^1(\beta, \Lambda) \cap \tilde{P} \right).$$

If the decomposition is properly subordinate to the stratum, we also put

$$\tilde{\mathfrak{J}}_{\tilde{P}} = \tilde{\mathfrak{H}}^1(\beta, \Lambda) \left(\tilde{\mathfrak{J}}(\beta, \Lambda) \cap \tilde{P} \right).$$

All these groups have Iwahori decompositions with respect to any parabolic subgroup with Levi component \tilde{M} .

5.2 Iwahori factorization of semisimple characters

We continue in the same situation, so $[\Lambda, n, 0, \beta]$ is a semisimple stratum and $V = \bigoplus_{j=1}^m W^{(j)}$ is a decomposition subordinate to the stratum. Let \tilde{M} be the Levi subgroup of \tilde{G} which is the stabilizer of this decomposition and let \tilde{P} be any parabolic subgroup with Levi component \tilde{M} . Write \tilde{U} for the unipotent radical of \tilde{P} .

Proposition 5.5 (cf. [8, Proposition 7.1.19]). *In this situation, let $\tilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$. Then the character $\tilde{\theta}|_{\tilde{\mathfrak{H}}^1(\beta, \Lambda) \cap \tilde{U}}$ is trivial. After identifying $\tilde{\mathfrak{H}}^1(\beta, \Lambda) \cap \tilde{M}$ with $\prod_{j=1}^m \tilde{\mathfrak{H}}^1(\beta^{(j)}, \Lambda^{(j)})$, we have*

$$\tilde{\theta}|_{\tilde{\mathfrak{H}}^1(\beta, \Lambda) \cap \tilde{M}} = \bigotimes_{j=1}^m \tilde{\theta}^{(j)},$$

where $\tilde{\theta}^{(j)} \in \mathcal{C}(\Lambda^{(j)}, 0, \beta^{(j)})$.

In this situation, we shall write $\tilde{\theta}^{(j)} = \tau_{\Lambda, \Lambda^{(j)}, \beta}(\tilde{\theta})$. By the definition of simple character in [10, §5], this is consistent with the notion of transfer between different algebras for simple characters. We do not pursue the functorial properties of this transfer map but note that it gives a map $\tau_{\Lambda, \Lambda^{(j)}, \beta} : \mathcal{C}(\Lambda, 0, \beta) \rightarrow \mathcal{C}(\Lambda^{(j)}, 0, \beta^{(j)})$ which is *not* in general injective; this happens if one of the subspaces $W^{(j)} \cap V^i = \{0\}$ but $\beta_i \neq 0$.

Proof We proceed by induction on $k_0(\beta, \Lambda)$, the base case again being the null case, when the only semisimple character is the trivial one and the result is clear. So we consider the induction step and put $r = -k_0(\beta, \Lambda)$. We will use again the notation of the proof of Proposition 5.4, so that $[\Lambda, n, r, \gamma]$ is the carefully chosen semisimple stratum equivalent to $[\Lambda, n, r, \beta]$ from there.

Recall that $\tilde{\theta}$ has the properties:

(i) we have $\tilde{\theta}|_{\tilde{H}^1 \cap \tilde{G}_i} = \tilde{\theta}_i$, for some simple character $\tilde{\theta}_i \in \mathcal{C}(\Lambda^i, 0, \beta_i)$, where $\tilde{G}_i = \text{Aut}_F(V^i)$;

(ii) $\tilde{\theta}|_{\tilde{H}^{[\frac{r}{2}]+1}(\beta, \Lambda)} = \tilde{\theta}_0|_{\tilde{H}^{[\frac{r}{2}]+1}(\beta, \Lambda)} \psi_{\beta-\gamma}$, for some semisimple character $\tilde{\theta}_0 \in \mathcal{C}(\Lambda, 0, \gamma)$.

Moreover, these properties characterize the semisimple characters in $\mathcal{C}(\Lambda, 0, \beta)$.

For the first assertion of the Lemma, let \tilde{M}_β denote the stabilizer of the splitting $V = \bigoplus_{i=1}^l V^i$ and let \tilde{U}_β be the unipotent radical of some parabolic subgroup \tilde{P}_β with Levi component \tilde{M}_β . According to [32, Lemma 3.15(i)], the restriction $\tilde{\theta}|_{\tilde{H}^1 \cap \tilde{U}_\beta}$ is trivial. Since $\tilde{H}^1 \cap \tilde{U}$ has an Iwahori decomposition with respect to $(\tilde{M}_\beta, \tilde{P}_\beta)$, we are reduced to showing that $\tilde{\theta}$ restricts trivially to $\tilde{H}^1 \cap \tilde{U} \cap \tilde{M}_\beta$. But here we may work block-by-block so the result follows from the simple case in [10, Proposition 5.2].

Now we turn to the second assertion. From Proposition 5.4 we have that $\tilde{H}^1(\beta, \Lambda) \cap \text{Aut}_F(W^{(j)}) = \tilde{H}^1(\beta^{(j)}, \Lambda^{(j)})$ so we may consider $\tilde{\theta}^{(j)} = \tilde{\theta}|_{\tilde{H}^1(\beta^{(j)}, \Lambda^{(j)})}$. To show that it is a semisimple character in $\mathcal{C}(\Lambda^{(j)}, 0, \beta^{(j)})$ we need only check that it satisfies the two properties (i), (ii) above.

The splitting of $[\Lambda^{(j)}, n^{(j)}, 0, \beta^{(j)}]$ is given by $W^{(j)} = \bigoplus_{i=1}^l W^{(i,j)}$, where $W^{(i,j)} = V^i \cap W^{(j)}$ and we ignore any trivial pieces of the splitting. We put $\tilde{G}^{(i,j)} = \text{Aut}_F(W^{(i,j)})$, let $\mathbf{1}^{(i,j)}$ be the projection from $W^{(j)}$ to $W^{(i,j)}$ with kernel $\bigoplus_{k \neq i} W^{(k,j)}$, and let $\mathbf{e}^{(i,j)}$ be the projection from V^i to $W^{(i,j)}$ with kernel $\bigoplus_{k \neq j} W^{(i,k)}$. We put $\beta_i^{(j)} = \mathbf{e}^{(i,j)} \beta_i \mathbf{e}^{(i,j)} = \mathbf{1}^{(i,j)} \beta^{(j)} \mathbf{1}^{(i,j)}$.

(i) We have $\tilde{H}^1(\beta^{(j)}, \Lambda^{(j)}) \cap \tilde{G}^{(i,j)} = \tilde{H}^1(\beta_i^{(j)}, \Lambda^{(i,j)})$ and

$$\tilde{\theta}^{(j)}|_{\tilde{H}^1(\beta^{(j)}, \Lambda^{(j)}) \cap \tilde{G}^{(i,j)}} = \left(\tilde{\theta}|_{\tilde{H}^1(\beta, \Lambda) \cap \tilde{G}^i} \right)|_{\tilde{H}^1(\beta_i^{(j)}, \Lambda^{(i,j)})} = \tilde{\theta}_i|_{\tilde{H}^1(\beta_i^{(j)}, \Lambda^{(i,j)})},$$

which, by [10, §5], is the simple character $\tau_{\Lambda^i, \Lambda^{(i,j)}, \beta_i}(\tilde{\theta}_i)$ in $\mathcal{C}(\Lambda^{(i,j)}, 0, \beta_i^{(j)})$.

(ii) The stratum $[\Lambda^{(j)}, n^{(j)}, 0, \gamma^{(j)}]$ is a semisimple stratum equivalent to $[\Lambda^{(j)}, n^{(j)}, 0, \beta^{(j)}]$ and it is straightforward that $\psi_{\beta-\gamma}|_{\tilde{H}^{[\frac{r}{2}]+1}(\beta^{(j)}, \Lambda^{(j)})} = \psi_{\beta^{(j)}-\gamma^{(j)}}$. Thus we have

$$\begin{aligned} \tilde{\theta}^{(j)}|_{\tilde{H}^{[\frac{r}{2}]+1}(\beta^{(j)}, \Lambda^{(j)})} &= \left(\tilde{\theta}|_{\tilde{H}^{[\frac{r}{2}]+1}(\beta, \Lambda)} \right)|_{\tilde{H}^{[\frac{r}{2}]+1}(\beta^{(j)}, \Lambda^{(j)})} \\ &= \left(\tilde{\theta}_0|_{\tilde{H}^{[\frac{r}{2}]+1}(\beta, \Lambda)} \psi_{\beta-\gamma} \right)|_{\tilde{H}^{[\frac{r}{2}]+1}(\beta^{(j)}, \Lambda^{(j)})} = \tilde{\theta}_0^{(j)}|_{\tilde{H}^{[\frac{r}{2}]+1}(\beta^{(j)}, \Lambda^{(j)})} \psi_{\beta^{(j)}-\gamma^{(j)}}, \end{aligned}$$

for some $\tilde{\theta}_0^{(j)} \in \mathcal{C}(\Lambda^{(j)}, 0, \gamma^{(j)})$, by the inductive hypothesis. ■

We continue with the same situation. From [32, Proposition 3.24], the pairing

$$\mathbf{k}_{\tilde{\theta}}(x, y) = \tilde{\theta}[x, y], \quad x, y \in \tilde{J}^1(\beta, \Lambda),$$

induces a nondegenerate alternating form on $\tilde{J}^1(\beta, \Lambda)/\tilde{H}^1(\beta, \Lambda)$. Likewise, for $j = 1, \dots, m$ we have a form $\mathbf{k}_{\tilde{\theta}^{(j)}}$ on $\tilde{J}^1(\beta^{(j)}, \Lambda^{(j)})/\tilde{H}^1(\beta^{(j)}, \Lambda^{(j)})$.

Let \tilde{U}_l denote the unipotent radical of the parabolic subgroup opposite \tilde{P} . As an immediate consequence of Proposition 5.5 and the Iwahori decompositions for $\tilde{H}^1(\beta, \Lambda)$ and $\tilde{J}^1(\beta, \Lambda)$, we get:

Lemma 5.6 (cf. [8, Proposition 7.2.3]). (i) *The subspaces $\tilde{J}^1 \cap \tilde{U}_l / \tilde{H}^1 \cap \tilde{U}_l$ and $\tilde{J}^1 \cap \tilde{U} / \tilde{H}^1 \cap \tilde{U}$ of $\tilde{J}^1 / \tilde{H}^1$ are both totally isotropic for the form $\mathbf{k}_{\tilde{\theta}}$ and orthogonal to $\tilde{J}^1 \cap \tilde{M} / \tilde{H}^1 \cap \tilde{M}$.*

(ii) The restriction of $\mathbf{k}_{\tilde{\theta}}$ to the group

$$\tilde{\mathbb{J}}^1 \cap \tilde{M} / \tilde{\mathbb{H}}^1 \cap \tilde{M} = \prod_{j=1}^m \tilde{\mathbb{J}}^1(\beta^{(j)}, \Lambda^{(j)}) / \tilde{\mathbb{H}}^1(\beta^{(j)}, \Lambda^{(j)})$$

is the orthogonal sum of the pairings $\mathbf{k}_{\tilde{\theta}^{(j)}}$, where $\tilde{\theta}^{(j)} = \tau_{\Lambda, \Lambda^{(j)}, \beta}(\tilde{\theta})$.

(iii) We have an orthogonal sum decomposition

$$\frac{\tilde{\mathbb{J}}^1}{\tilde{\mathbb{H}}^1} = \frac{\tilde{\mathbb{J}}^1 \cap \tilde{M}}{\tilde{\mathbb{H}}^1 \cap \tilde{M}} \perp \left(\frac{\tilde{\mathbb{J}}^1 \cap \tilde{U}_l}{\tilde{\mathbb{H}}^1 \cap \tilde{U}_l} \times \frac{\tilde{\mathbb{J}}^1 \cap \tilde{U}}{\tilde{\mathbb{H}}^1 \cap \tilde{U}} \right).$$

Recall that we have defined the group $\tilde{\mathbb{H}}_{\tilde{P}}^1 = \tilde{\mathbb{H}}^1(\beta, \Lambda) \left(\tilde{\mathbb{J}}^1(\beta, \Lambda) \cap \tilde{U} \right)$, and, similarly, $\tilde{\mathbb{J}}_{\tilde{P}}^1$. We define the character $\tilde{\theta}_{\tilde{P}}$ of $\tilde{\mathbb{H}}_{\tilde{P}}^1$ by

$$\tilde{\theta}_{\tilde{P}}(hj) = \tilde{\theta}(h), \quad \text{for } h \in \tilde{\mathbb{H}}^1(\beta, \Lambda), j \in \tilde{\mathbb{J}}^1(\beta, \Lambda) \cap \tilde{U}.$$

This is well-defined by Lemma 5.6. As in [8, §7.2], we immediately get

Corollary 5.7 (cf. [8, Propositions 7.2.4, 7.2.9]). *There is a unique irreducible representation $\tilde{\eta}_{\tilde{P}}$ of $\tilde{\mathbb{J}}_{\tilde{P}}^1$ such that $\tilde{\eta}_{\tilde{P}}|_{\tilde{\mathbb{H}}_{\tilde{P}}^1}$ contains $\tilde{\theta}_{\tilde{P}}$. Moreover, $\tilde{\eta} \simeq \text{Ind}_{\tilde{\mathbb{J}}_{\tilde{P}}^1}^{\tilde{\mathbb{J}}^1(\beta, \Lambda)} \tilde{\eta}_{\tilde{P}}$ and, for each $y \in \tilde{G}_E$, there is a unique $(\tilde{\mathbb{J}}_{\tilde{P}}^1, \tilde{\mathbb{J}}_{\tilde{P}}^1)$ -double coset in $\tilde{\mathbb{J}}^1(\beta, \Lambda) y \tilde{\mathbb{J}}^1(\beta, \Lambda)$ which intertwines $\tilde{\eta}_{\tilde{P}}$.*

Lemma 5.8 (cf. [16, Proposition 2.4]). *We have $I_{\tilde{G}}(\tilde{\theta}_{\tilde{P}}) = I_{\tilde{G}}(\tilde{\eta}_{\tilde{P}}) = \tilde{\mathbb{J}}_{\tilde{P}}^1 \tilde{G}_E \tilde{\mathbb{J}}_{\tilde{P}}^1$.*

Proof We have $I_{\tilde{G}}(\tilde{\theta}_{\tilde{P}}) = I_{\tilde{G}}(\tilde{\eta}_{\tilde{P}})$ so, by Corollary 5.7, we need only check that all of \tilde{G}_E intertwines $\tilde{\theta}_{\tilde{P}}$.

Let $\mathcal{B}_{i,j}$ be an E_i -basis for $W^{(i,j)} = V^i \cap W^{(j)}$ which splits the lattice sequence $\Lambda^{(i,j)}$ and consider the decomposition

$$V = \bigoplus_{i,j} \bigoplus_{\mathbf{v} \in \mathcal{B}_{i,j}} E_i \mathbf{v}. \quad (5.9)$$

This is subordinate to the stratum $[\Lambda, n, 0, \beta]$ and refines both the decomposition $V = \bigoplus_{j=1}^m W^{(j)}$ and the splitting $V = \bigoplus_{i=1}^l V^i$ of the stratum. Let \tilde{M}_0 denote the stabilizer of the decomposition (5.9) and let $\tilde{P}_0 \subseteq \tilde{P}$ be a parabolic subgroup with Levi component \tilde{M}_0 . We write \tilde{U}_0 for the unipotent radical of \tilde{P}_0 , so $\tilde{U}_0 \supseteq \tilde{U}$.

Let $\mathcal{P}_0 \subseteq A$ denote the Lie algebra of \tilde{P}_0 . Then $\mathcal{P}_0 \cap \mathfrak{b}_0(\Lambda) + \mathfrak{b}_1(\Lambda)$ is a minimal \mathfrak{o}_E -order in B , which must be of the form $\mathfrak{b}_0(\Lambda^{\mathfrak{m}})$, for some \mathfrak{o}_E -lattice sequence $\Lambda^{\mathfrak{m}}$ in V . Then, for a suitable integer $n_{\mathfrak{m}}$, the stratum $[\Lambda^{\mathfrak{m}}, n_{\mathfrak{m}}, 0, \beta]$ is semisimple and the decomposition (5.9) is subordinate to it. Moreover, $\tilde{\mathbb{P}}(\Lambda_{\mathfrak{o}_E}^{\mathfrak{m}})$ is an Iwahori subgroup of \tilde{G}_E contained in $\tilde{\mathbb{P}}(\Lambda_{\mathfrak{o}_E})$.

Now, by construction, $\tilde{\mathbb{P}}(\Lambda_{\mathfrak{o}_E}^{\mathfrak{m}}) = (\tilde{\mathbb{P}}(\Lambda_{\mathfrak{o}_E}) \cap \tilde{P}_0) \tilde{\mathbb{P}}^1(\Lambda_{\mathfrak{o}_E}) \subseteq \tilde{\mathbb{J}}_{\tilde{P}}$ and, since $\tilde{\mathbb{J}}_{\tilde{P}}$ normalizes $\tilde{\theta}_{\tilde{P}}$, we need only show that some set of double coset representatives for $\tilde{\mathbb{P}}(\Lambda_{\mathfrak{o}_E}^{\mathfrak{m}}) \backslash \tilde{G}_E / \tilde{\mathbb{P}}(\Lambda_{\mathfrak{o}_E}^{\mathfrak{m}})$ intertwines $\tilde{\theta}_{\tilde{P}}$. By the Bruhat decomposition, we may take these representatives to be in the \tilde{G}_E -normalizer of the maximal torus $\tilde{M}_0 \cap \tilde{G}_E$.

So suppose $y \in \tilde{G}_E$ normalizes \tilde{M}_0 . Since $\tilde{H}_{\tilde{P}}^1$ has Iwahori decompositions with respect to both $(\tilde{M}_0, \tilde{P}_0)$ and $(\tilde{M}_0, \tilde{P}_0^y)$, the group $\tilde{H}_{\tilde{P}}^1 \cap {}^y\tilde{H}_{\tilde{P}}^1$ has an Iwahori decomposition with respect to $(\tilde{M}_0, \tilde{P}_0)$. Moreover, $\tilde{\theta}_{\tilde{P}}$ is trivial on both $\tilde{H}_{\tilde{P}}^1 \cap \tilde{U}_0$ and $\tilde{H}_{\tilde{P}}^1 \cap \tilde{U}_0^y$. Hence, to check that y intertwines $\tilde{\theta}_{\tilde{P}}$, we need only check that it intertwines $\tilde{\theta}_{\tilde{P}}|_{\tilde{H}_{\tilde{P}}^1 \cap \tilde{M}_0}$. But, since $\tilde{M}_0 \subseteq \tilde{M}$ and $\tilde{H}_{\tilde{P}}^1 \cap \tilde{M} = \tilde{H}^1 \cap \tilde{M}$, this restriction is just $\tilde{\theta}|_{\tilde{H}^1(\beta, \Lambda) \cap \tilde{M}_0}$, which is surely intertwined by y , by Proposition 3.1. \blacksquare

We remark that, in the same way as in [8, §7.2], we can describe $\tilde{\eta}_{\tilde{P}}$ as the natural representation of $\tilde{J}_{\tilde{P}}^1$ on the space of $(\tilde{J}^1(\beta, \Lambda) \cap \tilde{U})$ -fixed vectors in $\tilde{\eta}$. It is also useful to note that the restriction of $\tilde{\eta}_{\tilde{P}}$ to $\tilde{J}_{\tilde{P}}^1 \cap \tilde{M} = \tilde{J}^1(\beta, \Lambda) \cap \tilde{M}$ is given by

$$\tilde{\eta}_{\tilde{P}}|_{\tilde{J}^1(\beta, \Lambda) \cap \tilde{M}} \simeq \bigotimes_{j=1}^m \tilde{\eta}^{(j)},$$

where $\tilde{\eta}^{(j)}$ is the unique irreducible representation of $\tilde{J}^1(\beta^{(j)}, \Lambda^{(j)})$ which contains the character $\tilde{\theta}^{(j)} = \tau_{\Lambda, \Lambda^{(j)}, \beta}(\tilde{\theta})$.

5.3 Glauberman transfer

In this section, we consider the case when we have a skew semisimple stratum $[\Lambda, n, 0, \beta]$. We say that a decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ is *self-dual* if, for $-m \leq j \leq m$, the orthogonal complement of $W^{(j)}$ is $\bigoplus_{k \neq -j} W^{(k)}$; we allow the possibility that $W^{(0)} = \{0\}$. Then, immediately from Proposition 5.2 we get:

Corollary 5.10. *Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum and let $V = \bigoplus_{j=-m}^m W^{(j)}$ be a self-dual decomposition as above. Let \tilde{M} be the Levi subgroup of \tilde{G} which is the stabilizer of this decomposition and let \tilde{P} be any σ -stable parabolic subgroup with Levi component \tilde{M} . Put $M^+ = \tilde{M} \cap G^+$, $P^+ = \tilde{P} \cap G^+$ and $M = \tilde{M} \cap G$, $P = \tilde{P} \cap G$.*

- (i) *If the decomposition is subordinate to $[\Lambda, n, 0, \beta]$, then the groups $H^1(\beta, \Lambda)$ and $J^1(\beta, \Lambda)$ have Iwahori decompositions with respect to (M, P) .*
- (ii) *If the decomposition is properly subordinate to $[\Lambda, n, 0, \beta]$, then the group $J^+(\beta, \Lambda)$ also has an Iwahori decomposition with respect to (M^+, P^+) and the groups $J(\beta, \Lambda)$, $J^0(\beta, \Lambda)$ have Iwahori decompositions with respect to (M, P) .*

Proof Note that, if $g \in G^+$ has Iwahori decomposition $g = lmu$, with respect to some self-dual (\tilde{M}, \tilde{P}) in \tilde{G} , then also $g = l^\sigma m^\sigma u^\sigma$ so, by uniqueness of decomposition, $l, m, u \in G^+$. Now, since $U \subseteq G$, if $g \in G$ then $l, m, u \in G$ also. \blacksquare

Notice that, in this situation, the involution swaps the blocks $\mathbf{e}^{(j)} \mathbf{Ae}^{(k)}$ and $\mathbf{e}^{(-k)} \mathbf{Ae}^{(-j)}$ so that the stratum $[\Lambda^{(0)}, n^{(0)}, 0, \beta^{(0)}]$ is skew semisimple. From Proposition 5.4 we get:

Corollary 5.11. *Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum and let $V = \bigoplus_{j=-m}^m W^{(j)}$ be a self-dual decomposition subordinate to the stratum. Let \tilde{M} be the Levi subgroup of \tilde{G} which is the stabilizer of this decomposition and put $M^+ = \tilde{M} \cap G^+$, $M = \tilde{M} \cap G$. Then*

$$H^1(\beta, \Lambda) \cap M \simeq H^1(\beta^{(0)}, \Lambda^{(0)}) \times \prod_{j=1}^m \tilde{H}^1(\beta^{(j)}, \Lambda^{(j)}),$$

and there is a similar decomposition for $J^1(\beta, \Lambda)$. If the decomposition is properly subordinate to the stratum then we also have

$$J^\circ(\beta, \Lambda) \cap M \simeq J^\circ(\beta^{(0)}, \Lambda^{(0)}) \times \prod_{j=1}^m \tilde{J}(\beta^{(j)}, \Lambda^{(j)}),$$

and similar decompositions for $J^+(\beta, \Lambda) \cap M^+$ and $J(\beta, \Lambda) \cap M$.

Note here that $\tilde{H}^1(\beta^{(j)}, \Lambda^{(j)})$ has been identified with

$$\left\{ (k, h) \in \tilde{H}^1(\beta^{(-j)}, \Lambda^{(-j)}) \times \tilde{H}^1(\beta^{(j)}, \Lambda^{(j)}) \text{ such that } h\bar{k} = 1 \right\}$$

(Recall that the involution $-$ swaps the blocks $\mathbf{e}^{(j)} A \mathbf{e}^{(j)}$ and $\mathbf{e}^{(-j)} A \mathbf{e}^{(-j)}$.)

We can also form the groups $H_P^1 = \tilde{H}_P^1 \cap G = H^1(\beta, \Lambda) (J^1(\beta, \Lambda) \cap U)$ and, likewise, J_P^1 and (when the decomposition is properly subordinate to the stratum) $J_P^+ = H^1(\beta, \Lambda) (J^+(\beta, \Lambda) \cap P)$, and J_P , J_P° .

Now, using the Glauberman correspondence, we transfer the results of §5.2 to the case of skew semisimple strata. We recall briefly the nature of the correspondence, for details of which (in our situation) we refer the reader to [30, §2].

Let \tilde{K} be a σ -stable pro- p subgroup of \tilde{G} and $K = \tilde{K}^\sigma$ the group of σ -fixed points. Then σ acts on the representations $\tilde{\rho}$ of \tilde{K} by $\tilde{\rho}^\sigma(k) = \rho(k^\sigma)$, for $k \in \tilde{K}$. There is a bijection, denoted $\tilde{\rho} \leftrightarrow \rho = \mathbf{g}(\tilde{\rho})$, between (equivalence classes of) irreducible representations of \tilde{K} with $\tilde{\rho}^\sigma \simeq \tilde{\rho}$ and (equivalence classes of) irreducible representations of K . Further, this correspondence commutes with irreducible restriction and irreducible induction. Recall also that the representation $\rho = \mathbf{g}(\tilde{\rho})$ is characterized as the unique component of $\tilde{\rho}|_K$ appearing with odd multiplicity; in particular, if $\tilde{\rho}$ is a character then $\rho = \tilde{\rho}|_K$. Moreover, for $g \in G^+$, we have that $\dim I_g(\tilde{\rho})$ is odd if and only if $\dim I_g(\rho)$ is odd; in particular, if $\tilde{\rho}$ is a character, or if all the intertwining spaces are one-dimensional (as is often the case for us), then $I_{G^+}(\rho|_K) = I_{\tilde{G}}(\tilde{\rho}|_{\tilde{K}}) \cap G^+$.

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum with a subordinate self-dual decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$. Let \tilde{M} be the stabilizer in \tilde{G} of this decomposition and let \tilde{P} be a σ -stable parabolic subgroup with Levi component \tilde{M} . Put $M = \tilde{M} \cap G$ and $P = \tilde{P} \cap G$.

Let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ be a skew semisimple character of $H^1(\beta, \Lambda)$, so that $\theta = \tilde{\theta}|_{H^1(\beta, \Lambda)}$, for some σ -stable semisimple character $\tilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$. From Corollary 5.11 and Proposition 5.5, we have

$$\theta|_{H^1(\beta, \Lambda) \cap M} = \theta^{(0)} \otimes \bigotimes_{j=1}^m \left(\tilde{\theta}^{(j)} \right)^2,$$

for some $\theta^{(0)} \in \mathcal{C}_-(\Lambda^{(0)}, 0, \beta^{(0)})$ and $\tilde{\theta}^{(j)} \in \mathcal{C}(\Lambda^{(j)}, 0, \beta^{(j)})$. Notice that we have $\tilde{H}^1(\beta^{(j)}, \Lambda^{(j)}) = \tilde{H}^1(2\beta^{(j)}, \Lambda^{(j)})$ and $\left(\tilde{\theta}^{(j)} \right)^2 \in \mathcal{C}(\Lambda^{(j)}, 0, 2\beta^{(j)})$, as in [3, §4.3 Lemma 1].

We define the character θ_P of $H_P^1 = H^1(\beta, \Lambda) (J^1(\beta, \Lambda) \cap U)$ by

$$\theta_P(hj) = \theta(h), \quad \text{for } h \in H^1(\beta, \Lambda), \quad j \in J^1(\beta, \Lambda) \cap U.$$

Then $\theta_P = \tilde{\theta}|_{H_P^1}$ and is the Glauberman transfer of $\tilde{\theta}$.

Let η be the unique irreducible representation of $J^1(\beta, \Lambda)$ containing θ and let $\tilde{\eta}$ be the unique irreducible representation of $\tilde{J}^1(\beta, \Lambda)$ containing $\tilde{\theta}$. Then $\eta = \mathbf{g}(\tilde{\eta})$. Let $\tilde{\eta}_{\tilde{P}}$ be the unique irreducible representation of $\tilde{J}_{\tilde{P}}^1$ containing $\tilde{\theta}_{\tilde{P}}$, given by Corollary 5.7, and define the representation η_P of $J_P^1 = H^1(\beta, \Lambda) (J^1(\beta, \Lambda) \cap P)$ by $\eta_P = \mathbf{g}(\tilde{\eta}_{\tilde{P}})$. Then Corollary 5.7 and Lemma 5.8, together with the properties of the Glauberman correspondence, give us

Lemma 5.12. *The representation η_P is the unique irreducible representation of J_P^1 such that $\eta_P|_{H_P^1}$ contains θ_P . We have $\eta \simeq \text{Ind}_{J_P^1}^{J^1(\beta, \Lambda)} \eta_P$. Moreover, for each $y \in G_E^+$, the coset $J_P^1 y J_P^1$ is the unique (J_P^1, J_P^1) -double coset in $J^1(\beta, \Lambda) y J^1(\beta, \Lambda)$ which intertwines η_P and*

$$\dim I_g(\eta_P) = \begin{cases} 1 & \text{if } g \in J_P^1 G_E^+ J_P^1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof The statement about the dimensions of the intertwining spaces comes from the corresponding statement for η , which is Proposition 3.5. \blacksquare

As above we can describe η_P as being the natural representation of J_P^1 on the space of $(J^1(\beta, \Lambda) \cap U)$ -fixed vectors in η and the restriction of η_P to $J_P^1 \cap M = J^1(\beta, \Lambda) \cap M$ is given by

$$\eta_P|_{J^1(\beta, \Lambda) \cap M} \simeq \eta^{(0)} \otimes \bigotimes_{j=1}^m \tilde{\eta}^{(j)},$$

where $\eta^{(0)}$ is the unique irreducible representation of $J^1(\beta^{(0)}, \Lambda^{(0)})$ which contains the character $\theta^{(0)}$ and, for j non-zero, $\tilde{\eta}^{(j)}$ is the unique irreducible representation of $\tilde{J}^1(\beta^{(j)}, \Lambda^{(j)}) = \tilde{J}^1(2\beta^{(j)}, \Lambda^{(j)})$ which contains the character $(\tilde{\theta}^{(j)})^2$.

We continue in the same situation and suppose from now on that the self-dual decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ is *properly* subordinate to the skew semisimple stratum $[\Lambda, n, 0, \beta]$. Let κ be a β -extension of η to $J^+(\beta, \Lambda)$ (relative to some Λ^M). We can form the natural representation κ_P of J_P^+ on the space of $(J^+(\beta, \Lambda) \cap U) = (J^1(\beta, \Lambda) \cap U)$ -fixed vectors in κ . Then $\kappa_P|_{J_P^1} = \eta_P$ so, in particular, κ_P is irreducible. As in the proof of [8, Proposition 7.2.15], the Mackey restriction formula gives:

Proposition 5.13 (cf. [8, Proposition 7.2.15]). *We have $\text{Ind}_{J_P^+}^{J^+(\beta, \Lambda)} \kappa_P \simeq \kappa$.*

As for η_P , we can describe the restriction of κ_P to $J_P^+ \cap M = J^+(\beta, \Lambda) \cap M$. Since $\kappa_P|_{J^+(\beta, \Lambda) \cap M}$ restricts further to $\eta_P|_{J^1(\beta, \Lambda) \cap M}$, it is irreducible and

$$\kappa_P|_{J^+(\beta, \Lambda) \cap M} = \kappa^{(0)} \otimes \bigotimes_{j=1}^m \tilde{\kappa}^{(j)},$$

where $\kappa^{(0)}$ is some irreducible representation of $J^+(\beta^{(0)}, \Lambda^{(0)})$ which extends $\eta^{(0)}$ and $\tilde{\kappa}^{(j)}$ is some irreducible representation of $\tilde{J}(\beta^{(j)}, \Lambda^{(j)})$ which extends $\tilde{\eta}^{(j)}$.

We will see below in Proposition 6.3 that, at least in certain circumstances, $\kappa^{(0)}$ is a $\beta^{(0)}$ extension and $\tilde{\kappa}^{(j)}$ is a $2\beta^{(j)}$ -extension for $1 \leq j \leq m$.

We can also make the same constructions for the restriction $\kappa|_{\mathbf{J}^\circ(\beta, \Lambda)}$, and we clearly obtain the restriction $\kappa_P|_{\mathbf{J}_P^\circ}$. Moreover, we have

$$\mathrm{Ind}_{\mathbf{J}_P^\circ}^{\mathbf{J}^\circ(\beta, \Lambda)}(\kappa_P|_{\mathbf{J}_P^\circ}) \simeq \kappa|_{\mathbf{J}^\circ(\beta, \Lambda)} \quad \text{and} \quad \mathrm{Ind}_{\mathbf{J}_P}^{\mathbf{J}(\beta, \Lambda)}(\kappa_P|_{\mathbf{J}_P}) \simeq \kappa|_{\mathbf{J}(\beta, \Lambda)}.$$

Indeed, these are special cases of Lemma 6.1 below. We also have

$$\kappa_P|_{\mathbf{J}^\circ(\beta, \Lambda) \cap M} = \kappa^{(0)}|_{\mathbf{J}^\circ(\beta^{(0)}, \Lambda^{(0)})} \otimes \bigotimes_{j=1}^m \tilde{\kappa}^{(j)}.$$

6 Intertwining and supercuspidal representations

The aim of this section is two-fold. First we seek to improve the lower bound for the intertwining of β -extensions from Corollary 4.6, in particular for standard β -extensions. In order to do this we need first to define certain Weyl group elements. Secondly, we seek to bound above the intertwining of representations of the form $\kappa|_{\mathbf{J}^\circ} \otimes \rho$, where κ is a standard β -extension and ρ is a *cuspidal* representation of the reductive quotient $\mathbf{J}^\circ/\mathbf{J}^1$. This will use both the lower bound for the intertwining of κ and, critically, the result of Morris, Lemma 1.1.

6.1 A Hecke algebra isomorphism

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum and let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ be a skew semisimple character. We use our usual notation. Let η be the unique irreducible representation of $\mathbf{J}^1(\beta, \Lambda)$ containing θ and let κ be a β -extension of η (relative to some Λ^M).

Let $V = \bigoplus_{j=-m}^m W^{(j)}$ be a self-dual decomposition which is properly subordinate to $[\Lambda, n, 0, \beta]$. As in §5, let M^+ be the Levi subgroup of G^+ which stabilizes the decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ and let P^+ be a parabolic subgroup with Levi component M^+ , and unipotent radical U . We use all the other related notation from §5.

The following Lemma will be useful in several situations.

Lemma 6.1. *Let \mathbf{K} be a compact open subgroup of G with $\mathbf{J}^1(\beta, \Lambda) \subseteq \mathbf{K} \subseteq \mathbf{J}^+(\beta, \Lambda)$, such that \mathbf{K} has an Iwahori decomposition with respect to (M^+, P^+) . We write*

$$\mathbf{K} \cap M^+ = \mathbf{K}^{(0)} \times \prod_{j=1}^m \tilde{\mathbf{K}}^{(j)}$$

Let ρ be the inflation to \mathbf{K} of an irreducible representation of $\mathbf{K}/\mathbf{J}^1(\beta, \Lambda)$, and put $\lambda = \kappa|_{\mathbf{K}} \otimes \rho$. Let λ_P denote the natural representation of $\mathbf{K}_P = \mathbf{H}^1(\beta, \Lambda)(\mathbf{K} \cap P)$ on the space of $\mathbf{J}^1(\beta, \Lambda) \cap U$ -fixed vectors in λ . Then

- (i) λ_P is irreducible and $\lambda \simeq \mathrm{Ind}_{\mathbf{K}_P}^{\mathbf{K}} \lambda_P$.
- (ii) We have $\lambda_P \simeq \kappa_P \otimes \rho$, considering ρ as a representation of $\mathbf{K}_P/\mathbf{J}_P^1 \cong \mathbf{K}/\mathbf{J}^1(\beta, \Lambda)$.
- (iii) $\lambda_P|_{\mathbf{K} \cap M} = \lambda^{(0)} \otimes \bigotimes_{j=1}^m \tilde{\lambda}^{(j)}$, where $\lambda^{(0)} = \kappa^{(0)}|_{\mathbf{K}^{(0)}} \otimes \rho^{(0)}$ is a representation of $\mathbf{K}^{(0)}$ and $\tilde{\lambda}^{(j)} = \tilde{\kappa}^{(j)}|_{\tilde{\mathbf{K}}^{(j)}} \otimes \tilde{\rho}^{(j)}$ is a representation of $\tilde{\mathbf{K}}^{(j)}$, for $1 \leq j \leq m$.

(iv) *There is an algebra isomorphism*

$$\mathcal{H}(G^+, \lambda_P) \cong \mathcal{H}(G^+, \lambda)$$

which preserves support: if $\phi \in \mathcal{H}(G^+, \lambda)$ has support KyK , for some $y \in G_E^+$, then the corresponding function $\phi_P \in \mathcal{H}(G^+, \lambda_P)$ has support $\text{K}_P y \text{K}_P$.

Proof (i), (ii) and (iii) are clear. We deduce from (ii) and [8, Proposition 4.1.3] that we have an algebra isomorphism $\mathcal{H}(G^+, \lambda_P) \cong \mathcal{H}(G^+, \lambda)$. Moreover, if $\phi \in \mathcal{H}(G^+, \lambda)$ has support KyK , for some $y \in G_E^+$, then [8, Corollary 4.1.5] shows that the corresponding function $\phi_P \in \mathcal{H}(G^+, \lambda_P)$ has support on a union of double cosets $\text{K}_P x \text{K}_P$ contained in KyK . Using the Iwahori decomposition for K , we may assume further that $x \in (\text{K} \cap U_l) y (\text{K} \cap U_l) \subseteq \text{J}^1(\beta, \Lambda) y \text{J}^1(\beta, \Lambda)$, where U_l is the unipotent radical of the parabolic subgroup opposite P^+ .

Since $\lambda_P|_{\text{J}_P^1}$ is a multiple of η_P , we also have that x intertwines η_P . Meanwhile, by Lemma 5.12, $\text{J}_P^1 y \text{J}_P^1$ is the unique $(\text{J}_P^1, \text{J}_P^1)$ -double coset in $\text{J}^1(\beta, \Lambda) y \text{J}^1(\beta, \Lambda)$ which intertwines η_P . Hence $x \in \text{J}_P^1 y \text{J}_P^1$ and, since $\text{J}_P^1 \subseteq \text{K}_P$, we have $\text{K}_P x \text{K}_P = \text{K}_P y \text{K}_P$. Hence ϕ_P has support $\text{K}_P y \text{K}_P$, as required. Since

$$\text{Supp } \mathcal{H}(G^+, \lambda) \subseteq I_{G^+}(\lambda|_{\text{J}^1(\beta, \Lambda)}) = I_{G^+}(\eta|_{\text{J}^1(\beta, \Lambda)}) = \text{J}^1(\beta, \Lambda) G_E^+ \text{J}^1(\beta, \Lambda) = \text{K} G_E^+ \text{K},$$

we are done. ■

Corollary 6.2. *Let κ be a β -extension of η to $\text{J}^+(\beta, \Lambda)$ (relative to Λ^M). Then $\text{P}^+(\Lambda_{\mathfrak{o}_E}^M) \subseteq I_G(\kappa_P|_{\text{J}_P^+})$.*

Proof Let $y \in \text{P}^+(\Lambda_{\mathfrak{o}_E}^M)$. By Corollary 4.6, y intertwines κ so the result follows by applying Lemma 6.1 with $\text{K} = \text{J}^+(\beta, \Lambda)$ and $\lambda = \kappa$. ■

We write

$$\kappa_P|_{\text{J}^+(\beta, \Lambda) \cap M} = \kappa^{(0)} \otimes \bigotimes_{j=1}^m \tilde{\kappa}^{(j)},$$

as before.

Proposition 6.3 (cf. [8, Corollary 7.2.16]). (i) *Suppose $\mathfrak{b}_0(\Lambda^{(0)})$ is a maximal self-dual \mathfrak{o}_E -order in $B^{(0)}$. Then $\kappa^{(0)}$ is a $\beta^{(0)}$ -extension of $\eta^{(0)}$ to $\text{J}^+(\beta^{(0)}, \Lambda^{(0)})$.*

(ii) *If $1 \leq j \leq m$, $W^{(j)} \subseteq V^i$, for some i , and $\mathfrak{b}_0(\Lambda^{(j)})$ is a maximal \mathfrak{o}_{E_i} -order in $B^{(j)}$, then $\tilde{\kappa}^{(j)}$ is a $2\beta^{(j)}$ -extension of $\tilde{\eta}^{(j)}$ to $\tilde{\text{J}}(\beta^{(j)}, \Lambda^{(j)})$.*

Note that this Proposition is clear when $|k_F| > 3$, since then *any* extension of $\eta^{(0)}$ to $\text{J}^+(\beta^{(0)}, \Lambda^{(0)})$ is a $\beta^{(0)}$ -extension in this maximal case and, for $j \neq 0$, any extension of $\tilde{\eta}^{(j)}$ to $\tilde{\text{J}}(\beta^{(j)}, \Lambda^{(j)})$ is a $2\beta^{(j)}$ -extension (the latter only requires $|k_F| > 2$).

Proof For $i = 1, \dots, l$, let Λ_m^i be an \mathfrak{o}_{E_i} -lattice sequence in V^i such that $\mathfrak{b}_0(\Lambda_m^i)$ is a minimal self-dual \mathfrak{o}_{E_i} -order contained in $\mathfrak{b}_0(\Lambda^i)$ and such that the self-dual decomposition

$$V^i = \bigoplus_{j=-m}^m V^i \cap W^{(j)}$$

splits Λ_m^i . Put $\Lambda^m = \bigoplus_{i=1}^l \Lambda_m^i$, a self-dual \mathfrak{o}_E -lattice sequence in V such that $P(\Lambda_{\mathfrak{o}_E}^m)$ is an Iwahori subgroup contained in $P(\Lambda_{\mathfrak{o}_E})$ and such that the self-dual decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ is (properly) subordinate to $[\Lambda^m, n_m, 0, \beta]$.

For $j = -m, \dots, m$, set $E^{(j)} = F[\beta^{(j)}]$. We also form the $\mathfrak{o}_{E^{(j)}}$ -lattice sequence $\Lambda^{m^{(j)}}$ in $W^{(j)}$, given by

$$\Lambda^{m^{(j)}}(r) = \Lambda^m(r) \cap W^{(j)}, \quad \text{for } r \in \mathbb{Z}.$$

Then $\mathfrak{b}_0(\Lambda^{m^{(j)}})$ is a minimal (self-dual, in the case $j = 0$) $\mathfrak{o}_{E^{(j)}}$ -order contained in $\mathfrak{b}_0(\Lambda^{(j)})$. In particular, we can think of the pair $(\Lambda^{m^{(j)}}, \Lambda^{(j)})$ in the same way as the pair (Λ^m, Λ) , and make the constructions analogous to those below in these cases too.

We form the group $J_{m,\Lambda}^1 = P_1(\Lambda_{\mathfrak{o}_E}^m)J^1(\beta, \Lambda)$ and let $\eta_{m,\Lambda}$ be the unique extension of η to $J_{m,\Lambda}^1$ which is intertwined by G_E^+ (using Corollary 3.11). Recall that, by Proposition 4.7, we have $\kappa|_{J_{m,\Lambda}^1} = \eta_{m,\Lambda}$. Notice that, since $P_1(\Lambda_{\mathfrak{o}_E}^m) \subseteq P(\Lambda_{\mathfrak{o}_E})$ and $P(\Lambda_{\mathfrak{o}_E}) \cap U = P_1(\Lambda_{\mathfrak{o}_E}) \cap U$, we have $J_{m,\Lambda}^1 \cap U = J^1(\beta, \Lambda) \cap U$. Then we can form the natural representation $\eta_P^{m,\Lambda}$ of $J_{m,\Lambda,P}^1 = H^1(\beta, \Lambda) \left(J_{m,\Lambda}^1 \cap P \right)$ on the space of $J^1(\beta, \Lambda) \cap U$ -fixed vectors in $\eta_{m,\Lambda}$. By applying Lemma 6.1, we have

$$\eta_{m,\Lambda} \simeq \text{Ind}_{J_{m,\Lambda,P}^1}^{J_{m,\Lambda}^1} \eta_P^{m,\Lambda}.$$

It is irreducible and we can describe its restriction to

$$J_{m,\Lambda}^1 \cap M = J_{m^{(0)},\Lambda^{(0)}}^1 \times \prod_{j=1}^m \tilde{J}_{m^{(j)},\Lambda^{(j)}}^1,$$

where $J_{m^{(0)},\Lambda^{(0)}}^1 = P_1(\Lambda_{\mathfrak{o}_E^{(0)}}^m)J^1(\beta^{(0)}, \Lambda^{(0)})$ and $\tilde{J}_{m^{(j)},\Lambda^{(j)}}^1 = \tilde{P}_1(\Lambda_{\mathfrak{o}_E^{(j)}}^m)J^1(\beta^{(j)}, \Lambda^{(j)})$, for $j = 1, \dots, m$. We have

$$\eta_P^{m,\Lambda}|_{J_{m,\Lambda}^1 \cap M} = \hat{\eta} = \hat{\eta}^{(0)} \otimes \bigotimes_{j=1}^m \hat{\eta}^{(j)},$$

where $\hat{\eta}^{(0)}$ is the restriction of $\kappa^{(0)}$, and $\hat{\eta}^{(j)}$ the restriction of $\tilde{\kappa}^{(j)}$, for $j = 1, \dots, m$.

By Lemma 6.1(iv), G_E^+ intertwines $\eta_P^{m,\Lambda}$ so certainly $G_E^+ \cap M$ intertwines $\hat{\eta} = \eta_P^{m,\Lambda}|_{J_{m,\Lambda}^1 \cap M}$. Then the uniqueness in Corollary 3.11 shows that $\hat{\eta}^{(0)}$ is the unique extension $\eta_{m^{(0)},\Lambda^{(0)}}$ of $\eta^{(0)}$ to $J_{m^{(0)},\Lambda^{(0)}}^1$ which is intertwined by all of $G_{E^{(0)}}^+ = G^+ \cap \text{Aut}_E(W^{(0)})$. But then, if $P^+(\Lambda_{\mathfrak{o}_E}^{(0)})$ is a maximal compact subgroup of $G_{E^{(0)}}^+$, the construction of β -extensions in Proposition 4.1 shows that $\kappa^{(0)}$ is a $\beta^{(0)}$ -extension of $\eta^{(0)}$, since it restricts to $\eta_{m^{(0)},\Lambda^{(0)}}$. The same argument works for $\tilde{\kappa}^{(j)}$, for $j = 1, \dots, m$, using the uniqueness in Proposition 3.12, whenever $W^{(j)} \subseteq V^i$ and $\tilde{P}(\Lambda_{\mathfrak{o}_E^i}^{(j)})$ is a maximal compact subgroup of $\tilde{G}_{E^{(j)}} = \text{Aut}_{E^i}(W^{(j)})$. \blacksquare

We also record the following for later use, which follows from the application of Lemma 6.1 in the proof above:

Corollary 6.4. *The intertwining of $\eta_P^{m,\Lambda}$ is given by*

$$\dim I_g(\eta_P^{m,\Lambda}) = \begin{cases} 1 & \text{if } g \in J_{m,\Lambda,P}^1 G_E^+ J_{m,\Lambda,P}^1; \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.3 motivates the following definition:

Definition 6.5. Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum. We say that a self-dual decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ is *exactly subordinate* to $[\Lambda, n, 0, \beta]$ if it is minimal amongst all self-dual decompositions which are properly subordinate to $[\Lambda, n, 0, \beta]$ – that is, no refinement of the decomposition is properly subordinate to the stratum.

A properly subordinate decomposition is exactly subordinate to $[\Lambda, n, 0, \beta]$ if and only if $\mathfrak{a}_0(\Lambda^{(0)}) \cap B^{(0)}$ is a maximal self-dual \mathfrak{o}_E -order in $B^{(0)}$ and, for each $j \neq 0$, there is an i such that $W^{(j)}$ is contained in V^i and $\mathfrak{a}_0(\Lambda^{(j)}) \cap B^{(j)}$ is a maximal \mathfrak{o}_{E_i} -order in $B^{(j)}$. In particular, Proposition 6.3 can be applied to all blocks when the decomposition is exactly subordinate.

6.2 Some Weyl group elements

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum and let $V = \bigoplus_{j=-m}^m W^{(j)}$ be a self-dual decomposition which is subordinate (but not necessarily exactly subordinate) to the stratum. We also assume, throughout this section, that each $W^{(j)}$ is contained in some V^i , for $j \neq 0$, and that $\mathfrak{a}_0(\Lambda^{(j)}) \cap B^{(j)}$ is a maximal \mathfrak{o}_{E_i} -order in $B^{(j)}$. In particular, this hypothesis is satisfied whenever the decomposition is exactly subordinate.

It will be useful to have a fixed ordering for the subspaces in the decomposition – in effect, this is giving a preferred choice of the parabolic subgroup P whose Levi component M is the stabilizer of the decomposition. For each $j \neq 0$, write $i = i_j$ for the unique integer such that $W^{(j)} \subseteq V^i$. Since $\mathfrak{a}_0(\Lambda^{(j)}) \cap B^{(j)}$ is a maximal \mathfrak{o}_{E_i} -order, for each $j \neq 0$ there is a unique integer $q_j = q_j(\Lambda)$ such that

- (i) $-e_i/2 \leq q_j \leq e_i/2$, where $e_i = e(\Lambda^i | \mathfrak{o}_{E_i})$;
- (ii) $\Lambda(q_j) \cap W^{(j)} \supsetneq \Lambda(q_j + 1) \cap W^{(j)}$;
- (iii) for $j > 0$, $q_j > -e_i/2$; for $j < 0$, $q_j < e_i/2$.

By duality, we have $i_{-j} = i_j$ and $q_{-j} = -q_j$, for $j \neq 0$. In particular, by swapping the numbering of $W^{(j)}$ and $W^{(-j)}$ if necessary, we may, and do, assume that $q_j \geq 0$ for $j > 0$. We put

$$\nu(j) = \nu(j, \Lambda) = (i_j, q_j) \in \mathbb{Z}^2.$$

Then, for $j > 0$ we order the subspaces $W^{(j)}$ lexicographically according to $\nu(j)$ – that is, by reordering if necessary, we assume that, for $0 < j < k \leq m$, either $i_j < i_k$ or $i_j = i_k$ and $q_j \leq q_k$. (If $\nu(j) = \nu(k)$ then we may order them either way.) The order for $j < 0$ is then determined by duality.

We remark also that this ordering depends on the ordering of the pieces in the splitting $V = \bigoplus_{i=1}^l V^i$, which may be put in any convenient order.

We now need to define certain elements of G_E^+ , which are essentially Weyl group elements. We use the block decomposition for A given by the decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$, ordering the blocks according to $\nu(j)$. For each $j > 0$ we choose an ordered \mathfrak{o}_{E_i} -basis $\mathcal{B}^{(j)} = \{\mathbf{v}_{j,1}, \dots, \mathbf{v}_{j,d_j}\}$ for the lattice $\Lambda^{(j)}(0)$ which splits the lattice sequence $\Lambda^{(j)}$, where $i = i_j$. (Indeed, the second condition follows from the first since $\mathfrak{b}_0(\Lambda^{(j)})$ is a maximal \mathfrak{o}_{E_i} -order.) Note that then $\mathcal{B}^{(j)} \subseteq \Lambda^{(j)}(q_j) \setminus \Lambda^{(j)}(q_j + 1)$.

Recall from §2.1 that we have a nondegenerate ϵ -hermitian $E_i/E_{i,0}$ -form f_i on V^i such that the duality of lattices in V^i induced by f_i coincides with that induced by the restriction of the form h .

On $W^{(j)} \oplus W^{(-j)}$, we have the nondegenerate ϵ -hermitian form which is the restriction of f_i , and we take the ordered basis $\mathcal{B}^{(-j)} = \{\mathbf{v}_{-j,1}, \dots, \mathbf{v}_{-j,d_j}\}$ of $W^{(j)}$ for which

$$f_i(\mathbf{v}_{j,s}, \mathbf{v}_{-j,t}) = \begin{cases} \varpi_i & \text{if } s = t, \\ 0 & \text{otherwise,} \end{cases}$$

where ϖ_i is a (fixed) uniformizer for E_i such that $\overline{\varpi_i} = (-1)^{e(E_i/E_{i,0})-1} \varpi_i$. By duality, this basis is an \mathfrak{o}_{E_i} -basis for the lattice $\Lambda^{(-j)}(1)$ and splits the lattice sequence $\Lambda^{(-j)}$; indeed $\mathcal{B}^{(-j)} \subseteq \Lambda^{(-j)}(e_i - q_j) \setminus \Lambda^{(-j)}(e_i - q_j + 1)$. Finally, for each $i = 1, \dots, l$, we choose a (possibly empty) self-dual E_i -basis $\mathcal{B}^{(i,0)}$ for $W^{(i,0)} = W^{(0)} \cap V^i$ which splits $\Lambda^{(i,0)} = \mathfrak{e}^{(0)} \Lambda^i$.

Suppose $j, k \neq 0$, are such that $i_j = i_k = i$ and $\dim_{E_i} W^{(j)} = \dim_{E_i} W^{(k)}$. Then we write $I_{j,k}$ for the element of B^i which sends the ordered basis $\mathcal{B}^{(k)}$ to the ordered basis $\mathcal{B}^{(j)}$ and all other basis elements to 0. Note that, in the case $j = k$, we have $I_{j,j} = \mathfrak{e}^{(j)}$, the projection onto $W^{(j)}$. We also have the following properties, which are immediate from the definitions:

Lemma 6.6. *For $j, k, s, t \neq 0$ such that $I_{j,k}$ and $I_{s,t}$ are defined, we have*

$$(i) \quad I_{j,k} I_{s,t} = \begin{cases} I_{j,t} & \text{if } k = s; \\ 0 & \text{otherwise.} \end{cases}$$

$$(ii) \quad \text{if } j, k > 0 \text{ then } \overline{I_{j,k}} = I_{-k,-j} \text{ and } \overline{I_{j,-j}} = \epsilon(-1)^{e(E_i/E_{i,0})-1} I_{j,-j}.$$

$$(iii) \quad I_{j,k} \in \begin{cases} \mathfrak{b}_{q_j - q_k}(\Lambda) \setminus \mathfrak{b}_{q_j - q_k + 1}(\Lambda) & \text{if } j, k > 0, \\ \mathfrak{b}_{q_j - q_k - e_i}(\Lambda) \setminus \mathfrak{b}_{q_j - q_k - e_i + 1}(\Lambda) & \text{if } j > 0, k < 0. \end{cases}$$

We remark that, if the decomposition is exactly subordinate then: for $j > 0$, we have $0 < q_j < e_i/2$: and, for $j \neq k$, if $i_j = i_k$ then $q_j \neq q_k$.

We are now ready to define our Weyl group elements. If $j, k > 0$ are distinct, such that $i_j = i_k = i$ and $\dim_{E_i} W^{(j)} = \dim_{E_i} W^{(k)}$, then we put

$$s_{j,k} = I_{j,k} + I_{k,j} + I_{-k,-j} + I_{-j,-k} + \sum_{t \neq \pm j, \pm k} I_{t,t}.$$

Then $s_{j,k}$ exchanges the block $\mathfrak{e}^{(j)} \mathbf{Ae}^{(j)}$ with $\mathfrak{e}^{(k)} \mathbf{Ae}^{(k)}$ and the block $\mathfrak{e}^{(-j)} \mathbf{Ae}^{(-j)}$ with $\mathfrak{e}^{(-k)} \mathbf{Ae}^{(-k)}$. Note also that Lemma 6.6 implies that $\overline{s_{j,k}} = s_{j,k} = s_{j,k}^{-1}$, so that $s_{j,k} \in G_E^+$; indeed, we have $s_{j,k} \in G_E$ since $\det_{B/E}(s_{j,k}) = 1$.

For $j > 0$, we put

$$\begin{aligned} s_j &= I_{-j,j} + \epsilon(-1)^{e(E_i/E_{i,0})-1} I_{j,-j} + \sum_{t \neq \pm j} I_{t,t}, \\ s_j^{\overline{\varpi}} &= \varpi_i^{-1} I_{-j,j} + \epsilon \varpi_i I_{j,-j} + \sum_{t \neq \pm j} I_{t,t}. \end{aligned}$$

These elements exchange the blocks $\mathfrak{e}^{(j)} \mathbf{Ae}^{(j)}$ and $\mathfrak{e}^{(-j)} \mathbf{Ae}^{(-j)}$. Again, Lemma 6.6 implies that $s_j, s_j^{\overline{\varpi}} \in G_E^+$. Indeed, except in the case when $E_i = F = F_0$, $\epsilon = 1$ and $\dim_F(W^{(j)})$ is odd, they lie in G_E .

From Lemma 6.6(iii), we also get:

Lemma 6.7. *If $j, k > 0$ are distinct, such that $i_j = i_k = i$ and $\dim_{E_i} W^{(j)} = \dim_{E_i} W^{(k)}$, then:*

- (i) $s_{j,k} \in P^+(\Lambda_{\mathfrak{o}_E})$ if and only if $q_j(\Lambda) = q_k(\Lambda)$;
- (ii) $\begin{cases} s_j \in P^+(\Lambda_{\mathfrak{o}_E}) & \text{if and only if } q_j(\Lambda) = e_i/2; \\ s_j^\varpi \in P^+(\Lambda_{\mathfrak{o}_E}) & \text{if and only if } q_j(\Lambda) = 0. \end{cases}$

We continue in the same situation. Let Λ^M be an \mathfrak{o}_E -lattice sequence in V such that $\mathfrak{b}_0(\Lambda^M)$ is a self-dual \mathfrak{o}_E -order containing $\mathfrak{b}_0(\Lambda)$. Then the decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ is subordinate to Λ^M and, for $j \neq 0$, $\mathfrak{a}_0(\Lambda^M) \cap B^{(j)} = \mathfrak{a}_0(\Lambda) \cap B^{(j)}$ is a maximal \mathfrak{o}_{E_i} -order in $B^{(j)}$, where $i = i_j$; hence we can define the integers $q_j(\Lambda^M)$, as above. In particular, this has the following consequences:

- (i) if $j > 0$ then $\Lambda^M(0) \cap W^{(j)} = \Lambda(0) \cap W^{(j)}$;
- (ii) if $j > 0$ then $q_j(\Lambda^M) \geq 0$;
- (iii) if $j > k > 0$ then $q_j(\Lambda^M) \geq q_k(\Lambda^M)$.

Then (ii)–(iii) imply that the ordering of the terms in the decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ is also a good ordering for Λ^M , and, having fixed such an ordering, (i) implies that the elements $s_{j,k}$, s_j and s_j^ϖ are the same for Λ and Λ^M .

Definition 6.8. We say that two blocks $W^{(j)}$ and $W^{(k)}$, with $j, k \neq 0$, are *companion with respect to Λ^M* if both are contained in the same subspace V^i , $\dim_{E_i} W^{(j)} = \dim_{E_i} W^{(k)}$ and, for $r \in \mathbb{Z}$,

$$\Lambda^M(r) \cap W^{(j)} = \Lambda^M(r+1) \cap W^{(j)} \iff \Lambda^M(r) \cap W^{(k)} = \Lambda^M(r+1) \cap W^{(k)}.$$

So, in the notation above, $W^{(j)}$ and $W^{(k)}$ are companion with respect to Λ^M if and only if $i_j = i_k = i$, $\dim_{E_i} W^{(j)} = \dim_{E_i} W^{(k)}$ and $q_j(\Lambda^M) \equiv q_k(\Lambda^M) \pmod{e_i^M}$, where $e_i^M = e(\Lambda^M, i | \mathfrak{o}_{E_i})$.

Lemma 6.7 then says that, for $j, k > 0$ distinct, $W^{(j)}$ and $W^{(k)}$ are companion with respect to Λ^M if and only if $s_{j,k} \in P^+(\Lambda_{\mathfrak{o}_E}^M)$, while $W^{(j)}$ and $W^{(-j)}$ are companion with respect to Λ^M if and only if either s_j or s_j^ϖ lies in $P^+(\Lambda_{\mathfrak{o}_E}^M)$.

Finally, suppose that Λ^M is such that $\mathfrak{b}_0(\Lambda^{M,i})$ is a maximal self-dual \mathfrak{o}_{E_i} -order, for some (fixed) i . Then, with our numbering (in particular, $\Lambda^M(r)^\# = \Lambda^M(1-r)$), the lattices $\Lambda^M(0) \cap V^i$ and $\Lambda^M(1) \cap V^i$ determine all the lattices in the image of $\Lambda^{M,i}$ (they are all E_i -multiples of these two lattices). Then we see that

$$\Lambda^{M,i}(0) \supseteq \Lambda^{M,i}(1) = \dots = \Lambda^{M,i}(e_i^M/2) \supseteq \Lambda^{M,i}(e_i^M/2 + 1) = \dots = \Lambda^{M,i}(e_i^M).$$

[Note that the converse of this is not quite true: if precisely one of these containments is proper then $\mathfrak{b}_0(\Lambda^{M,i})$ is necessarily maximal, but if both are proper then $\mathfrak{b}_0(\Lambda^{M,i})$ can be the ‘‘Siegel’’ order.]

In particular, for $j > 0$ with $i_j = i$, we have $q_j(\Lambda^M) \in \{0, e_i^M/2\}$ so $W^{(j)}$ and $W^{(-j)}$ are companion with respect to Λ^M , and

$$s_j^\varpi \in P^+(\Lambda_{\mathfrak{o}_E}^M) \iff q_j(\Lambda^M) = 0; \quad s_j \in P^+(\Lambda_{\mathfrak{o}_E}^M) \iff q_j(\Lambda^M) = e_i^M/2.$$

6.3 Intertwining of beta extensions

We continue with the notation of the previous section and, throughout this section and the next, we suppose that the self-dual decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ is *exactly* subordinate to the skew semisimple stratum $[\Lambda, n, 0, \beta]$, with the numbering as in the previous section. In particular, we are in a special case of the situation of §6.2.

Using the Weyl group element s_j , we can define an involution σ_j on $\tilde{G}^{(j)} = \text{Aut}_F(W^{(j)})$ as follows. Identifying $\tilde{G}^{(j)}$ with the subgroup $\{(\bar{g}^{-1}, g) \in \tilde{G}^{(-j)} \times \tilde{G}^{(j)}\} \subseteq M \cap G$, we put, for $g \in \tilde{G}^{(j)}$,

$$\sigma_j(g) = s_j g (s_j)^{-1}.$$

Lemma 6.9. (i) *If $1 \leq j < k \leq m$ and $W^{(j)} \cong W^{(k)}$ as E_i -vector spaces (for some $i = i_j = i_k$) then conjugation by $s_{j,k}$ induces an isomorphism $\tilde{J}(\beta^{(j)}, \Lambda^{(j)}) \cong \tilde{J}(\beta^{(k)}, \Lambda^{(k)})$.*

(ii) *For $1 \leq j \leq m$, the group $\tilde{J}(\beta^{(j)}, \Lambda^{(j)})$ is stable under the involution σ_j .*

Proof Suppose $1 \leq j \leq m$. We can identify $\beta^{(j)}$ with β_i , for $i = i_j$. Since the decomposition is exactly subordinate, the \mathfrak{o}_{E_i} -order $\mathfrak{b}_0(\Lambda^{(j)})$ is maximal in $\text{End}_{E_i}(W^{(j)})$. Moreover, $\Lambda^{(j)}$ is the unique (upto translation of index) \mathfrak{o}_{E_i} -lattice sequence of period $e(\Lambda^i | \mathfrak{o}_{E_i})$ whose associated \mathfrak{o}_{E_i} -order is $\mathfrak{b}_0(\Lambda^{(j)})$.

(i) If $W^{(j)} \cong W^{(k)}$ as E_i -vector spaces then conjugation by $s_{j,k}$ identifies an \mathfrak{o}_{E_i} -basis for $\Lambda^{(j)}(0)$ with an \mathfrak{o}_{E_i} -basis for $\Lambda^{(k)}(0)$, so we get an isomorphism $\mathfrak{b}_0(\Lambda^{(j)}) \cong \mathfrak{b}_0(\Lambda^{(k)})$ of \mathfrak{o}_{E_i} -orders. By the uniqueness above, this must then identify $\Lambda^{(j)}$ with a translate of $\Lambda^{(k)}$. In particular $\mathfrak{a}_r(\Lambda^{(j)})$ is identified with $\mathfrak{a}_r(\Lambda^{(k)})$, for all $r \in \mathbb{Z}$, and, since $\beta^{(j)}$ is identified with $\beta^{(k)}$ via β_i , we have an isomorphism $\tilde{J}(\beta^{(j)}, \Lambda^{(j)}) \cong \tilde{J}(\beta^{(k)}, \Lambda^{(k)})$.

(ii) As in (i), conjugation by s_j stabilizes $\tilde{J}(\beta^{(j)}, \Lambda^{(j)}) \times \tilde{J}(\beta^{(-j)}, \Lambda^{(-j)})$, swapping the two factors. Since $s_j \in G^+$, taking intersections with G^+ gives the result. \blacksquare

Let Λ^M be an \mathfrak{o}_E -lattice sequence in V such that $\mathfrak{b}_0(\Lambda^M)$ is a maximal self-dual \mathfrak{o}_E -order containing $\mathfrak{b}_0(\Lambda)$ and let κ be a β -extension relative to Λ^M . Let M be the Levi subgroup of G which is the stabilizer of the decomposition and let P be a parabolic subgroup with Levi component M . We form the representation κ_P of $J_P^{\mathfrak{p}}$ and write

$$\kappa_P|_{J_P^{\mathfrak{p}} \cap M} = \kappa^{(0)} \otimes_{j=1}^m \tilde{\kappa}^{(j)}.$$

We recall that, since the decomposition is *exactly* subordinate to the stratum, $\kappa^{(0)}$ is a $\beta^{(0)}$ -extension and $\tilde{\kappa}^{(j)}$ is a $2\beta^{(j)}$ -extension, for $j = 1, \dots, m$, by Proposition 6.3.

Corollary 6.10. (i) *If $1 \leq j < k \leq m$ and $W^{(j)}$ is companion to $W^{(k)}$ with respect to Λ^M then conjugation by $s_{j,k}$ induces an equivalence $\tilde{\kappa}^{(j)} \simeq \tilde{\kappa}^{(k)}$.*

(ii) *For $1 \leq j \leq m$, conjugation by s_j induces an equivalence $\tilde{\kappa}^{(j)} \circ \sigma_j \simeq \tilde{\kappa}^{(j)}$.*

Proof (i) In this situation $s_{j,k} \in P^+(\Lambda_{\mathfrak{o}_E}^M)$ so, by Corollary 6.2, $s_{j,k}$ intertwines κ_P . Moreover, it normalizes $J_P^{\mathfrak{p}} \cap M$ so, since the restriction of κ_P to this group is irreducible, it normalizes $\kappa_P|_{J_P^{\mathfrak{p}} \cap M} = \kappa_0 \otimes \bigotimes_{j=1}^m \tilde{\kappa}^{(j)}$. The result is then clear, from Lemma 6.9.

(ii) If $s_j \in P^+(\Lambda_{\mathfrak{o}_E}^M)$ then the proof is the same as for (i). Otherwise $s_j^{\varpi} \in P^+(\Lambda_{\mathfrak{o}_E}^M)$ so we get an equivalence $\tilde{\kappa}^{(j)} \circ \sigma_j \simeq \varpi_i \tilde{\kappa}^{(j)}$. But $\tilde{\kappa}^{(j)}$ is normalized by ϖ_i and the result follows. \blacksquare

Corollary 6.11. *For $1 \leq j \leq m$, there exists a $2\beta^{(j)}$ -extension $\tilde{\kappa}^{(j)}$ which is σ_j -stable.*

Remark 6.12. Once we have such a σ_j -stable $\tilde{\kappa}^{(j)}$, it follows (using [8, Theorem 5.2.2]) that all σ_j -stable $2\beta^{(j)}$ -extensions are of the form $\tilde{\kappa}^{(j)} \otimes \tilde{\chi} \circ \det^{(j)}$, where:

- $\det^{(j)} : \text{End}_{E_i}(W^{(j)}) \rightarrow E_i$ is the determinant; and
- $\tilde{\chi}$ is the inflation of a σ_i -invariant character of $\kappa_{E_i}^\times$, where σ_i is the involution $x \mapsto (\bar{x})^{-1}$ on $\kappa_{E_i}^\times$ and $\bar{}$ is the Galois involution on k_{E_i} with fixed field $k_{E_{i,0}}$.

If $E_i/E_{i,0}$ is ramified, then there are only two such characters: the trivial character and the quadratic character. If $E_i/E_{i,0}$ is unramified, then there are $|k_{E_{i,0}}| + 1$ such characters.

Since $\kappa_P|_{J_P^\circ \cap M} = \kappa^{(0)} \otimes_{j=1}^m \tilde{\kappa}^{(j)}$ and κ_P is trivial on both $J_P^\circ \cap U$ and $J_P^\circ \cap U_l$, this observation gives us the following useful corollary:

Corollary 6.13. *Let $\Lambda^M, \Lambda^{M'}$ be \mathfrak{o}_E -lattice sequences in V such that $\mathfrak{b}_0(\Lambda^M)$ and $\mathfrak{b}_0(\Lambda^{M'})$ are maximal self-dual \mathfrak{o}_E -orders containing $\mathfrak{b}_0(\Lambda)$. Let κ be a β -extension of η relative to Λ^M , and let κ' be a β -extension of η relative to $\Lambda^{M'}$. There are σ_i -invariant characters $\tilde{\chi}^{(j)}$ of $\kappa_{E_i}^\times$ and a character $\chi^{(0)}$ of $J^\circ(\beta^{(0)}, \Lambda^{(0)})/J^1(\beta^{(0)}, \Lambda^{(0)})$ such that, writing $\chi = \chi^{(0)} \otimes \bigotimes_{j=1}^m \tilde{\chi}^{(j)} \circ \det^{(j)}$, we have*

$$\kappa' = \text{Ind}_{J_P^\circ}^{J^\circ(\beta, \Lambda)}(\kappa_P \otimes \chi).$$

Moreover, if $W^{(j)}$ is companion to $W^{(k)}$ with respect to both Λ^M and $\Lambda^{M'}$, then $\tilde{\chi}^{(j)} = \tilde{\chi}^{(k)}$.

Corollary 6.10 can also be applied to get more information about the intertwining of κ_P (and hence of κ). Suppose that κ is a standard β -extension, so $\Lambda^M = \mathfrak{M}_\Lambda$. If $j, k \neq 0$ and $W^{(j)} \simeq W^{(k)}$ as E_i -spaces, for some i , then $W^{(j)}$ and $W^{(k)}$ are companion with respect to \mathfrak{M}_Λ . Moreover, by Lemma 6.7(ii), we have $s_j \in P(\mathfrak{M}_{\Lambda, \mathfrak{o}_E})$.

For each i , write $\mathcal{B}^i = \mathcal{B}^{(i,0)} \cup \bigcup_{j:i_j=i} \mathcal{B}^{(j)}$, where $\mathcal{B}^{(j)}$ is the E_i -basis for $W^{(j)}$ chosen in the previous section and $\mathcal{B}^{(i,0)}$ is the basis for $V^i \cap W^{(0)}$. Then \mathcal{B}^i is an E_i -basis for V^i which splits the lattice sequence Λ^i . We write \tilde{T}_{E_i} for the maximal split torus in \tilde{G}_{E_i} which corresponds to the basis \mathcal{B}^i – that is, \tilde{T}_{E_i} is the stabilizer in \tilde{G}_{E_i} of the decomposition $V^i = \bigoplus_{\mathbf{v} \in \mathcal{B}^i} E_i \mathbf{v}$.

We write $\tilde{T}_E = \prod_{i=1}^l \tilde{T}_{E_i}$ and put $T_E^+ = \tilde{T}_E \cap G_E^+$ and $T_E = \tilde{T}_E \cap G_E$. We also put $N^+ = N_{G_E^+}(T_E^+)$ and $N = N_{G_E}(T_E) = N^+ \cap G$.

Proposition 6.14. *Let $w \in N^+$ be such that w normalizes $P^\circ(\Lambda_{\mathfrak{o}_E}) \cap M$. Then w normalizes $\kappa_P|_{J^\circ(\beta, \Lambda) \cap M}$ and intertwines $\kappa_P|_{J_P^\circ}$.*

Proof Suppose $w \in N$ normalizes $P^\circ(\Lambda_{\mathfrak{o}_E}) \cap M$. In particular, it then permutes the blocks of the decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$. Moreover, since w is fixed by the involution σ , we have that $wW^{(j)} = W^{(k)}$ if and only if $wW^{(-j)} = W^{(-k)}$. In particular, this implies that w stabilizes $W^{(0)}$ and its orthogonal complement $\bigoplus_{j \neq 0} W^{(j)}$. But then the 0-component $w^{(0)} = \mathbf{e}^{(0)} w \mathbf{e}^{(0)}$ of w normalizes $P^\circ(\Lambda_{\mathfrak{o}_E}^{(0)})$ so, since $\mathfrak{b}_0(\Lambda^{(0)})$ is a maximal self-dual \mathfrak{o}_E -order in $B^{(0)}$, we have $w^{(0)} \in P(\Lambda_{\mathfrak{o}_E}^{(0)})$, and $w^{(0)}$ certainly normalizes $\kappa^{(0)}$.

The remaining part of w can be written as the product of:

- a block diagonal element z with $z^{(j)} = \mathbf{e}^{(j)} z \mathbf{e}^{(j)}$ a non-zero scalar in E_{i_j} for $1 \leq j \leq m$; and

- a block permutation matrix s which, in turn, can be expressed as a product of the matrices $s_{j,k}$ and s_j , for $1 \leq j < k \leq m$.

Since κ is a standard β -extension, Corollary 6.10 implies that s normalizes $\kappa_P|_{J^\circ(\beta,\Lambda)\cap M} = \kappa^{(0)} \otimes \bigotimes_{j=1}^m \tilde{\kappa}^{(j)}$. On the other hand, $z^{(j)}$ certainly normalizes $\tilde{\kappa}^{(j)}$, since it is a $2\tilde{\beta}^{(j)}$ -extension. Hence w normalizes $\kappa_P|_{J^\circ(\beta,\Lambda)\cap M}$, as required.

To show that w also intertwines $\kappa_P|_{J_P^\circ}$, we imitate [27, Théorème 2.19]. Since $J^\circ(\beta,\Lambda)$ has an Iwahori decomposition with respect to both (M, P) and (M, P^w) , the group $J_P^\circ \cap {}^w J_P^\circ$ also has an Iwahori decomposition with respect to (M, P) . Now let Φ be a non-zero intertwining operator of $\kappa_P|_{J^\circ(\beta,\Lambda)\cap M}$ with ${}^w \kappa_P|_{J^\circ(\beta,\Lambda)\cap M}$. For $g \in J_P^\circ \cap {}^w J_P^\circ$ we write $g = lmu$, with $l \in U_l$, $m \in M$ and $u \in U$. Then

$$\Phi \circ \kappa_P(g) = \Phi \circ \kappa_P(m) = {}^w \kappa_P(m) \circ \Phi.$$

It remains only to check that $l, u \in \ker({}^w \kappa_P)$ or, equivalently, that $l^w, u^w \in \ker(\kappa_P)$.

By elementary row and column operations (cf. [27, Lemme 2.20]) we have $\tilde{U}^w = (\tilde{U}^w \cap \tilde{U})(\tilde{U}^w \cap \tilde{U}_l)$ so, by uniqueness of Iwahori decomposition, $U^w = (U^w \cap U)(U^w \cap U_l)$ and $u^w \subseteq UU_l$. Moreover, since $u^w \in J_P^\circ$, the uniqueness of the Iwahori decomposition again implies $u^w \subseteq (J_P^\circ \cap U)(J_P^\circ \cap U_l) \subseteq \ker(\kappa_P)$. The same argument applies for l^w . \blacksquare

Write $N_\Lambda = \{w \in N : w \text{ normalizes } P^\circ(\Lambda_{\mathfrak{o}_E}) \cap M\}$. It will be useful to understand this group in terms of the notation of Morris in §1. If $P^\circ(\Lambda_{\mathfrak{o}_E})$ is a maximal parahoric subgroup of G_E (so the decomposition is just $V = W^{(0)}$ and $M = P = G$) then we have $N_\Lambda \subseteq P(\Lambda_{\mathfrak{o}_E})$, since this is the normalizer of $P^\circ(\Lambda_{\mathfrak{o}_E})$ in G_E . Otherwise, the group called \mathfrak{M}_J in §1 is just $M \cap G_E$ (because the decomposition is exactly subordinate) and the group $\mathfrak{M}_J \cap P_J$ is just $P^\circ(\Lambda_{\mathfrak{o}_E}) \cap M$. In particular, our group N_Λ is just the group written $N_N(\mathfrak{M}_J \cap P_J)$.

6.4 Intertwining and supercuspidal types

We continue with the notation of the previous section so that we have: a self-dual decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ exactly subordinate to a skew semisimple stratum $[\Lambda, n, 0, \beta]$, numbered as usual; a semisimple character $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$; a standard β -extension κ to $J^\circ(\beta, \Lambda)$ of the unique irreducible representation η of $J^1(\beta, \Lambda)$ containing θ ; and a parabolic subgroup P of G whose Levi component M is the stabilizer of the decomposition.

The following proposition bounds the intertwining of a pair (J_P°, λ_P) , where $\lambda_P = \kappa_P \otimes \rho$ and ρ is (the inflation of) a cuspidal irreducible representation of $J_P^\circ/J_P^1 \cong P^\circ(\Lambda_{\mathfrak{o}_E})/P^1(\Lambda_{\mathfrak{o}_E})$.

Proposition 6.15. *We have $I_G(\lambda_P|_{J_P^\circ}) \subseteq J_P^\circ N_\Lambda J_P^\circ$.*

Remark The proof, which is a variant of the proof of [8, Proposition 5.3.2], is inspired by [3, page 553] (using Proposition 1.1 in place of *op. cit.* Lemma 4.2).

Proof Suppose $g \in G$ intertwines $\lambda_P = \kappa_P \otimes \rho$, so that $g \in I_G(\eta_P|_{J_P^1}) = J_P^\circ G_E J_P^\circ$, as ρ is trivial on J_P^1 . Thus we may assume g lies in G_E .

Now let Λ^m be a self-dual \mathfrak{o}_E -lattice sequence such that $\mathfrak{b}_0(\Lambda_{\mathfrak{o}_E}^m)$ is a minimal self-dual \mathfrak{o}_E -order contained in $\mathfrak{b}_0(\Lambda_{\mathfrak{o}_E})$ and put $J_{m,\Lambda,P}^\circ = P^\circ(\Lambda_{\mathfrak{o}_E}^m)J_P^1$ and $J_{m,\Lambda,P}^1 = P_1(\Lambda_{\mathfrak{o}_E}^m)J_P^1$. Then $G_E \cap J_P^\circ = P^\circ(\Lambda_{\mathfrak{o}_E})$ is a parahoric subgroup of G_E containing the Iwahori subgroup $P^\circ(\Lambda_{\mathfrak{o}_E}^m)$. Thus we may further assume that g is a distinguished double coset representative for $P^\circ(\Lambda_{\mathfrak{o}_E}) \backslash G_E / P^\circ(\Lambda_{\mathfrak{o}_E})$ (see §1).

Since $\dim I_g(\eta_P, J_P^1) = 1$, we can imitate the proof of [8, Proposition 5.3.2] to show that any non-zero intertwining operator in $I_g(\kappa_P \otimes \rho, J_P^0)$ has the form $S \otimes T$, with $S \in I_g(\eta_P, J_P^1)$ and T an endomorphism of the space of ρ .

Now, by Lemma 5.12 and Corollary 6.4, the operator S also intertwines the restriction $\kappa_P|_{J_{m,\Lambda,P}^1}$ so, again as in the proof of [8, Proposition 5.3.2], it follows that T belongs to $I_g(\rho|_{J_{m,\Lambda,P}^1})$. In particular, g intertwines the restriction of ρ to $J_{m,\Lambda,P}^1 \cap G_E = P_1(\Lambda_{\mathfrak{o}_E}^m)$. But $P_1(\Lambda_{\mathfrak{o}_E}^m)$ is the radical of an Iwahori subgroup of G_E so, by Proposition 1.1 and the remarks following it (see also the end of the previous section for a translation of this into our notations here), $g \in N_\Lambda$, as required. \blacksquare

Corollary 6.16. *Put $N_\Lambda(\rho) = \{g \in N_\Lambda : {}^g\rho \simeq \rho\}$. Then $I_G(\lambda_P|_{J_P^0}) \subseteq J_P^0 N_\Lambda(\rho) J_P^0$.*

Proof By Proposition 6.15, we need only show that $I_{N_\Lambda}(\lambda_P|_{J_P^0}) \subseteq J_P^0 N_\Lambda(\rho) J_P^0$. So suppose $g \in N_\Lambda$ intertwines λ_P . As in the proof of Proposition 6.15, any non-zero intertwining operator in $I_g(\kappa_P \otimes \rho, J_P^0)$ has the form $S \otimes T$, with $S \in I_g(\eta_P, J_P^1)$ and T an endomorphism of the space of ρ . Now, by Proposition 6.14, g intertwines κ_P so, by Lemma 5.12, the operator S also intertwines κ_P and it follows that T belongs to $I_g(\rho|_{J_P^0})$. In particular, g intertwines the restriction of ρ to $J_P^0 \cap G_E = P^\circ(\Lambda_{\mathfrak{o}_E})$. Then, by Theorem 1.2, $g \in P^\circ(\Lambda_{\mathfrak{o}_E}) N_\Lambda(\rho) P^\circ(\Lambda_{\mathfrak{o}_E})$, as required. \blacksquare

In the case when $P(\Lambda_{\mathfrak{o}_E}^M)$ is a maximal compact subgroup of G_E we can now obtain supercuspidal representations from this construction:

Let $[\Lambda^M, n, 0, \beta]$ be a skew semisimple stratum with $\mathfrak{b}_0(\Lambda^M)$ a maximal self-dual \mathfrak{o}_E -order in B . Let θ_M be a skew semisimple character in $\mathcal{C}_-(\Lambda^M, 0, \beta)$, let η_M be the unique irreducible representation of $J_M^1 = J^1(\beta, \Lambda^M)$ and let κ_M be a β -extension of η_M to $J_M = J(\beta, \Lambda^M)$.

Put $\lambda_M = \kappa_M \otimes \tau$, for τ an irreducible cuspidal representation of $J_M/J_M^1 \simeq P(\Lambda_{\mathfrak{o}_E}^M)/P_1(\Lambda_{\mathfrak{o}_E}^M)$. Recall that $P(\Lambda_{\mathfrak{o}_E}^M)/P_1(\Lambda_{\mathfrak{o}_E}^M)$ is not, in general, connected: by an irreducible cuspidal representation we mean an irreducible representation whose restriction to the connected component contains an irreducible cuspidal representation ρ . Write $J_M^0 = P^\circ(\Lambda_{\mathfrak{o}_E}^M) J_M^1$.

Definition 6.17. We call a pair (J_M, λ_M) as above a *maximal simple type for G* .

Proposition 6.18. *Let (J_M, λ_M) be a maximal simple type for G . Then $I_G(\lambda_M|_{J_M}) = J_M$.*

Proof Suppose $g \in G$ intertwines $\lambda_M = \kappa_M \otimes \tau$, so that $g \in I_G(\eta_M|_{J_M^1}) = J_M G_E J_M$, as τ is trivial on J_M^1 . Thus we may assume g lies in G_E . Now $\kappa_M|_{J_M^0}$ is irreducible and is normalized by J_M . If ρ is an irreducible cuspidal component of $\tau|_{J_M^0}$ then, by Clifford Theory, the restriction of λ_M to J_M^0 takes the form

$$\lambda_M|_{J_M^0} = m \sum_p \kappa_M \otimes \tau^p \simeq m \sum_p (\kappa_M \otimes \rho)^p,$$

where $m \in \mathbb{N}$ is a multiplicity and the sum is taken over a set of representatives for $P(\Lambda_{\mathfrak{o}_E}^M)/N_{P(\Lambda_{\mathfrak{o}_E}^M)}(\tau)$. Since g certainly intertwines $\lambda_M|_{J_M^0}$, we see that there exist $p_1, p_2 \in P(\Lambda_{\mathfrak{o}_E}^M)$ such that $p_1 g p_2$ intertwines $\kappa_M \otimes \rho$ and, since $p_i \in J_M$, we may assume that g intertwines $\kappa_M \otimes \rho$. But then, since $N_\Lambda(\rho) \subseteq N_{G_E}(P^\circ(\Lambda_{\mathfrak{o}_E}^M)) = P(\Lambda_{\mathfrak{o}_E}^M)$, Corollary 6.16 implies that $g \in J_M$, as required. \blacksquare

From Proposition 6.18, together with [13, Proposition 1.5] and [9, Proposition 5.4], we immediately get:

Corollary 6.19. *Let (J_M, λ_M) be a maximal simple type in G . Then $\pi = \text{c-Ind}_{J_M}^G \lambda_M$ is an irreducible supercuspidal representation of G and (J_M, λ_M) is a $[G, \pi]_G$ -type.*

We will show in the following section that all irreducible supercuspidal representations of G arise from this construction.

7 Exhaustion

In this final section, we show that any irreducible supercuspidal representation of G contains a maximal simple type. In particular, every irreducible supercuspidal representation of G arises from our constructions. We do not, however, address any unicity issues – for example, we do not have an intertwining implies conjugacy result for maximal simple types, so we do not know when two apparently different maximal simple types might induce to equivalent irreducible supercuspidal representations of G .

7.1 A Hecke algebra injection

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum and let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ be a skew semisimple character. We use our usual notation: $V = \bigoplus_{i=1}^l V^i$ is the splitting associated to $[\Lambda, n, 0, \beta]$, $E = F[\beta] = \bigoplus_{i=1}^l E_i$, $B = \text{End}_E(V) = \bigoplus_{i=1}^l B^i$, $G_E = B \cap G = \prod_{i=1}^l G_{E_i}$, etc.

Let $[\Lambda', n', 0, \beta]$ be another skew semisimple stratum, with $\mathfrak{b}_0(\Lambda) \subseteq \mathfrak{b}_0(\Lambda')$ and let $\theta' = \tau_{\Lambda, \Lambda', \beta}(\theta)$. Let κ (respectively κ') be a β -extension of $J^\circ(\beta, \Lambda)$ (respectively $J^\circ(\beta, \Lambda')$) such that κ, κ' are compatible. Let ρ be the inflation of some irreducible representation of $J^\circ(\beta, \Lambda)/J^1(\beta, \Lambda)$ and put $\vartheta = \kappa \otimes \rho$. Set

$$\mathcal{G}^\circ = J^\circ(\beta, \Lambda)/J^1(\beta, \Lambda) \cong \prod_{i=1}^l \text{P}^\circ(\Lambda_{\mathfrak{o}_{E_i}}^i)/\text{P}_1(\Lambda_{\mathfrak{o}_{E_i}}^i)$$

and write $\rho = \otimes_{i=1}^l \rho_i$, for ρ_i the inflation of an irreducible representation of the finite reductive group $\mathcal{G}_i^\circ = \text{P}^\circ(\Lambda_{\mathfrak{o}_{E_i}}^i)/\text{P}_1(\Lambda_{\mathfrak{o}_{E_i}}^i)$.

We put $J_{\Lambda, \Lambda'}^\circ = \text{P}^\circ(\Lambda_{\mathfrak{o}_E})J^1(\beta, \Lambda')$; then we can identify $J_{\Lambda, \Lambda'}^\circ/J^1(\beta, \Lambda')$ with $\text{P}^\circ(\Lambda_{\mathfrak{o}_E})/\text{P}_1(\Lambda_{\mathfrak{o}_E}')$, which has $\text{P}^\circ(\Lambda_{\mathfrak{o}_E})/\text{P}_1(\Lambda_{\mathfrak{o}_E})$ as a quotient. Hence we can view ρ as a representation of $J_{\Lambda, \Lambda'}^\circ$. We define ϑ' to be the (irreducible) representation of $J_{\Lambda, \Lambda'}^\circ$ given by

$$\vartheta' = \left(\kappa'|_{J_{\Lambda, \Lambda'}^\circ} \right) \otimes \rho.$$

Proposition 7.1 (cf. [8, Proposition 5.5.13]). *There is a canonical algebra isomorphism*

$$\mathcal{H}(G, \vartheta) \cong \mathcal{H}(G, \vartheta')$$

which preserves support: if $\phi \in \mathcal{H}(G, \vartheta)$ is supported on $J^\circ(\beta, \Lambda)yJ^\circ(\beta, \Lambda)$, for some $y \in G_E$, then the corresponding function $\phi' \in \mathcal{H}(G, \vartheta')$ has support $J_{\Lambda, \Lambda'}^\circ y J_{\Lambda, \Lambda'}^\circ$.

Proof Suppose first that $\mathfrak{a}_0(\Lambda) \subseteq \mathfrak{a}_0(\Lambda')$. Given the simple intersection property from Lemma 2.6, the proof in this case is identical to that of [8, Proposition 5.5.13].

In general, by Lemma 2.8, there is an \mathfrak{o}_E -lattice sequence Λ'' with $\mathfrak{b}_0(\Lambda'') = \mathfrak{b}_0(\Lambda)$ and $\mathfrak{a}_0(\Lambda'') \subseteq \mathfrak{a}_0(\Lambda')$. By Lemma 2.10, we have a sequence $\Lambda_0 = \Lambda, \Lambda_1, \dots, \Lambda_t = \Lambda''$ of \mathfrak{o}_E -lattice sequences such

that, for each i , $\mathfrak{b}_0(\Lambda_i) = \mathfrak{b}_0(\Lambda)$ and either $\mathfrak{a}_0(\Lambda_i) \subseteq \mathfrak{a}_0(\Lambda_{i-1})$ or $\mathfrak{a}_0(\Lambda_i) \supseteq \mathfrak{a}_0(\Lambda_{i-1})$. For each i , let κ_i be the β -extension of $J^\circ(\beta, \Lambda_i)$ compatible with κ and put $\vartheta_i = \kappa_i \otimes \rho$, where we use the fact that $J^\circ(\beta, \Lambda_i)/J^1(\beta, \Lambda_i) \cong J^\circ(\beta, \Lambda)/J^1(\beta, \Lambda)$ to view ρ as a representation of $J^\circ(\beta, \Lambda_i)$.

Applying the first case with $\{\Lambda_{i-1}, \Lambda_i\}$ in place of $\{\Lambda, \Lambda'\}$, we see that we have support-preserving Hecke algebra isomorphisms

$$\mathcal{H}(G, \vartheta_{i-1}) \cong \mathcal{H}(G, \vartheta_i),$$

for $1 \leq i \leq t$. Again applying the first case, with $\Lambda' = \Lambda_t$ in place of Λ , we have a support-preserving Hecke algebra isomorphism

$$\mathcal{H}(G, \vartheta_t) \cong \mathcal{H}(G, \vartheta').$$

Putting these together gives the required isomorphism. \blacksquare

Proposition 7.2 (cf. [8, (5.6.2), Lemma 5.6.3]). *Thinking of ρ as an irreducible representation of $P^\circ(\Lambda_{\mathfrak{o}_E})$, there is an algebra isomorphism*

$$\mathcal{H}(J(\beta, \Lambda'), \vartheta') \cong \mathcal{H}(P(\Lambda'_{\mathfrak{o}_E}), \rho)$$

which preserves support: if $\phi \in \mathcal{H}(J(\beta, \Lambda'), \vartheta')$ is supported on $J_{\Lambda, \Lambda'}^\circ y J_{\Lambda, \Lambda'}^\circ$, for some $y \in P(\Lambda'_{\mathfrak{o}_E})$, then the corresponding function ϕ_E has support $P^\circ(\Lambda_{\mathfrak{o}_E}) y P^\circ(\Lambda_{\mathfrak{o}_E})$.

Proof As in [8, Lemma 5.6.3], the irreducibility of $\kappa'|_{P(\Lambda_{\mathfrak{o}_E})J^1(\beta, \Lambda')}$ implies that we have an algebra isomorphism $\mathcal{H}(J(\beta, \Lambda'), \rho) \cong \mathcal{H}(J(\beta, \Lambda'), \vartheta')$, given as follows: map $\phi \in \mathcal{H}(J(\beta, \Lambda'), \rho)$ to $\kappa'^{\vee} \otimes \phi \in \mathcal{H}(J(\beta, \Lambda'), \vartheta')$, where \vee denotes the contragredient. The isomorphism with $\mathcal{H}(P(\Lambda'_{\mathfrak{o}_E}), \rho)$ follows by reduction to $J(\beta, \Lambda')/J^1(\beta, \Lambda') \cong P(\Lambda'_{\mathfrak{o}_E})/P^1(\Lambda'_{\mathfrak{o}_E})$. \blacksquare

Let $V = \bigoplus_{j=-m}^m W^{(j)}$ be a self-dual decomposition which is properly subordinate to $[\Lambda, n, 0, \beta]$. As in §5, let M be the Levi subgroup of G which stabilizes the decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ and let P be a parabolic subgroup with Levi component M , and unipotent radical U . We use all the other related notation from §5.

By definition, the representation ρ is trivial on $J^1(\beta, \Lambda)$ so we can apply Lemma 6.1 with $K = J^\circ(\beta, \Lambda)$ and $\lambda = \vartheta = \kappa \otimes \rho$ to obtain Iwahori factorization results for ϑ . In particular, we get a support-preserving Hecke algebra isomorphism

$$\mathcal{H}(G, \vartheta_P) \cong \mathcal{H}(G, \vartheta).$$

Putting all this together, we see that there is an injective algebra map

$$\mathcal{H}(P(\Lambda'_{\mathfrak{o}_E}), \rho) \cong \mathcal{H}(J(\beta, \Lambda'), \vartheta') \hookrightarrow \mathcal{H}(G, \vartheta') \cong \mathcal{H}(G, \vartheta) \cong \mathcal{H}(G, \vartheta_P) \quad (7.3)$$

which preserves support: if $\phi_E \in \mathcal{H}(P(\Lambda'_{\mathfrak{o}_E}), \rho)$ has support $P^\circ(\Lambda_{\mathfrak{o}_E}) y P^\circ(\Lambda_{\mathfrak{o}_E})$, for some $y \in P(\Lambda'_{\mathfrak{o}_E})$, then the corresponding function $\phi_P \in \mathcal{H}(G, \vartheta_P)$ has support $J_P^\circ y J_P^\circ$.

7.2 Covers

Let π be an irreducible representation of G and suppose that there is a pair $([\Lambda, n, 0, \beta], \theta)$, consisting of a skew semisimple stratum $[\Lambda, n, 0, \beta]$ and a skew semisimple character $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$, such that $\pi|_{H^1(\beta, \Lambda)}$ contains θ .

Among all such pairs, for fixed β , we choose one for which the order $\mathfrak{b}_0(\Lambda) = \mathfrak{a}_0(\Lambda) \cap B$ is minimal. Since there is a unique irreducible representation η of $J^1 = J^1(\beta, \Lambda)$ containing θ , π must also contain η and hence some irreducible representation ϑ of $J^\circ = J^\circ(\beta, \Lambda)$ containing η . Since η extends to J° , we see that $\vartheta = \kappa \otimes \rho$, where κ is some standard β -extension of η and ρ is the inflation of some irreducible representation of J°/J^1 . As above, we put

$$\mathcal{G}^\circ = J^\circ/J^1 \cong \prod_{i=1}^l \mathrm{P}^\circ(\Lambda_{\mathfrak{o}_{E_i}}^i)/\mathrm{P}_1(\Lambda_{\mathfrak{a}_i}^i)$$

and write $\rho = \otimes_{i=1}^l \rho_i$, for ρ_i the inflation of an irreducible representation $\bar{\rho}_i$ of the finite reductive group $\mathcal{G}_i^\circ = \mathrm{P}^\circ(\Lambda_{\mathfrak{o}_{E_i}}^i)/\mathrm{P}_1(\Lambda_{\mathfrak{a}_i}^i)$.

Lemma 7.4. *In the situation above, all the representations $\bar{\rho}_i$ are cuspidal.*

Proof Suppose, for contradiction, that $\bar{\rho}_1$ is not cuspidal. Then there exists a proper parabolic subgroup \mathcal{P}_1° of \mathcal{G}_1° , with unipotent radical \mathcal{U}_1 , such that $\bar{\rho}_1|_{\mathcal{U}_1}$ contains the trivial character. Let Λ'^1 be a self-dual \mathfrak{o}_1 -lattice sequence in V^1 such that $\mathfrak{b}_0(\Lambda'^1) \subseteq \mathfrak{b}_0(\Lambda^1)$ and the image of $\mathrm{P}^\circ(\Lambda_{\mathfrak{o}_1}^1)$ under the quotient map $\mathrm{P}^\circ(\Lambda_{\mathfrak{o}_1}^1) \rightarrow \mathcal{G}_1$ is \mathcal{P}_1° . Such a lattice sequence Λ'^1 is obtained as a refinement of Λ^1 : \mathcal{P}_1° is defined by a self-dual k_{E_1} -flag in $\Lambda^1(0)/\Lambda^1(1)$, k_{E_1} -flags in $\Lambda^1(r)/\Lambda^1(r+1)$, for $1 \leq r < \frac{e}{2}$, where $e = e(\Lambda^1|\mathfrak{o}_1)$, and a self-dual k_{E_1} -flag in $\Lambda^1(\lfloor \frac{e+1}{2} \rfloor)/\Lambda^1(\lfloor \frac{e}{2} \rfloor + 1)$; then the image of Λ'^1 consists of all the lattices which are inverse images of these flags, their duals, and their E_1 -scalar multiples. We put $\Lambda' = \Lambda'^1 \oplus \bigoplus_{i=2}^l \Lambda^i$ so that $\mathfrak{b}_0(\Lambda') \subseteq \mathfrak{b}_0(\Lambda)$. By Lemma 2.8, we can choose Λ'' an \mathfrak{o}_E -lattice sequence so that $\mathfrak{b}_0(\Lambda'') = \mathfrak{b}_0(\Lambda')$ and $\mathfrak{a}_0(\Lambda'') \subseteq \mathfrak{a}_0(\Lambda)$. Then $[\Lambda'', n'', 0, \beta]$ is a skew semisimple stratum in A , for a suitable integer n'' . Let $\theta'' = \tau_{\Lambda, \Lambda'', \beta}(\theta)$ be the transfer of θ to $H^1(\beta, \Lambda'')$ and let η'' be the unique irreducible representation of $J^1(\beta, \Lambda'')$ which contains θ'' . We show that π contains η'' , and hence also θ'' , contradicting the minimality of $\mathfrak{b}_0(\Lambda)$.

Since $\mathrm{P}_1(\Lambda''_{\mathfrak{o}_E})$ projects onto \mathcal{U}_1 under the projection map $J^\circ \mapsto \mathcal{G}$, $\rho|_{\mathrm{P}_1(\Lambda''_{\mathfrak{o}_E})J^1}$ contains the trivial character so π contains the restriction of κ to $\mathrm{P}_1(\Lambda''_{\mathfrak{o}_E})J^1$. The lemma now follows from:

Lemma 7.5 (cf. [8, Lemma 8.1.6]). *The representations of $\mathrm{P}_1(\Lambda'')$ induced by η'' and by $\kappa|_{\mathrm{P}_1(\Lambda''_{\mathfrak{o}_E})J^1}$ are irreducible and equivalent to each other.*

The proof of Lemma 7.5 is identical to that of [8, Lemma 8.1.6]. This also finishes the proof of Lemma 7.4. \blacksquare

We continue with the same situation, so that the semisimple character θ contained in π has $\mathfrak{b}_0(\Lambda)$ minimal for this property and hence the representations ρ_i are all cuspidal. For the remainder of this section, we also make the following hypothesis:

- (H) There is no skew semisimple stratum $[\Lambda', n', 0, \beta]$ with $\mathfrak{b}_0(\Lambda')$ a maximal self-dual \mathfrak{o}_E -order in B such that π contains a representation of the form $\kappa' \otimes \rho'$, where κ' is a standard β -extension and ρ' is a cuspidal representation of $\mathrm{P}^\circ(\Lambda'_{\mathfrak{o}_E})$.

In particular, hypothesis (H) implies that $\mathfrak{b}_0(\Lambda)$ is *not* a maximal self-dual \mathfrak{o}_E -order in B . Let $[\Lambda', n', 0, \beta]$ be another skew semisimple stratum with $\mathfrak{M}_{\Lambda'} = \mathfrak{M}_\Lambda$ and $\mathfrak{b}_0(\Lambda') \supseteq \mathfrak{b}_0(\Lambda)$. We let $\theta' = \tau_{\Lambda, \Lambda', \beta} \theta$ be the transfer of θ to $H^1(\beta, \Lambda')$ and let η' be the unique irreducible representation of $J^1(\beta, \Lambda')$ containing θ' . Let κ' be the unique standard β -extension of η' compatible with κ and let ρ' be the inflation to $J^\circ(\beta, \Lambda')$ of an irreducible component of

$$\mathrm{Ind}_{\mathrm{P}^\circ(\Lambda'_{\mathfrak{o}_E})/\mathrm{P}_1(\Lambda'_{\mathfrak{o}_E})}^{\mathrm{P}^\circ(\Lambda'_{\mathfrak{o}_E})/\mathrm{P}_1(\Lambda'_{\mathfrak{o}_E})} \bar{\rho}'.$$

Definition 7.6. With notation as above, we say that a representation of the form $\vartheta' = \kappa' \otimes \rho'$ lies over $\vartheta = \kappa \otimes \rho$.

Note that, thinking of ρ as a representation of $J_{\Lambda, \Lambda'}^{\circ} = P^{\circ}(\Lambda_{\mathfrak{o}_E})J^1(\beta, \Lambda')$ trivial on $J^1(\beta, \Lambda')$, the representations lying over ϑ are precisely the irreducible constituents of

$$\mathrm{Ind}_{J_{\Lambda, \Lambda'}^{\circ}}^{J^{\circ}(\beta, \Lambda')} \left(\kappa' |_{J_{\Lambda, \Lambda'}^{\circ}} \otimes \rho \right) = \kappa' \otimes \mathrm{Ind}_{J_{\Lambda, \Lambda'}^{\circ}}^{J^{\circ}(\beta, \Lambda')} \rho.$$

Lemma 7.7 (cf. [8, Proposition 8.3.5]). *Let ϑ and Λ' be as above and let π be an irreducible smooth representation of G . Then π contains ϑ if and only if it contains some ϑ' lying over ϑ .*

Proof We can think of ρ as a representation of $P^{\circ}(\Lambda_{\mathfrak{o}_E})P_1(\Lambda)$ trivial on $P_1(\Lambda)$. Notice also that $\mathrm{Ind}_{J^{\circ}(\beta, \Lambda)}^{P^{\circ}(\Lambda_{\mathfrak{o}_E})P_1(\Lambda)} \kappa$ is irreducible and restricts irreducibly to the representation $\mathrm{Ind}_{J^1(\beta, \Lambda)}^{P_1(\Lambda)} \eta$ of $P_1(\Lambda)$. Since ρ is irreducible, but trivial on $P_1(\Lambda)$, we deduce that $\left(\mathrm{Ind}_{J^{\circ}(\beta, \Lambda)}^{P^{\circ}(\Lambda_{\mathfrak{o}_E})P_1(\Lambda)} \kappa \right) \otimes \rho$ is also irreducible.

Suppose first that we also have $\mathfrak{a}_0(\Lambda') \supseteq \mathfrak{a}_0(\Lambda)$. Then, by the compatibility of κ and κ' , we have

$$\begin{aligned} \mathrm{Ind}_{J^{\circ}(\beta, \Lambda)}^{P^{\circ}(\Lambda_{\mathfrak{o}_E})P_1(\Lambda)} \vartheta &\simeq \left(\mathrm{Ind}_{J^{\circ}(\beta, \Lambda)}^{P^{\circ}(\Lambda_{\mathfrak{o}_E})P_1(\Lambda)} \kappa \right) \otimes \rho \\ &\simeq \left(\mathrm{Ind}_{J_{\Lambda, \Lambda'}^{\circ}}^{P^{\circ}(\Lambda_{\mathfrak{o}_E})P_1(\Lambda)} \kappa' |_{J_{\Lambda, \Lambda'}^{\circ}} \right) \otimes \rho \\ &\simeq \mathrm{Ind}_{J_{\Lambda, \Lambda'}^{\circ}}^{P^{\circ}(\Lambda_{\mathfrak{o}_E})P_1(\Lambda)} \left(\kappa' |_{J_{\Lambda, \Lambda'}^{\circ}} \otimes \rho \right). \end{aligned}$$

Since these representations are irreducible, π contains ϑ if and only if it contains $\kappa' |_{J_{\Lambda, \Lambda'}^{\circ}} \otimes \rho$. But this is the case if and only if π contains some representation lying over ϑ .

For the general case, by Lemma 2.8, there is an \mathfrak{o}_E -lattice sequence Λ'' with $\mathfrak{b}_0(\Lambda'') = \mathfrak{b}_0(\Lambda)$ and $\mathfrak{a}_0(\Lambda'') \subseteq \mathfrak{a}_0(\Lambda')$. By Lemma 2.10, we have a sequence $\Lambda_0 = \Lambda, \Lambda_1, \dots, \Lambda_t = \Lambda''$ of \mathfrak{o}_E -lattice sequences such that, for each i , $\mathfrak{b}_0(\Lambda_i) = \mathfrak{b}_0(\Lambda)$ and either $\mathfrak{a}_0(\Lambda_i) \subseteq \mathfrak{a}_0(\Lambda_{i-1})$ or $\mathfrak{a}_0(\Lambda_i) \supseteq \mathfrak{a}_0(\Lambda_{i-1})$. For each i , let κ_i be the β -extension of $J^{\circ}(\beta, \Lambda_i)$ compatible with κ and put $\vartheta_i = \kappa_i \otimes \rho$.

Applying the first case with $\{\Lambda_i, \Lambda_{i-1}\}$ in place of $\{\Lambda, \Lambda'\}$, we see that π contains ϑ_i if and only if it contains ϑ_{i-1} . (Note that, when $\mathfrak{a}_0(\Lambda_i) \subseteq \mathfrak{a}_0(\Lambda_{i-1})$, we have $J_{\Lambda_i, \Lambda_{i-1}}^{\circ} = J^{\circ}(\beta, \Lambda_{i-1})$ so ϑ_{i-1} is the only representation lying over ϑ_i , and vice versa.)

Applying the first case again with $\Lambda'' = \Lambda_t$ in place of Λ , we see that π contains ϑ_t if and only if it contains some representation ϑ' lying over ϑ_t . Since $P^{\circ}(\Lambda_{\mathfrak{o}_E}) = P^{\circ}(\Lambda''_{\mathfrak{o}_E})$, these are precisely the representations lying over $\vartheta = \vartheta_0$, and the result follows. \blacksquare

We continue under the hypothesis (H), so that $\mathfrak{b}_0(\Lambda)$ is not a maximal self-dual \mathfrak{o}_E -order in B . Indeed, we can suppose moreover that $P(\Lambda_{\mathfrak{o}_E})$ is not a maximal compact subgroup of G_E ; for, if it were, then there would exist a self-dual \mathfrak{o}_E -lattice sequence Λ' such that $\mathfrak{b}_0(\Lambda')$ is a maximal self-dual \mathfrak{o}_E -order in B containing $\mathfrak{b}_0(\Lambda)$ and $P(\Lambda'_{\mathfrak{o}_E}) = P(\Lambda_{\mathfrak{o}_E})$; then, by Lemma 7.7 and using the notation there, π would also contain the unique representation ϑ' of $J^{\circ}(\beta, \Lambda')$ which lies over ϑ , and we have $\vartheta' = \kappa' \otimes \rho$, contradicting (H).

We may also make another assumption from now on, namely that the subspaces V^i are numbered so that $\beta_i \neq 0$, for $i > 1$ – that is, if any block of the semisimple stratum $[\Lambda, n, 0, \beta]$ is a null stratum then it is the 1st block.

Let $V = \bigoplus_{j=-m}^m W^{(j)}$ be a self-dual decomposition exactly subordinate to the stratum $[\Lambda, n, 0, \beta]$. Since the order $\mathfrak{b}_0(\Lambda)$ is not maximal self-dual, this decomposition is non-trivial (that is $m \geq 1$). We use the usual numbering for exactly subordinate decompositions (see §6.2). Then we have

$$\mathbf{J}^\circ(\beta, \Lambda)/\mathbf{J}^1(\beta, \Lambda) \cong \mathbf{P}^\circ(\Lambda_{\mathfrak{o}_E}^{(0)})/\mathbf{P}_1(\Lambda_{\mathfrak{o}_E}^{(0)}) \times \prod_{j=1}^m \tilde{P}(\Lambda_{\mathfrak{o}_E}^{(j)})/\tilde{P}_1(\Lambda_{\mathfrak{o}_E}^{(j)}) \quad (7.8)$$

and we can write

$$\bar{\rho} = \bar{\rho}^{(0)} \otimes \bigotimes_{j=1}^m \tilde{\rho}^{(j)},$$

where we can interpret $\tilde{\rho}^{(j)}$ as an irreducible representation of $\tilde{P}(\Lambda_{\mathfrak{o}_E}^{(j)})$ or $\tilde{J}(\beta^{(j)}, \Lambda^{(j)})$, as appropriate.

Suppose that $0 < j \leq m-1$ is such that $i_j = i_{j+1} = i$ and $\tilde{\rho}^{(j)} \not\cong \tilde{\rho}^{(j+1)}$. Since the decomposition is exactly subordinate, we have $0 < q_j(\Lambda) < q_{j+1}(\Lambda)$ (see the remarks following Lemma 6.6). We write $e_i = e(\Lambda^i |_{\mathfrak{o}_{E_i}})$ and define another self-dual \mathfrak{o}_E -lattice sequence Λ' in V , of the same \mathfrak{o}_F -period as Λ , by

$$\Lambda'(ke_i + r) = \begin{cases} \Lambda(ke_i + r) & \text{if } -e_i/2 < r \leq -q_{j+1}, \text{ or } -q_j < r \leq q_j, \text{ or } q_{j+1} < r \leq e_i/2; \\ \varpi_i^k \Lambda^{(j)}(q_j) \oplus \varpi_i^k \Lambda^{(j+1)}(q_{j+1}) \oplus \bigoplus_{t \neq j, j+1} \Lambda^{(t)}(ke_i + r) & \text{if } q_j < r \leq q_{j+1}; \\ \varpi_i^k \Lambda^{(-j)}(1 - q_j) \oplus \varpi_i^k \Lambda^{(-j-1)}(1 - q_{j+1}) \oplus \bigoplus_{t \neq -j, -j-1} \Lambda^{(t)}(ke_i + r) & \text{if } -q_{j+1} < r \leq -q_j. \end{cases}$$

Then $[\Lambda', n, 0, \beta]$ is a skew semisimple stratum, the decomposition is exactly subordinate to this stratum, and

$$\begin{aligned} q_t(\Lambda') &= q_t(\Lambda) \quad \text{for } 0 < t < j \text{ or } j+1 < t < m, \\ q_j(\Lambda') &= q_{j+1}(\Lambda), \quad q_{j+1}(\Lambda') = q_j(\Lambda). \end{aligned}$$

In particular, for the standard numbering relative to Λ' , the j^{th} and $(j+1)^{\text{th}}$ blocks have been exchanged (and likewise for $-j$ and $-j-1$). Note also that we have $\mathfrak{M}_\Lambda = \mathfrak{M}_{\Lambda'}$ so there is a unique standard β -extension κ' to $\mathbf{J}^\circ(\beta, \Lambda')$ compatible with κ .

We have

$$\mathbf{J}^\circ(\beta, \Lambda')/\mathbf{J}^1(\beta, \Lambda') \cong \mathbf{J}^\circ(\beta, \Lambda)/\mathbf{J}^1(\beta, \Lambda)$$

(where the j^{th} and $(j+1)^{\text{th}}$ terms in the decomposition (7.8) have been interchanged) and we denote by ρ' the inflation to $\mathbf{J}^\circ(\beta, \Lambda')$ of $\bar{\rho}$. We put $\vartheta' = \kappa' \otimes \rho'$.

Lemma 7.9 (cf. [8, Proposition 8.3.4]). *The representation π contains ϑ' .*

Proof We define a third self-dual \mathfrak{o}_E -lattice sequence Λ'' in V , of the same \mathfrak{o}_F -period as Λ , by

$$\Lambda''(ke_i + r) = \begin{cases} \Lambda(ke_i + r) & \text{if } -e_i/2 < r \leq -q_{j+1}, \text{ or } -q_j < r \leq q_j, \text{ or } q_{j+1} < r \leq e_i/2; \\ \varpi_i^k \Lambda^{(j)}(q_j) \oplus \varpi_i^k \Lambda^{(j+1)}(q_j) \oplus \bigoplus_{t \neq j, j+1} \Lambda^{(t)}(ke_i + r) & \text{if } q_j < r \leq q_{j+1}; \\ \varpi_i^k \Lambda^{(-j)}(1 - q_j) \oplus \varpi_i^k \Lambda^{(-j-1)}(1 - q_j) \oplus \bigoplus_{t \neq -j, -j-1} \Lambda^{(t)}(ke_i + r) & \text{if } -q_{j+1} < r \leq -q_j. \end{cases}$$

Then $[\Lambda'', n, 0, \beta]$ is a skew semisimple stratum but the decomposition is no longer exactly (or even properly) subordinate. We have

$$\begin{aligned} q_t(\Lambda'') &= q_t(\Lambda) \quad \text{for } 0 < t < j, \quad j+1 < t \leq m, \\ q_j(\Lambda'') &= q_{j+1}(\Lambda'') = q_{j+1}(\Lambda). \end{aligned}$$

Now the self-dual \mathfrak{o}_E -order $\mathfrak{b}_0(\Lambda'')$ contains both $\mathfrak{b}_0(\Lambda)$ and $\mathfrak{b}_0(\Lambda')$. Moreover, $P^\circ(\Lambda_{\mathfrak{o}_E})/P_1(\Lambda''_{\mathfrak{o}_E})$ is a maximal parabolic subgroup in $P^\circ(\Lambda''_{\mathfrak{o}_E})/P_1(\Lambda''_{\mathfrak{o}_E})$, and $P^\circ(\Lambda'_{\mathfrak{o}_E})/P_1(\Lambda''_{\mathfrak{o}_E})$ is the opposite parabolic.

Since $\tilde{\rho}^{(j)} \not\cong \tilde{\rho}^{(j+1)}$, the induced representation

$$\text{Ind}_{P^\circ(\Lambda_{\mathfrak{o}_E})/P_1(\Lambda''_{\mathfrak{o}_E})}^{P^\circ(\Lambda''_{\mathfrak{o}_E})/P_1(\Lambda''_{\mathfrak{o}_E})} \tilde{\rho}$$

is irreducible and there is a unique irreducible representation ϑ'' of $J^\circ(\beta, \Lambda'')$ lying over ϑ . By Lemma 7.7, π therefore contains ϑ'' . But this representation also lies over ϑ' so, again by Lemma 7.7, π contains ϑ' . \blacksquare

In particular, this means that we may repeat this process several times and hence we may (and do) assume that:

$$(A) \quad \text{if } k > j > 0 \text{ and } i_j = i_k \text{ (so } q_j < q_k \text{) then } \dim_{E_i}(W^{(j)}) \leq \dim_{E_i}(W^{(k)}).$$

7.2.1 The case $\rho \not\cong \rho \circ \sigma$

We suppose first that there is some index k such that the representation $\tilde{\rho}^{(k)}$ of $\tilde{J}(\beta^{(k)}, \Lambda^{(k)})$ is not fixed by the involution σ_k (see §6.3). For $j > 0$ we define $\tilde{\rho}^{(-j)} = \tilde{\rho}^{(j)} \circ \sigma_j$ (a representation of $\tilde{J}(\beta^{(j)}, \Lambda^{(j)})$) and set

$$J = \{-m \leq j \leq m : \tilde{\rho}^{(j)} \simeq \tilde{\rho}^{(k)}\}.$$

Note that, since $\tilde{\rho}^{(k)}$ is not fixed by σ_k , if $j \in J$ then $-j \notin J$. Put $-J = \{j : -j \in J\}$ and $J_0 = \{j : \pm j \notin J\}$; then we set

$$Y_1 = \bigoplus_{j \in J} W^{(j)}, \quad Y_0 = \bigoplus_{j \in J_0} W^{(j)}, \quad Y_{-1} = \bigoplus_{j \in -J} W^{(j)}.$$

The decomposition $V = Y_1 \oplus Y_0 \oplus Y_{-1}$ is then properly subordinate to the stratum $[\Lambda, n, 0, \beta]$. Let M' be the Levi subgroup of G which is the stabilizer of this decomposition, and let $P' = M'U'$ be a parabolic subgroup with Levi factor M' . Let M be the Levi subgroup of G which stabilizes the decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ and let $P = MU$ be a parabolic subgroup with Levi factor M such that $P \subseteq P'$. We can then form the representation ϑ_P of J_P° , as in Lemma 6.1.

Proposition 7.10. *The pair (J_P°, ϑ_P) is a cover of $(J_P^\circ \cap M', \vartheta_P|_{J_P^\circ \cap M'})$. In particular, any smooth irreducible representation π of G which contains ϑ is not supercuspidal.*

Proof By the choice of the subspaces Y_1, Y_0, Y_{-1} , we have $N_\Lambda(\rho) \subseteq M'$, where we recall $N_\Lambda(\rho) = \{g \in N_\Lambda : g \text{ normalizes } \rho\}$. By Corollary 6.16, we get $I_G(\vartheta_P) \subseteq J_P^\circ M' J_P^\circ$ and it follows from [9, Theorem 7.2] that any smooth irreducible representation π of G which contains ϑ_P is not supercuspidal. However, π contains ϑ if and only if it contains ϑ_P , by Lemma 6.1. \blacksquare

7.2.2 The case $\rho \simeq \rho \circ \sigma$

We suppose now that $\tilde{\rho}^{(j)} \circ \sigma_j \simeq \tilde{\rho}^{(j)}$, for all $1 \leq j \leq m$. Let M be the Levi subgroup of G which stabilizes the decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$. We fix, throughout this section, a choice of parabolic subgroup P with Levi component M , namely the stabilizer in G of the self-dual flag

$$\{0\} \subseteq W^{(-m)} \subseteq \dots \subseteq \bigoplus_{j=-m}^k W^{(j)} \subseteq \dots \subseteq \bigoplus_{j=-m}^m W^{(j)} = V.$$

Let U be the unipotent radical of P , and U_l its opposite relative to M .

According to the numbering of §6.2, $i_m = l$ so that $W^{(m)} \subseteq V^l$. For $t = 0, 1$, we define a self-dual \mathfrak{o}_{E_l} -lattice sequence \mathfrak{M}_t^l in V^l as follows: if $t = 1$,

$$\mathfrak{M}_1^l(2k+r) = \begin{cases} \varpi_l^k \Lambda^l(1 - e_l/2) & \text{for } r = 0, \\ \varpi_l^k \Lambda^l(e_l/2) & \text{for } r = 1. \end{cases}$$

if $t = 0$ then recall that, again according to the numbering of §6.2, $q_m \geq 1$ is the greatest integer such that $\Lambda^l(q_m) \supseteq \Lambda^l(e_l/2)$ and we put

$$\mathfrak{M}_0^l(2k+r) = \begin{cases} \varpi_l^k \Lambda^l(1 - q_m) & \text{for } r = 0, \\ \varpi_l^k \Lambda^l(q_m) & \text{for } r = 1; \end{cases}$$

Then the $\mathfrak{b}_0(\mathfrak{M}_t^l)$ are maximal self-dual \mathfrak{o}_{E_l} -orders containing $\mathfrak{b}_0(\Lambda^l)$. For each t , we put $\mathfrak{M}_t = \mathfrak{M}_t^i \oplus \bigoplus_{k \neq i} \Lambda^k$.

We recall the elements s_m and s_m^ϖ of §6.2. Note that $s_m \in P^+(\mathfrak{M}_0)$ and $s_m^\varpi \in P^+(\mathfrak{M}_1)$, by Lemma 6.7. We define $U^m = s_m U s_m^{-1}$ and $U_l^m = s_m U_l s_m^{-1}$.

Lemma 7.11. *We have:*

- (i) $J_P^\circ \cap M$, $J_P^\circ \cap U \cap U^m$ and $J_P^\circ \cap U_l \cap U_l^m$ are stable under conjugation by s_m and by s_m^ϖ ;
- (ii) $s_m(J_P^\circ \cap U_l \cap U^m)s_m^{-1} \subseteq J_P^\circ \cap U \cap U_l^m$;
- (iii) $(s_m^\varpi)^{-1}(J_P^\circ \cap U \cap U_l^m)s_m^\varpi \subseteq J_P^\circ \cap U_l \cap U^m$;

In particular, $J_P^\circ s_m J_P^\circ s_m^\varpi J_P^\circ = J_P^\circ s_m s_m^\varpi J_P^\circ$.

Proof We prove the corresponding statements in \tilde{G} and the Lemma follows by taking fixed points under σ . Recall that, since the decomposition is properly subordinate, $\mathfrak{J}(\beta, \Lambda)$ can be decomposed as

$$\mathfrak{J}(\beta, \Lambda) = \bigoplus_{-m \leq j, k \leq m} \mathbf{e}^{(j)} \mathfrak{J}(\beta, \Lambda) \mathbf{e}^{(k)},$$

and similarly for $\mathfrak{H}(\beta, \Lambda)$. We can also write, for example,

$$\tilde{U} \cap s_m \tilde{U} s_m^{-1} = 1 + \sum_{-m < j < k < m} \mathbf{e}^{(j)} A \mathbf{e}^{(k)}.$$

Then the properties from Lemma 6.6 imply that the conjugations by s_m and by s_m^ϖ are trivial on $\tilde{U} \cap s_m \tilde{U} s_m^{-1}$ and (i) follows easily.

Parts (ii) and (iii) are rather similar so we will only address (iii). We have

$$\tilde{J}_{\tilde{P}} \cap \tilde{U} \cap s_m \tilde{U}_l s_m^{-1} = 1 + \sum_{-m < k \leq m} \mathbf{e}^{(-m)} \mathfrak{J}(\beta, \Lambda) \mathbf{e}^{(k)} + \sum_{-m < j < m} \mathbf{e}^{(j)} \mathfrak{J}(\beta, \Lambda) \mathbf{e}^{(m)}.$$

For $-m < k < m$, we have

$$\begin{aligned} (s_m^{\varpi})^{-1} \mathbf{e}^{(-m)} \mathfrak{J}(\beta, \Lambda) \mathbf{e}^{(k)} s_m^{\varpi} &= \varpi_l I_{m, -m} \mathbf{e}^{(-m)} \mathfrak{J}(\beta, \Lambda) \mathbf{e}^{(k)} I_{k, k} \\ &= \mathbf{e}^{(m)} \varpi_l I_{m, -m} \mathfrak{J}(\beta, \Lambda) \mathbf{e}^{(k)} \\ &\subseteq \mathbf{e}^{(m)} \mathfrak{H}(\beta, \Lambda) \mathbf{e}^{(k)}, \end{aligned}$$

since, by Lemma 6.6, $\varpi_l I_{m, -m} \in \mathfrak{b}_1(\Lambda)$ and, by [32, Lemma 3.11(ii)], $\mathfrak{b}_1(\Lambda) \mathfrak{J}(\beta, \Lambda) \subseteq \mathfrak{H}(\beta, \Lambda)$. But $1 + \mathbf{e}^{(m)} \mathfrak{H}(\beta, \Lambda) \mathbf{e}^{(k)} \subseteq \tilde{J}_{\tilde{P}}$, as required. The situation for the other blocks is similar.

The final assertion is immediate by the Iwahori decomposition of J_P° : we have

$$s_m J_P^\circ s_m^{\varpi} = (s_m (J_P^\circ \cap U_l) s_m^{-1}) (s_m (J_P^\circ \cap M) s_m^{-1}) s_m s_m^{\varpi} ((s_m^{\varpi})^{-1} (J_P^\circ \cap U) s_m^{\varpi}) \subseteq J_P^\circ s_m s_m^{\varpi} J_P^\circ,$$

by (i)–(iii). ■

We put $\zeta = s_m s_m^{\varpi}$, so that $J_P^\circ s_m J_P^\circ s_m^{\varpi} J_P^\circ = J_P^\circ \zeta J_P^\circ$.

Corollary 7.12. *For $k \geq 0$, we have:*

- (i) $\zeta^k (J_P^\circ \cap U) \zeta^{-k} \subseteq J_P^\circ \cap U$;
- (ii) $\zeta^{-k} (J_P^\circ \cap U_l) \zeta^k \subseteq J_P^\circ \cap U_l$;
- (iii) $\zeta^k (J_P^\circ \cap M) \zeta^{-k} = J_P^\circ \cap M$.

In particular, for $k_1, k_2 \geq 0$,

$$J_P^\circ \zeta^{k_1} J_P^\circ \zeta^{k_2} J_P^\circ = J_P^\circ \zeta^{k_1 + k_2} J_P^\circ.$$

Proof This is immediate from Lemma 7.11. For example, we have $J_P^\circ \cap U = (J_P^\circ \cap U \cap U^m) (J_P^\circ \cap U \cap U_l^m)$ and certainly ζ^k normalizes the first factor here. But, noting that s_m, s_m^{ϖ} differ from their inverses only by an element of $J_P^\circ \cap M$, by Lemma 7.11(iii),(ii) we have

$$\zeta (J_P^\circ \cap U \cap U_l^m) \zeta^{-1} \subseteq s_m (J_P^\circ \cap U_l \cap U^m) s_m^{-1} \subseteq (J_P^\circ \cap U \cap U_l^m),$$

as required. ■

Now we will construct certain invertible elements of $\mathcal{H}(G, \vartheta_P)$. We assume for now that s_m, s_m^{ϖ} are elements of G . We will treat the other case below.

For $t = 0, 1$, let κ_t denote a β -extension of η compatible with some standard β -extension of $J^\circ(\beta, \mathfrak{M}_t)$. By Corollary 6.13, we have

$$\kappa_t \simeq \text{Ind}_{J_P^\circ}^{J^\circ(\beta, \Lambda)} \kappa_P \otimes \chi_t,$$

for some self-dual character $\chi_t = \chi_t^{(0)} \otimes \bigotimes_{j=1}^m \chi_t^{(j)} \circ \det^{(j)}$. We write $\rho_t = \rho \otimes \chi_t^{-1}$ so that $\vartheta = \kappa_t \otimes \rho_t$. Since the character $\chi_t^{(m)}$ is self-dual, we still have that $\tilde{\rho}_t^{(m)} \simeq \tilde{\rho}_t^{(m)} \circ \sigma_m$ and, since it factors through the determinant $\det^{(m)}$, the representation $\tilde{\rho}_t^{(m)}$ is still cuspidal.

By (7.3), we have support-preserving injective algebra maps

$$\mathcal{H}(P(\mathfrak{M}_{t, \mathfrak{o}_E}), \rho_t) \hookrightarrow \mathcal{H}(G, \vartheta_P).$$

The former has structure understood by the results of Morris [21, Theorem 7.12]. In particular:

- when $t = 0$, it has an invertible element with support $P^\circ(\Lambda_{\mathfrak{o}_E})s_m P^\circ(\Lambda_{\mathfrak{o}_E})$ and we denote by T the image in $\mathcal{H}(G, \vartheta_P)$ of this, so that T has support $J_P^\circ s_m J_P^\circ$.
- when $t = 1$, it has an invertible element with support $P^\circ(\Lambda_{\mathfrak{o}_E})s_m^\varpi P^\circ(\Lambda_{\mathfrak{o}_E})$ and we denote by T^ϖ the image in $\mathcal{H}(G, \vartheta_P)$ of this, so that T^ϖ has support $J_P^\circ s_m^\varpi J_P^\circ$;

Now we define $S = T * T^\varpi$. By Lemma 7.11, it is supported on the single double coset $J_P^\circ \zeta J_P^\circ$ and it is invertible, since both T and T^ϖ are.

Finally, put $S' = S^{e(E_l/F)}$. Then S' is invertible and, by Lemma 7.12, it has support $J_P^\circ \zeta^{e(E_l/F)} J_P^\circ$. We also put $Y_1 = W^{(m)}$, $Y_0 = \bigoplus_{j \neq \pm m} W^{(j)}$ and $Y_{-1} = W^{(-m)}$.

Now we turn to the case when s_m, s_m^ϖ are not elements of G . Recall that this means that $E_l = F = F_0$, that $\epsilon = 1$, and that $\dim_F W^{(m)}$ is odd. In particular, this means we have $\beta_l = 0$ so $i = i_m = 1$ and $V^k \subseteq W^{(0)}$, for $k > 1$, by our ordering of the blocks V^i ; in particular, $P(\Lambda_{\mathfrak{o}_{E_k}}^k)$ is a maximal compact subgroup of G_{E_k} , for $k > 1$.

Moreover, $\tilde{J}(\beta^{(m)}, \Lambda^{(m)}) = P(\Lambda^{(m)})$ and $\kappa^{(m)}$ is just the trivial character (not the quadratic character, since that does not extend to $P^+(\Lambda^{(m)})$). The involution σ_m is transpose-inverse for the basis \mathcal{B}_m chosen to define s_m . In particular, the conditions that $\rho^{(m)}$ be cuspidal and stable under σ_m imply that $\dim_F W^{(m)} = 1$ (see for example [1, Theorem 7.1]). But then, by our ordering of the subspaces $W^{(j)}$, (see assumption (A) above) we have $\dim_F(W^{(j)}) = 1$, for all $j = 1, \dots, m$.

We have two distinct cases here:

- $V^1 \cap W^{(0)} \neq \{0\}$. In this case there is an element $p \in P^+(\Lambda_{\mathfrak{o}_E}^{(0)}) \setminus P(\Lambda_{\mathfrak{o}_E}^{(0)})$ such that $p^2 = 1$. (In fact, $p \in P^+(\Lambda_{\mathfrak{o}_E}^{(l,0)})$, in the notation of §5.1–2.) Then we replace s_m and s_m^ϖ by ps_m and ps_m^ϖ respectively, which lie in G . (Note that p commutes with s_m and s_m^ϖ .) The argument is then as above and we obtain an invertible element S' of $\mathcal{H}(G, \vartheta_P)$ supported on $J_P^\circ \zeta J_P^\circ$ (we have $e(E_1/F) = 1$). We define the subspaces $Y_1 = W^{(m)}$, $Y_0 = \bigoplus_{j \neq \pm m} W^{(j)}$ and $Y_{-1} = W^{(-m)}$ as above also.
- $V^1 \cap W^{(0)} = \{0\}$. In this case, the first observation to make is that $m > 1$; for if $m = 1$ then G_{E_1} is a 2-dimensional special orthogonal group, which is just F^\times , so $P(\Lambda_{\mathfrak{o}_1}^1)$ is certainly a maximal compact subgroup of G_{E_1} . But then, $P(\Lambda_{\mathfrak{o}_E})$ is a maximal compact subgroup of G_E , contradicting hypothesis (H).

Hence $m \geq 2$ and we may imitate the constructions above but with the elements $s_m s_{m-1}$ and $s_m^\varpi s_{m-1}^\varpi$ in place of s_m and s_m^ϖ respectively. (Note that the element ζ does change in this case.) Then the analogue of Lemma 7.11 holds in this situation (with U^m replaced by $s_m s_{m-1} U s_{m-1}^{-1} s_m^{-1}$) and the argument is again as above. So we obtain an invertible element S' of $\mathcal{H}(G, \vartheta_P)$ supported on $J_P^\circ \zeta J_P^\circ$. In this case we put $Y_1 = W^{(m)} \oplus W^{(m-1)}$, $Y_0 = \bigoplus_{j \neq \pm m, \pm(m-1)} W^{(j)}$ and $Y_{-1} = W^{(-m)} \oplus W^{(1-m)}$.

Finally, we suppose we are in any of the cases – that is $s_m \in G$ or not. The decomposition $V = Y_1 \oplus Y_0 \oplus Y_{-1}$ is properly subordinate to the stratum $[\Lambda, n, 0, \beta]$. Let M' be the Levi subgroup of G which is the stabilizer of this decomposition, and let $P' = M'P$, a parabolic subgroup with Levi factor M' . We again form the representation ϑ_P of J_P° .

Proposition 7.13. *The pair (J_P°, ϑ_P) is a G -cover of $(J_P^\circ \cap M', \vartheta_P|_{J_P^\circ \cap M'})$. In particular, any smooth irreducible representation π of G which contains ϑ is not supercuspidal.*

Proof Certainly (J_P°, ϑ_P) has the Iwahori factorization properties required, since $U' \subseteq U$. It remains only to observe $S' \in \mathcal{H}(G, \vartheta_P)$ is invertible with support $J_P^\circ \zeta^{e(E_i/F)} J_P^\circ = J_P^\circ \zeta' J_P^\circ$, for ζ' a strongly (P, J_P°) -positive element of the centre of M' . The result now follows as in Proposition 7.10, by appealing to [9, Theorem 7.2] and Lemma 6.1. \blacksquare

7.3 Conclusions

Let π be any irreducible supercuspidal representation of G . We know from [32, Theorem 5.1] that π contains some semisimple character $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$, where $[\Lambda, n, 0, \beta]$ is a skew semisimple stratum. As in §7.2, it contains some representation $\vartheta = \kappa \otimes \rho$ of $J^\circ(\beta, \Lambda)$, with κ a standard β -extension and ρ cuspidal.

If hypothesis (H) were satisfied then either Proposition 7.10 or Proposition 7.13 would give us a contradiction. Hence π contains some such $\vartheta = \kappa \otimes \rho$ with $\mathfrak{b}_0(\Lambda)$ a maximal self-dual \mathfrak{o}_E -order in B . Then π contains some irreducible component λ of $\text{Ind}_{J^\circ(\beta, \Lambda)}^{J(\beta, \Lambda)} \vartheta$, and we have $\lambda = \kappa \otimes \tau$, for κ a β -extension and τ some irreducible component of $\text{Ind}_{J^\circ(\beta, \Lambda)}^{J(\beta, \Lambda)} \rho$. Such a τ is cuspidal so $(J(\beta, \Lambda), \lambda)$ is a maximal simple type and, by Corollary 6.19, $\text{c-Ind}_J^G \lambda$ is irreducible with $\pi \simeq \text{c-Ind}_J^G \lambda$. We have proved our main result:

Theorem 7.14. *Let π be an irreducible supercuspidal representation of G . Then there exists a maximal simple type (J_M, λ_M) , in the sense of Definition 6.17, such that $\pi \simeq \text{c-Ind}_{J_M}^G \lambda_M$.*

In particular, this verifies, for the group G , the folklore conjecture that all irreducible supercuspidal representations are irreducibly compact-induced from compact (mod-centre) subgroups.

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Shaun Stevens
School of Mathematics
University of East Anglia
Norwich NR4 7TJ
United Kingdom
email: Shaun.Stevens@uea.ac.uk