MTH-2B71 Vector Calculus: Problem sheet (1)

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Please hand in the questions marked (*) by 3pm on Monday Week 7, 6th November

1. Find $\nabla\phi$ for the scalar field $\phi(x, y, z) = xyz + x^2 \cos y$.

2. Find the unit normal vector at the general point (x, y, z) on the surface

$$(i) \quad z = \cos y + \cos x, \qquad\qquad (ii)(*) \quad x^2 - 2yz = 0.$$

3.$(*)$ Find a unit tangent vector to the curve C described by

$$x(t) = t^2, \quad y(t) = t, \quad z(t) = t^3,$$

at a point with parameter t. Find also the rate of change of the function $F(x, y, z) = x^2 + yz + 3xz^2$ in the direction of the curve C at the point with coordinates $(1, 1, 1)$.

4. What kind of surfaces are the level surfaces of the following scalar fields?

$$(i) \quad f = 4x + 3y - z, \qquad (ii) \quad f = x^2 + y^2 + z^2, \qquad (iii) \quad f = z - \sqrt{x^2 + y^2}.$$

5. Calculate the divergence $\nabla \cdot \mathbf{u}$ of the following vector fields $\mathbf{u}$:

$$(i) \quad \mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \qquad (ii) \quad \mathbf{u} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k},$$

$$(iii) \quad \mathbf{u} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}, \qquad (iv) \quad \mathbf{u} = u_1(y, z)\mathbf{i} + u_2(x, z)\mathbf{j} + u_3(x, y)\mathbf{k}.$$

6.$(*)$ Show that

$$\nabla \cdot (r^n \mathbf{r}) = (n + 3)r^n$$

for any number n, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

7. Find the Laplacian $\nabla^2\phi$ of the scalar field

$$\phi = 3x^2 + 2y^2 - cz^2.$$

For what value of c is Laplace's equation satisfied?

8. Prove that for any scalar fields f and g,

$$\nabla \cdot (f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g.$$

9. Find the curl $\nabla \times \mathbf{u}$ for the following vector fields $\mathbf{u}$:

$$(i)\quad \mathbf{u} = \sin y\,\mathbf{i} + \cos z\,\mathbf{j}, \qquad\qquad (ii)\quad \mathbf{u} = x^2 y\,\mathbf{i} + y^2 x\,\mathbf{j} + z\,\mathbf{k},$$

$$(iii)\quad \sin(zy)\,\mathbf{i} + \cos(zy)\,\mathbf{j} + \sin(xy)\,\mathbf{k}, \qquad\qquad (iv)\quad \mathbf{u} = u_1(x)\mathbf{i} + u_2(y)\mathbf{j} + u_3(z)\mathbf{k}.$$

10. Prove the following identities:

$$(i)\quad \nabla \cdot (\nabla \times \mathbf{u}) = 0, \qquad (ii)(*)\quad \nabla \times (\nabla \phi) = \mathbf{0},$$

$$(iii)\quad \nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}.$$

11. For a smooth vector field $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$, the vector operator $(\mathbf{a} \cdot \nabla)$ is defined by

$$\mathbf{a} \cdot \nabla = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}.$$

Prove the identity

$$(\mathbf{a} \cdot \nabla)\mathbf{a} = \frac{1}{2}\nabla\left(|\mathbf{a}|^2\right) - \mathbf{a} \times (\nabla \times \mathbf{a}).$$