Computational Stability of Bryan-Semtner-Cox type models (Draft)

David P. Stevens, School of Mathematics, University of East Anglia, Norwich, NR4 7J, England.

The purpose of this short note is to list a multitude of stability critera for Bryan-Semtner-Cox [Bryan (1969), Semtner (1974), Cox (1984)] type ocean general circulation models. None of the results listed here are new, rather this note brings together results (but not their derivation) into what is hopefully a useful form.

The method of leapfrogging for time differencing tends to cause 'time splitting' of the solution into two modes, the true physical mode and a computational mode at alternate timesteps (Mesinger and Arakawa, 1976, p14). This can lead to problems in all the prediction equations studied above. The splitting is usually alleviated by some form of mixing of the two solutions. Bryan (1969) and Semtner (1974) use a forward timestep of length Δt periodically. Cox (1984) makes provision for using an Euler backward timestep, which is an improvement as unlike the forward timestep it has a damping effect on high frequency noise, although at a greater computational cost. Another alternative is the Robert time filter (Asselin, 1972). The variable μ being time stepped is operated on as follows

$$\overline{\mu^n} = \mu^n + \frac{1}{2}\nu(\overline{\mu^{n-1}} - 2\mu^n + \mu^{n+1}), \tag{1}$$

to obtain its filtered counterpart $\overline{\mu}$. This operation is carried out every timestep with a small filter parameter $\nu = 0.01$. The Robert time filter has a similar damping effect on high frequency noise as a periodic Euler backward time step, but at a lesser computational cost. (** Killworth *et al.* on stability of mixing steps)

Models can be made more efficient by taking different length timesteps for the barotropic, baroclinic and tracer equations, that is by taking longer timesteps for the slowly evolving tracer fields. Killworth *et al.* (1984) discuss the use and restrictions of this method for various grid sizes. They conclude that the greatest gains can be made when using a coarse resolution grid of O(300km). The tracer timestep can then be up to 25 times the length of that in the momentum equations. In fact Bryan *et al.* (1975) use timesteps in the ratio of 23.03:8.64:1 for the tracer, internal mode and external mode equations respectively. However the method can only be used with confidence when equilibrium solutions are being sought.

Certain stability criteria need to be met when choosing grid spacing, timesteps, eddy viscosity and eddy diffusivity for a model. Linear stability can be determined by examining balances between various terms in the governing equations. In the following criteria conditions on Δt apply to all equations, conditions on Δt^T and Δt^v apply to the tracer equation and the momentum equations respectively. The variables $\Delta_{\min}$ and $\Delta_{\max}$ refer to the minimum and maximum horizontal grid spacing. The maximum advective speed is v and c is the fastest wave speed for the variable being calculated.

The first condition is the well known CFL condition (named after Courant, Friedrich and Lewy)

$$\Delta t < \frac{\Delta_{\min}}{v + c} \tag{2}$$

on the timestep. Physically the CFL condition states that information should not propagate more than one grid box in one timestep. Recently the increase in computer power has led to the introduction of fine vertical grids. In some extreme cases this has resulted in vertical motion violating the CFL condition.

The timestep limitation due to the horizontal diffusive terms is well known and can be found in almost any text on numerical analysis, for example O'Brien (1986). The criteria for the tracer equation is

$$\Delta t^T < \frac{\Delta_{\min}^2}{8A_h}, \tag{3}$$

with the following similar condition for the momentum equations

$$\Delta t^v < \frac{\Delta^2_{\min}}{8A_m}. \tag{4}$$

Due to the use of a no slip condition at lateral boundaries a viscous boundary layer will be formed. The thickness of this layer according to Munk (1950) is $O(A_m/\beta)^{\frac{1}{3}}$ and thus

$$A_m > \beta\Delta^3_{\max} \quad \text{where} \;\; \beta = \frac{\partial f}{\partial \phi} \tag{5}$$

for the boundary layer to be properly resolved.

The next criteria to be satisfied is the grid point Reynolds number which arises from the balance between advection and diffusion. Bryan *et al.* (1975) show the criteria to be

$$A_m \;\; \text{or} \;\; A_h > \frac{u\Delta_{\max}}{2}, \tag{6}$$

but as they point out this is not a necessary condition for stability. However once an instability develops it will not disappear unless the criteria is satisfied.

In coarse resolution studies the above criteria dictate unrealistically large values of A_m and A_h. However, as Bryan *et al.* (1975) point out, the computational mode in the density field is suppressed by the high viscosity on the velocity field because of the strong coupling between the two through the pressure term. Thus lateral diffusion of temperature and salinity A_h can be made much less than the value required for A_m.

Problems with the U.K. Fine Resolution Antarctic Model (FRAM) led Killworth (1987) to investigate topographically induced instabilities. For changes in depth from H_1 to H_2 ($H_1 << H_2$)

$$\frac{A_m \Delta t}{\Delta^2} < \frac{2H_1}{H_2}. \tag{7}$$

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