

THE κ -STRONGLY PROPER FORCING AXIOM

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ABSTRACT. We study methods to obtain the consistency of forcing axioms, and particularly higher forcing axioms. We first force over a model with a supercompact cardinal $\theta > \kappa$ to get the consistency of the forcing axiom for κ -strongly proper forcing notions which are also κ -lattice, and then eliminate the need for large cardinals. The proof goes through a natural reflection property for κ -strongly proper forcings. We also produce a model of this forcing axiom with 2^κ arbitrarily large, and prove the inconsistency of certain natural strengthenings of the axiom.

1. INTRODUCTION

Forcing axioms are set-theoretic axioms which state that the universe is “rich with filters” for forcing notions in a particular class. More technically, forcing axioms are statements saying that given a forcing notion in a particular class, and a “relatively small” collection of dense open subsets, there is a filter which meets all dense open sets in the collection. Martin’s Axiom states that if \mathbb{P} is c.c.c. (i.e., if \mathbb{P} satisfies the countable chain condition) and $\{D_\alpha \mid \alpha < \kappa\}$ is a family of dense open subsets of \mathbb{P} , where $\kappa < 2^{\aleph_0}$, then there is a filter G such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. ZFC trivially proves that for $\kappa = \aleph_0$ such filters exist for any forcing \mathbb{P} , regardless of its combinatorial properties.

When we assume forcing axioms hold in the universe V , we can prove that there are objects in V which exhibit “somewhat generic properties”. For instance, if we assume Martin’s Axiom, and $\{f_\alpha \mid \alpha < \kappa\} \subseteq \omega^\omega$ for $\kappa < 2^{\aleph_0}$, then we may consider $\mathbb{P} = \omega^{<\omega}$ and $D_n^\alpha = \{s \in \omega^{<\omega} \mid f_\alpha(m) < s(m) \text{ for some } m \in |s|, m > n\}$, for $\alpha < \kappa$ and $n < \omega$, as our dense open sets. If $G \cap D_n^\alpha \neq \emptyset$ for all α and n , then $\bigcup G = g \neq f_\alpha$ for all α , and in fact for each α , $g(m) > f_\alpha(m)$ holds infinitely often. Therefore, Martin’s Axiom implies that \mathfrak{d} , the dominating number, equals 2^{\aleph_0} .

In the classical case, forcing axioms are phrased around \aleph_1 as the main cardinal of interest. In this context, the forcing notions themselves somehow revolve around this (e.g., properness is defined with models of size less than \aleph_1 , i.e., countable). Recent work on extensions of classical forcing axioms such as the Proper Forcing Axiom, relative to collections of more than \aleph_1 dense sets, deals with subclasses of proper forcing notions, and tries to push the size of 2^{\aleph_0} to \aleph_3 or higher. This is difficult, since amongst these “somewhat generic properties” we can find, for example, closed and unbounded subsets of ω_2 contradicting club guessing on $\omega_2 \cap \text{cf}(\omega)$,¹ or

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¹The existence, in ZFC, of a club-sequence $\langle C_\alpha \mid \alpha < \omega_2, \text{cf}(\alpha) = \omega \rangle$ guessing clubs of ω_2 —i.e., such that every club $C \subseteq \omega_2$ includes some C_α —is a well-known result of Shelah.

functions $\omega_2 \rightarrow \{0, 1\}$ uniformising colourings for which there is no uniformisation (see [6]).

Moving to higher cardinals is harder also because we lose our iteration theorems. Iterating c.c.c. forcing notions with finite support is still c.c.c., and iterating proper forcing notions with countable support is still proper. But moving to higher cardinals, even if we require the forcings to be very closed, might result in unwanted cardinal collapsing (see [6]).

James Cummings, Mirna Džamonja, and Itay Neeman proved in [1] the consistency of a forcing axiom of this flavour by replacing c.c.c. by a more restrictive form of the κ^+ -c.c. known as the strong κ^+ -c.c. In this note we deal with κ -strong properness, a more general notion than the strong κ^+ -c.c. from [1]. We show that Neeman’s consistency proof of PFA using finite conditions can be generalised quite easily to this context even when κ is uncountable. We then prove that κ -strongly proper forcings satisfy a weak reflection property: to prove that enough filters exist for any κ -strongly proper forcing, it is enough to prove that enough filters exist for κ -strongly proper subforcings of size 2^κ . Using this reflection property, together with an argument involving the fact that all κ -sequences of ordinals added by a κ -lattice and κ -strongly proper forcing come from adding a Cohen subset of κ (Proposition 2.3), we show that the assumption of a supercompact cardinal (or any large cardinal) is in fact unnecessary. We then modify our construction and show the consistency of the forcing axiom together with 2^κ being arbitrarily large. This modified construction actually shows the consistency of a slightly stronger form of our forcing axiom and does not need the fact that κ -sequences of ordinals added by a κ -lattice and κ -strongly proper forcing come from a κ -Cohen extension.

Our main result is thus the consistency relative to ZFC, for any given regular cardinal κ with $\kappa^{<\kappa} = \kappa$, of the forcing axiom, for families of κ^+ -many dense sets, for the family of forcing notions which are both κ -lattice and κ -strongly proper. This is a rather small class, containing κ -Cohen forcing and the natural forcing for adding a club of κ^+ with conditions of size less than κ , but not much more. One consequence of the corresponding forcing axiom, due to the inclusion in the class of the above forcing for adding a club of κ^+ , is the failure of tail club-guessing on κ^+ for ordinals of cofinality κ ; in other words, the forcing axiom implies that for every sequence $\langle C_\alpha \mid \alpha \in \kappa^+, \text{cf}(\alpha) = \kappa \rangle$, where each C_α is a club of α , there is a club $C \subseteq \kappa^+$ such that $C_\alpha \setminus C$ is unbounded in α for every $\alpha < \kappa^+$ of cofinality κ .² One could try to obtain a more useful forcing axiom by considering a slightly broader class of forcing notions. We finish the paper observing that certain natural moves in this direction lead to inconsistent principles.

Throughout the paper we work in ZFC + GCH for the sake of simplicity, although many of these results can be proved without GCH if one is willing to collapse cardinals, as long as one assumes that $\kappa^{<\kappa} = \kappa$ where appropriate.

The structure of the first part of the paper is what we may call an “onion proof”. We start by sketching Neeman’s consistency proof of PFA in the present context, using a supercompact cardinal. We then prove the weak reflection lemma, which allows us to “peel off” the consistency strength of the proof to a mere inaccessible cardinal, and then we show that this too can be reduced to nothing more than ZFC. In the second part of the paper we modify (in section 7) our final construction from the first part so as to obtain a model of our forcing axiom with 2^κ arbitrarily large, and in section 8 we prove that various natural strengthenings of the forcing axiom are actually false.

²When $\kappa = \omega$, the consistency of the above club-guessing failure is of course well-known. For $\kappa > \omega$, the consistency of the corresponding club-guessing failure is due to Shelah (e.g., [7]).

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2. PRELIMINARIES

We say that a relation \prec is a weak total order on X if the transitive closure of \prec is a total order on X . We say that a set M is κ -closed if for every $\alpha < \kappa$, every function $f: \alpha \rightarrow M$ is already in M . In the case of a forcing \mathbb{P} , we say that \mathbb{P} is κ -closed if every decreasing sequence of length less than κ has a lower bound, and that \mathbb{P} is κ -directed closed if every directed set of size less than κ has a lower bound. We will say that a forcing is κ -lattice if every set of size less than κ of pairwise compatible conditions has a greatest lower bound.³

2.1. Strong properness.

Definition 2.1. Let M be a set and \mathbb{P} a forcing in M . We say that a condition $q \in \mathbb{P}$ is *strongly M -generic* (for \mathbb{P}) if for every $q' \leq q$ there is $\pi_M(q') \in \mathbb{P} \cap M$ such that every condition in $\mathbb{P} \cap M$ extending $\pi_M(q')$ is compatible with q' .

Definition 2.2. Let \mathbb{Q} be a forcing notion and let \mathcal{K} be a class of models. We say that \mathbb{Q} is *strongly proper for \mathcal{K}* if for every cardinal χ and every $M \in \mathcal{K}$ such that $M \prec H(\chi)$ and $\mathbb{Q} \in M$, every $p \in \mathbb{Q} \cap M$ can be extended to a strongly M -generic condition.

When \mathcal{K} is the class of all κ -closed models M of size κ , we simply say that \mathbb{Q} is κ -strongly proper.

The following is a generalisation of an observation of Mitchell in [2].

Proposition 2.3. *Suppose that $\kappa^{<\kappa} = \kappa$, and let \mathbb{P} be a κ -lattice and κ -strongly proper forcing notion. Any κ -sequence of ordinals added by \mathbb{P} is κ -Cohen generic.*

Proof. Let \dot{f} be a \mathbb{P} -name and $p \in \mathbb{P}$ such that $p \Vdash \dot{f}$ is a $\check{\kappa}$ -sequence of ordinals. Let M be a κ -closed elementary submodel of $H(\chi)$, for some large enough χ , such that $\mathbb{P}, \dot{f}, p \in M$. We let $\mathbb{Q} = \mathbb{P} \cap M$. By elementarity of M and its κ -closedness we have that \mathbb{Q} is a κ -lattice forcing of size κ , and therefore by a back-and-forth argument, using $\kappa^{<\kappa} = \kappa$, we have that \mathbb{Q} is isomorphic to $\kappa^{<\kappa}$.

Let $p^* \leq p$ be a strongly M -generic condition. Let G be a V -generic filter with $p^* \in G$. Then $G \cap \mathbb{Q}$ is V -generic for \mathbb{Q} , and this is forced by p^* : given any $q_0 \leq p^*$ and any dense subset $D \subseteq \mathbb{Q}$, we may extend the projection of q_0 into M , $\pi_M(q_0)$, to a condition $q \in D$, and since $\mathbb{Q} \subseteq M$, $q \in M$ and therefore compatible with q_0 in \mathbb{P} . Also, for every $\alpha < \kappa$ the set D_α of \mathbb{P} -conditions deciding $\dot{f}(\check{\alpha})$ is in M , and by elementarity of M , $D_\alpha \cap M$ is a dense subset of \mathbb{Q} . This means that \dot{g} defined by $\{\langle q, \langle \check{\alpha}, \check{\beta} \rangle \mid q \Vdash_{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{\beta}, q \in M \rangle\}$ is a \mathbb{Q} -name such that $p^* \Vdash_{\mathbb{P}} \dot{f} = \dot{g}$, and therefore $\dot{f}^G = \dot{g}^{G \cap \mathbb{Q}} \in V[G \cap \mathbb{Q}]$. \square

The following is clear.

Proposition 2.4. *The κ -support iteration of κ -lattice forcings is κ -lattice.* \square

It will be convenient to adopt the following notation: Given a class Γ of forcing notions and a cardinal λ , $\text{FA}_\lambda(\Gamma)$ is the assertion that for every $\mathbb{P} \in \Gamma$ and every collection $\mathcal{D} = \{D_\alpha \mid \alpha < \lambda\}$ of dense subsets of \mathbb{P} there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \lambda$. We also say that G is a \mathcal{D} -generic filter (of \mathbb{P}).

³It would perhaps be more appropriate to call such a forcing notion a κ -lower semi-lattice. However, in the interest of keeping things simple, we will not use this terminology.

Definition 2.5. The κ -Strongly Proper Forcing Axiom (κ -StrPFA) is

$$\text{FA}_{\kappa^+}(\{\mathbb{P} \mid \mathbb{P} \text{ is } \kappa\text{-lattice and } \kappa\text{-strongly proper}\}).$$

We note that unlike the case with MA, where we allow \mathcal{D} to have any size $< 2^{\aleph_0}$, here we regard our forcing axiom as an analogue of PFA and therefore consider only families \mathcal{D} of size at most κ^+ .

3. THE BASIC INGREDIENTS: SUPERCOMPACT CARDINALS

Theorem 3.1. *Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$ and let $\theta > \kappa$ be a supercompact cardinal. Then there is a κ -lattice and κ -strongly proper forcing \mathbb{P} which forces $\theta = \kappa^{++}$ together with κ -StrPFA.*

We prove this theorem by almost entirely repeating the consistency proof of PFA by finite conditions given by Neeman in [4], to the point that the authors cannot take credit for this theorem. We will omit most of the proofs of the subclaims, as they are essentially the same as those of Neeman; instead we will indicate, at the appropriate places, what the relevant claims from [4] are. The rest of this section is devoted to the proof of this theorem.

Let $F: \theta \rightarrow V_\theta$ be a Laver function for θ and let E denote the set of strong limit cardinals in θ of cofinality at least κ^+ . Let \mathcal{S} be the set of κ -closed $M \prec H(\theta)$ such that $|M| = \kappa$ and let $\mathcal{T} = \{H(\alpha) \mid \alpha \in E\}$. We define for each $\alpha \in E \cup \{\theta\}$ a forcing \mathbb{P}_α such that \mathbb{P}_α is a complete subforcing of \mathbb{P}_β for all $\alpha \leq \beta$ in $E \cup \{\theta\}$. Our forcing \mathbb{P} will be \mathbb{P}_θ .

Given $\beta \in E \cup \{\theta\}$, we define \mathbb{P}_β as the collection of all the pairs $\langle p, s \rangle$ such that:

- (1) $s \in [(\mathcal{S} \cup \mathcal{T}) \cap H(\beta)]^{<\kappa}$ and \in is a weak total order on s .
- (2) p is a function with $\text{dom}(p) \in [E \cap \beta]^{<\kappa}$ such that for all $\alpha \in \text{dom}(p)$,
 - (a) $F(\alpha)$ is a \mathbb{P}_α -name such that $\Vdash_\alpha F(\alpha)$ is a κ -lattice, κ -strongly proper forcing notion whose conditions are ordinals,⁴
 - (b) $H(\alpha) \in s$, and
 - (c) $p(\alpha)$ is a nice \mathbb{P}_α -name such that $\Vdash_\alpha p(\alpha) \in F(\alpha)$.
- (3) For every $\alpha \in \text{dom}(p)$ and every $M \in s \cap \mathcal{S}$ such that $\alpha \in M$, the pair $\langle p \upharpoonright \alpha, s \cap H(\alpha) \rangle$ is a condition in \mathbb{P}_α which forces in \mathbb{P}_α that $p(\alpha)$ is a strong $F(\alpha)$ -master condition for $M[\dot{G}_\alpha]$.

We define $\langle p_1, s_1 \rangle \leq_\beta \langle p_0, s_0 \rangle$ if the following conditions hold:

- $s_0 \subseteq s_1$,
- $\text{dom}(p_0) \subseteq \text{dom}(p_1)$, and
- for all $\alpha \in \text{dom}(p_0)$, $\langle p_1 \upharpoonright \alpha, s_1 \cap H(\alpha) \rangle \Vdash_\alpha p_1(\alpha) \leq_{F(\alpha)} p_0(\alpha)$.

To simplify the notation, if $\alpha \in E$ and $\langle p, s \rangle \in \mathbb{P}_\beta$ for some $\beta > \alpha$, we will write $\langle p, s \rangle \upharpoonright \alpha$ to denote $\langle p \upharpoonright \alpha, s \cap H(\alpha) \rangle$.

Given $\beta \in E \cup \{\theta\}$, we denote by $\mathbb{P} \upharpoonright \beta$ the partial order $\{\langle p, s \rangle \in \mathbb{P} \mid \text{dom}(p) \subseteq \beta\}$. Note that there is no restriction on s .

Claim 3.2 (Claim 6.5 in [4]). *Condition (3) in the definition of \mathbb{P}_β is equivalent to, instead of considering (in the hypothesis) those M such that $\alpha \in M \in s \cap \mathcal{S}$, considering (in the hypothesis) those M such that $M \in s \cap \mathcal{S}$ occurs above $H(\alpha)$ in s and such that no model between $H(\alpha)$ and M is transitive.*

Claim 3.3 (Claim 6.6 in [4]). *Let $\alpha < \beta$ be two ordinals in $E \cup \{\theta\}$. Suppose that $\langle p, s \rangle \in \mathbb{P} \upharpoonright \beta$ with $H(\alpha) \in s$, and let $\langle q, t \rangle \in \mathbb{P} \cap H(\alpha)$ be a condition extending $\langle p, s \rangle \upharpoonright \alpha$. Then $\langle p, s \rangle$ and $\langle q, t \rangle$ are compatible in $\mathbb{P} \upharpoonright \beta$, as witnessed by $\langle r, s \cup t \rangle$, where $r = q \cup p \upharpoonright [\alpha, \beta)$.*

⁴It is not really necessary to require conditions in the forcing named by $F(\alpha)$ to be ordinals, but it simplifies things to do so.

Claim 3.4 (Claim 6.7 in [4]). Let $\beta \in E \cup \{\theta\}$.

- (1) Let $\langle p, s \rangle \in \mathbb{P} \upharpoonright \beta$ and $\alpha \in E$ be such that $H(\alpha) \in s$. Then $\langle p, s \rangle$ is a strongly $H(\alpha)$ -generic condition for $\mathbb{P} \upharpoonright \beta$.
- (2) Let $\langle p, s \rangle \in \mathbb{P} \upharpoonright \beta$ and $\alpha \in E$, and suppose that $\langle p, s \rangle \in H(\alpha)$. Then $\langle p, s \cup \{H(\alpha)\} \rangle \in \mathbb{P} \upharpoonright \beta$.
- (3) $\mathbb{P} \upharpoonright \beta$ is strongly proper for \mathcal{T} .

Claim 3.5 (Claim 6.8 in [4]). Let \mathbb{Q} be a κ -lattice κ -strongly proper forcing. Fix λ such that $\mathbb{Q} \in H(\lambda)$ and an \in -chain $\langle M_i \mid i < \mu \rangle$ of κ -closed elementary submodels of $H(\lambda)$ with $\mathbb{Q} \in M_i$, with $\mu < \kappa$. Suppose that $i^* < \mu$ and $q \in M_{i^*}$ is a strongly M_{i^*} -generic condition for all $i < i^*$. Then there is some $q' \leq q$ which is a strongly M_i -generic for all $i < \mu$. In particular, there is a condition $q \in \mathbb{Q}$ extending any given $q_0 \in M_0$ which is strongly M_i -generic for all $i < \mu$.

Sketch of Proof of Claim. We build a decreasing sequence, q_j for $i^* < j < \mu$, of conditions extending q and such that $q_j \in M_j$ is a strongly M_j -generic for all $j > i^*$. At limit steps we use the assumption that \mathbb{Q} is κ -lattice and take q_j to be the greatest lower bound of $\langle q_i \mid i < j \rangle$. At successor steps we simply use the fact that there is an extension of q_j to a strongly M_j -generic condition q_{j+1} . We apply elementarity to find q_{j+1} in M_{j+1} . Finally, by taking $i^* = 0$, the last part of the claim follows immediately. \square

Claim 3.6 (Claim 6.9 in [4]). Let $\langle p, s \rangle \in \mathbb{P}$ such that for some α , $H(\alpha) \in s$ but $\alpha \notin \text{dom}(p)$. Moreover, let $M \in s \cap \mathcal{S}$ and $\langle q, t \rangle \in \mathbb{P} \cap M$ be such that $\alpha \in \text{dom}(q)$ and $\langle p, s \rangle \leq \langle q \upharpoonright \theta \setminus \{\alpha\}, t \rangle$. If $(s \cap M) \setminus H(\alpha) \subseteq t$, then there is a function p' which extends p , with $\text{dom}(p') = \text{dom}(p) \cup \{\alpha\}$, and such that $\langle p', s \rangle \in \mathbb{P}$ and $\langle p', s \rangle \leq \langle q, t \rangle$.

Claim 3.7 (Claim 6.10 in [4]). Let $\langle p, s \rangle$ and $\langle q, t \rangle$ be conditions in \mathbb{P} . Let $M \in s \cap \mathcal{S}$ such that $\langle q, t \rangle \in M$. Suppose there is some $\delta < \theta$ such that:

- (1) $\langle p, s \rangle \leq \langle q \upharpoonright \delta, t \rangle$ and $(\text{dom}(p) \cap \text{dom}(q)) \setminus \delta = \emptyset$, and
- (2) $(s \cap M) \setminus H(\delta) \subseteq t$.

Then there is a function p' extending p such that $\text{dom}(p') = \text{dom}(p) \cup (\text{dom}(q) \setminus \delta)$ and such that $\langle p', s \rangle \in \mathbb{P}$ extends $\langle q, t \rangle$.

Claim 3.8 (Corollary 6.11 in [4]). Let $M \in \mathcal{S}$ and let $\langle p, s \rangle \in \mathbb{P} \cap M$. Then there is condition $\langle q, t \rangle \in \mathbb{P}$ that extends $\langle p, s \rangle$ and is such that $M \in t$.

Claim 3.9. \mathbb{P} is κ -lattice.

Proof. Suppose that $\langle \langle p_i, s_i \rangle \mid i < \mu \rangle$ is a directed system of conditions with $\mu < \kappa$. Let p be the function with domain $\bigcup_{i < \mu} \text{dom}(p_i)$ such that for each $i < \mu$ and $\alpha \in \text{dom}(p_i)$, $p(\alpha)$ is a canonical \mathbb{P}_α -name for a condition forced to be the greatest lower bound of $\{p_j(\alpha) \mid i \leq j < \mu\}$ provided $\{p_j(\alpha) \mid i \leq j < \mu\}$ is a directed set of conditions in $F(\alpha)$. Let also s be the closure of $\bigcup_{i < \mu} s_i$ under intersections. It is then immediate to verify that $\langle p, s \rangle$ is a lower bound of $\{\langle p_i, s_i \rangle \mid i < \mu\}$, and it is indeed the greatest lower bound by construction. \square

Claim 3.10. Let $\beta \in E$, $\langle p, s \rangle \in \mathbb{P}$, and $M \in s \cap \mathcal{S}$ be such that $\beta \in M$. Suppose that $\langle p', s' \rangle \in \mathbb{P}_\beta$ extends $\langle p, s \rangle \upharpoonright \beta$. Then given any $\langle \bar{p}, \bar{s} \rangle \in \mathbb{P}_\beta \cap M$ such that $\langle p', s' \rangle \leq \langle \bar{p}, \bar{s} \rangle$, there is $\pi_M(p', s') \in \mathbb{P}_\beta \cap M$ such that $\pi_M(p', s')$ extends $\langle \bar{p}, \bar{s} \rangle$ and such that every $\langle q, t \rangle \in M$ which extends $\pi_M(p', s')$ is compatible with $\langle p', s' \rangle$. In particular, $\langle p, s \rangle \upharpoonright \beta$ is a strongly M -generic condition for \mathbb{P}_β whenever $\beta \in M \in s$.

Proof. The case where $\kappa = \omega$, i.e. when we deal with the usual notion of a strongly proper forcing, was proved by Neeman in [4]. We therefore assume $\kappa > \omega$. We prove the claim by induction on β . Let $\langle \bar{p}, \bar{s} \rangle \in M$ be a condition such that

$\langle p', s' \rangle \leq \langle \bar{p}, \bar{s} \rangle$. Let $\langle \alpha_i \mid i < \mu \rangle$, for some $\mu < \kappa$, be the strictly increasing enumeration of $\text{dom}(p') \cap M$. Without loss of generality we may assume $\mu > 0$, as otherwise the conclusion is immediate.

Using the previous claim that \mathbb{P} is κ -lattice and suitable bookkeeping, we may build a \leq -decreasing sequence in \mathbb{P} , $\langle \langle p_i, s_i \rangle \mid i \leq \mu \cdot \omega \rangle$, where $\langle p_0, s_0 \rangle = \langle p', s' \rangle$ and for every $i < \mu \cdot \omega$ and $\alpha \in \text{dom}(p_i) \cap M$ there is some $j > i$ such that $\langle p_j, s_j \rangle \upharpoonright \alpha \in \mathbb{P}_\alpha$ decides, for some name $\dot{\xi}_j^\alpha \in M$ for an ordinal, that $\dot{\xi}_j^\alpha$ is a condition in $F(\alpha)$ such that every $F(\alpha)$ -condition in $M \cap F(\alpha)$ extending $\dot{\xi}_j^\alpha$ is $F(\alpha)$ -compatible with $p_j(\alpha)$.

By suitable applications of the induction hypothesis we can make sure that for every α and every increasing sequence of indices $\langle j_\eta \mid \eta < \nu \rangle$ such that $\dot{\xi}_{j_\eta}^\alpha$ is defined, $\langle \dot{\xi}_{j_\eta}^\alpha \mid \eta < \nu \rangle$ is forced to be a decreasing sequence of conditions in $F(\alpha)$.

Given any $\alpha < \theta$, if a limit stage i of the construction is such that we have dealt with α (i.e., $\dot{\xi}_j^\alpha$ has been defined) cofinally often below i , then we let $\dot{\xi}_i^\alpha$ be a \mathbb{P}_α -name for the greatest lower bound of $\{\dot{\xi}_j^\alpha \mid j \in I\}$ in $F(\alpha)$ —where I is the cofinal subset of $j \in i$ for which $\dot{\xi}_j^\alpha$ is defined. Since $\dot{\xi}_i^\alpha$ is forced to be the greatest lower bound of $\{\dot{\xi}_j^\alpha \mid j \in I\}$, rather than an arbitrary lower bound of this set, the greatest lower bound of $\{p_j(\alpha) \mid j \in I\}$ is forced to be compatible with $\dot{\xi}_i^\alpha$, and so the construction can keep going. This is the only place where we use the fact that the forcings $F(\alpha)$ are forced to be κ -lattice, rather than just κ -closed or even κ -directed closed.

Let $\langle p^*, s^* \rangle = \langle p_{\mu \cdot \omega}, s_{\mu \cdot \omega} \rangle$. We may—and we do—set up our bookkeeping in such a way that $\langle \bar{p}^*, s^* \cap M \rangle \in M$ is a condition in \mathbb{P} extending $\langle \bar{p}, \bar{s} \rangle$. $\langle \bar{p}^*, s^* \cap M \rangle$ will be our $\pi_M(p', s')$.

Suppose now that $\langle q, t \rangle \in M$ is a condition in \mathbb{P} extending $\langle \bar{p}^*, s^* \cap M \rangle$. It is enough to prove that $\langle q, t \rangle$ is compatible with $\langle p^*, s^* \rangle$. For this, we let $\langle \alpha_i \mid i < \mu^* \rangle$, for some $\mu^* < \kappa$, be the strictly increasing enumeration of $\text{dom}(p^*) \cup \text{dom}(q)$. We may assume for simplicity that μ^* is a limit ordinal. We build a decreasing sequence $\langle \langle q_i, t_i \rangle \mid i \leq \mu^* \rangle$ of \mathbb{P} -conditions such that each $\langle q_{i+1}, t_{i+1} \rangle$ is a condition in $\mathbb{P} \upharpoonright \alpha_i$ extending $\langle p^* \upharpoonright \alpha_i, s^* \rangle$ and $\langle q \upharpoonright \alpha_i, t \rangle$, taking greatest lower bounds at limit stages. The desired common extension of $\langle q, t \rangle$ and $\langle p^*, s^* \rangle$ will be $\langle q_{\mu^*}, t_{\mu^*} \rangle$. At successor stages $i+1$ for which $\alpha_i \in \text{dom}(q) \cap \text{dom}(p^*)$ we apply the fact that $F(\alpha_i)$ is forced to be κ -directed closed to find $p_i(\alpha_i)$ which is forced to extend $q_i(\alpha_i)$ and $p^*(\alpha_i)$ in $F(\alpha_i)$, noting that $p^*(\alpha_i)$ is, by construction, a name forced to be the greatest lower bound of a decreasing sequence in $F(\alpha_i)$ of $|i|$ -many conditions compatible with $q(\alpha_i)$. This completes the proof. \square

To complete the proof of [Theorem 3.1](#) we observe that the following is a corollary from the above (an analogous corollary appears in [\[4\]](#)).

Corollary 3.11. *For every $\alpha \leq \theta$, \mathbb{P}_α is κ -strongly proper.* \square

Finally, by standard reflection arguments using the Laver function and the fact that unboundedly often we choose $\text{Col}(\kappa^+, \alpha)$ and $\text{Add}(\kappa, 1)$ as $F(\alpha)$, we get the following corollary.

Corollary 3.12. *\mathbb{P} forces $\theta = \kappa^{++}$ together with κ -StrPFA.* \square

The following fact is not needed for the proof of [Theorem 3.1](#), but it will be needed for the proofs of [Theorem 5.1](#), [Theorem 6.1](#), and [Theorem 7.1](#).

Claim 3.13. *\mathbb{P} has the θ -c.c.*

Proof. Let $\langle p_\xi, s_\xi \rangle \in \mathbb{P}$ for $\xi < \theta$. By a standard Δ -system argument we may find $I \subseteq \theta$ of size θ , together with $\bar{\theta} < \theta$, a function \bar{p} , and $\bar{s} \in [(\mathcal{S} \cup \mathcal{T}) \cap H(\bar{\theta})]^{< \kappa}$ such that for all $\xi_0 < \xi_1$ in I :

- (1) $\text{dom}(p_{\xi_0}) \cap \text{dom}(p_{\xi_1}) = \text{dom}(\bar{p})$ and $p_{\xi_0} \upharpoonright \bar{\theta} = p_{\xi_1} \upharpoonright \bar{\theta} = \bar{p}$,
- (2) $s_{\xi_0} \cap H(\bar{\theta}) = s_{\xi_1} \cap H(\bar{\theta}) = \bar{s}$, and
- (3) there is $\alpha \in E$ such that $\langle p_{\xi_0}, s_{\xi_0} \rangle \in H(\alpha)$, $p_{\xi_1} \upharpoonright \alpha = \bar{p}$, and $s_{\xi_1} \cap H(\alpha) = \bar{s}$.

Let us fix $\xi_0 < \xi_1$ in I and let $\alpha \in E$ as in (3) for this pair. We may assume that $s_{\xi_1} \setminus H(\alpha) \neq \emptyset$ as otherwise the proof is simpler. If the member of minimal rank in $s_{\xi_1} \setminus H(\alpha)$ is in \mathcal{T} , then $\langle p_{\xi_0} \cup p_{\xi_1}, s_{\xi_0} \cup s_{\xi_1} \rangle$ is a condition in \mathbb{P} extending both $\langle p_{\xi_0}, s_{\xi_0} \rangle$ and $\langle p_{\xi_1}, s_{\xi_1} \rangle$ thanks to the fact that the members of \mathcal{T} form an \in -chain.

Let us now consider the case that the member of minimal rank in $s_{\xi_1} \setminus H(\alpha)$ is a model M in \mathcal{S} . Let $\alpha^* = \min((M \cap \text{Ord}) \setminus \alpha)$. Then $\alpha^* \in E$. To see this, we first note that $\text{cf}(\alpha^*) \geq \kappa^+$. We then note that if $\alpha \notin M$, then α^* is a limit of members of E as $\alpha < \alpha^*$ is above $\text{sup}(M \cap \alpha^*)$, and therefore it is a strong limit.

It now follows that $\langle p_{\xi_0} \cup p_{\xi_1}, s_{\xi_0} \cup \{H(\alpha^*)\} \cup s_{\xi_1} \rangle$ is a condition in \mathbb{P} extending both $\langle p_{\xi_0}, s_{\xi_0} \rangle$ and $\langle p_{\xi_1}, s_{\xi_1} \rangle$ (again using the fact that the members of \mathcal{T} form an \in -chain). \square

4. WEAK REFLECTION OF κ -STRONGLY PROPER FORCINGS

Lemma 4.1 (Weak Reflection Lemma). *Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Suppose that \mathbb{P} is κ -lattice and κ -strongly proper forcing and let $\mathcal{D} = \{D_\alpha \mid \alpha < \kappa^+\}$ be a family of dense open sets. Then there is a κ -lattice and κ -strongly proper forcing $\mathbb{P}^* \subseteq \mathbb{P}$ of size 2^κ and a family of dense subsets of \mathbb{P}^* , $\mathcal{D}^* = \{D_\alpha^* \mid \alpha < \kappa^+\}$, such that there is a \mathcal{D} -generic filter of \mathbb{P} if and only if there is a \mathcal{D}^* -generic filter of \mathbb{P}^* .*

Proof. Let θ be a large enough regular cardinal and let $N \prec H(\theta)$ be a κ^+ -closed elementary submodel such that $\mathbb{P}, \mathcal{D} \in N$ and $|N| = 2^\kappa$. Let $\mathbb{P}^* = \mathbb{P} \cap N$ and $\mathcal{D}^* = \{D_\alpha \cap N \mid \alpha < \kappa^+\}$.

Claim 4.2. \mathbb{P}^* is κ -lattice and κ -strongly proper.

Proof. The fact that \mathbb{P}^* is κ -lattice follows immediately from the closedness of N and elementarity. We now prove that \mathbb{P}^* is κ -strongly proper. Let $\lambda \in H(\theta)$ be a large enough regular cardinal, which exists if we choose θ to be sufficiently large,⁵ and let $M \prec H(\lambda)$ be κ -closed, of cardinality κ , and such that $\mathbb{P} \in M$.

By κ^+ -closedness of N , we get that $M \cap N \in N$, and of course $|M \cap N| = \kappa$ and $M \cap N$ is κ -closed. Also, we may assume that $H(\lambda) \in N$, and therefore $M \cap N$ is an elementary submodel of $H(\lambda)$. Since λ was large enough, by elementarity of N it follows that whenever $p \in \mathbb{P} \cap M \cap N$, there is an extension of p to a strongly $M \cap N$ -generic condition q for \mathbb{P} . By elementarity, we can find such a q in N . But this implies in particular that q is also strongly $M \cap N$ -generic for \mathbb{P}^* (as witnessed by the restriction of the projection function π_M to $\mathbb{P}^* \upharpoonright q$), which of course means that q is strongly M -generic for \mathbb{P}^* . \square

It is now trivial to see that there is a \mathcal{D}^* -generic filter for \mathbb{P}^* if and only if there is a \mathcal{D} -generic filter for \mathbb{P} . \square

The above lemma should be compared with the well-known fact that if \mathbb{P} is a c.c.c. partial order, $\kappa \leq |\mathbb{P}|$, and $\{D_\alpha \mid \alpha < \kappa\}$ is a collection of dense subsets of \mathbb{P} , then there is a c.c.c. suborder \mathbb{Q} of \mathbb{P} such that $|\mathbb{Q}| = \kappa$ and such that $D_\alpha \cap \mathbb{Q}$ is a dense subset of \mathbb{Q} for every $\alpha < \kappa$. This reflection property for c.c.c. forcings is of course what enables one to force MA_κ , for a given infinite cardinal κ , without any large cardinals. As we will soon see, the present weak reflection lemma is one of the two main ingredients that will allow us to force κ -StrPFA without any use of large cardinals.

⁵By which we mean $\theta > 2^{2^\kappa}$, and of course we may assume to have chosen θ this way.

5. PEELING OFF SUPERCOMPACTNESS TO INACCESSIBILITY

Given regular cardinals $\kappa < \theta$, we write S_κ^θ to denote $\{\alpha < \theta \mid \text{cf}(\alpha) = \kappa\}$. We define $S_{>\kappa}^\theta$ similarly.

Theorem 5.1. *Assume GCH holds in V . Suppose that κ is a regular cardinal and $\theta > \kappa$ is an inaccessible cardinal such that $\diamond(S_{>\kappa}^\theta)$ holds. Then there is a κ -lattice and κ -strongly proper forcing \mathbb{P} which forces that $\theta = \kappa^{++} = 2^\kappa$ and that the κ -StrPFA holds.*

Proof. We repeat the same argument as in the proof of [Theorem 3.1](#) with $\mathbb{P} = \mathbb{P}_\theta$ as described in that proof. The main difference is that here we use the diamond sequence to guess the names for our partial orders. To be more precise, we fix a bijection $\varphi: \theta \rightarrow V_\theta$ and a diamond sequence $\langle A_\alpha \mid \alpha \in \theta, \text{cf}(\alpha) > \kappa \rangle$ on $S_{>\kappa}^\theta$, and let $F: S_{>\kappa}^\theta \rightarrow V_\theta$ be the function defined by $F(\alpha) = \varphi \text{``} A_\alpha \subseteq V_\theta$ for each α . We then proceed as before with this function F in place of the Laver function. It is not difficult to see that all relevant claims from [section 3](#) apply to the present construction.

Suppose now that \mathbb{Q} is a κ -lattice and κ -strongly proper forcing in $V[G]$, and \mathcal{D} is a sequence of length κ^+ of dense open sets. By the weak reflection lemma we can reduce \mathbb{Q} to \mathbb{Q}^* of size $\kappa^{++} = \theta = 2^\kappa$. Let $\dot{\mathbb{Q}}^*$ and $\dot{\mathcal{D}}^*$ be \mathbb{P} -names for \mathbb{Q}^* . Since \mathbb{P} has the θ -chain condition ([Claim 3.13](#)), we may assume that both $\dot{\mathbb{Q}}^*$ and $\dot{\mathcal{D}}^*$ are included in V_θ . By the choice of F , there is some large enough α such that $F(\alpha) = \dot{\mathbb{Q}}^* \cap V_\alpha$, and for a large enough χ we can fix $R \prec H(\chi)$ which is κ^+ -closed, $R \cap V_\theta = V_\alpha$, and such that R contains all the relevant objects. The rest of Neeman's argument will be as before, and hence the proof will be complete, provided we can show that \mathbb{P}_α forces $F(\alpha)$ to be κ -lattice and κ -strongly proper.

The fact that $\Vdash_\alpha F(\alpha)$ is κ -lattice is straightforward, using that R is κ^+ -closed: Given $\mu < \kappa$ and a sequence $\sigma = \langle \dot{r}_\alpha \mid \alpha < \mu \rangle$ of \mathbb{P} -names for $\dot{\mathbb{Q}}^*$ -conditions in R , σ is in R , and therefore, by elementarity of R and the fact that $\dot{\mathbb{Q}}^*$ is forced to be κ -lattice, we may fix a \mathbb{P} -name in R for a condition which is forced to be the greatest lower bound of $\{\dot{r}_\alpha \mid \alpha < \mu\}$ provided this set is directed in $\dot{\mathbb{Q}}^*$.

It remains to prove that $F(\alpha)$ is also forced to be κ -strongly proper. For this, let \dot{N} be a \mathbb{P}_α -name for a κ -closed elementary submodel of some large enough $H(\lambda)$ such that $\Vdash_\alpha F(\alpha) \in \dot{N}$ and $|\dot{N}| = \kappa$. We may assume for simplicity that $\lambda \in R$. Let \dot{N}' be a \mathbb{P}_α -name for $\dot{N} \cap R[G_\alpha]$, and let \dot{r} be a \mathbb{P}_α -name for a condition in $F(\alpha) \cap \dot{N}'$. It suffices to show that there is a name \dot{r}^* , of an extension of \dot{r} in $F(\alpha)$, forced to be a strongly \dot{N}' -generic condition for $F(\alpha)$.

The key point⁶ is that \dot{N}' may be identified with a \mathbb{P}_α -name \dot{N}^\dagger for a κ -sequence of ordinals,⁷ and since \mathbb{P}_α is κ -lattice (by [Claim 3.9](#) and the fact $\text{cf}(\alpha) > \kappa$) and κ -strongly proper (by [Corollary 3.11](#)), this means that \dot{N}^\dagger may be taken as a \mathbb{P} -name in a complete suborder $\bar{\mathbb{P}}$ of \mathbb{P}_α isomorphic to $\kappa^{<\kappa}$. But by $\bar{\mathbb{P}} \subseteq R$ and the κ^+ -closedness of R , this means that $\dot{N}^\dagger \in R$ and therefore also $\dot{N}' \in R$, and since $R \cap V_\theta = V_\alpha$, R thinks that \dot{N}' is a \mathbb{P} -name for a relevant model. Since $\dot{\mathbb{Q}}^*$ is a \mathbb{P} -name of a κ -strongly proper forcing, the same holds in R , and therefore \dot{r} can be extended to a condition \dot{r}^* as wanted. \square

6. REDUCING THE CONSISTENCY STRENGTH TO ZFC

The next step is to remove the inaccessible cardinal from our hypotheses, thereby arriving at our first main result.

⁶This may be a key point in the present proof, but it is not needed in general for this type of constructions (see [Remark 7.4](#)).

⁷Working in the \mathbb{P}_α -extension of V , we may fix an ordinal λ_0 for which there is a bijection $\varphi: H(\lambda) \rightarrow \lambda_0$. But then we may identify \dot{N}' with an enumeration in length κ of $\varphi \text{``} \dot{N}'$.

Theorem 6.1. *Assume GCH, and let $\kappa < \kappa^+ < \theta$ be regular cardinals. Then there is a κ -lattice and κ -strongly proper forcing \mathbb{P} which forces $2^\kappa = \kappa^{++} = \theta$ together with κ -StrPFA.*

Since we can start by forcing with $\text{Col}(\kappa^+, < \theta)$, we may as well assume that $\theta = \kappa^{++}$, and that no cardinals are collapsed. More importantly, after this preliminary forcing we may fix a $\diamond(S_{\kappa^+}^\theta)$ -sequence \vec{A} .

The proof of the theorem is the same as in the inaccessible case, but we need to find a substitute for the models $H(\alpha)$ ($= V_\alpha$) from the filtration $\langle V_\alpha \mid \alpha \in E \rangle$ used in the side conditions. For this we simply take a filtration $\vec{N} = \langle N_\alpha \mid \alpha < \theta \rangle$ of $H(\theta)$ into transitive models such that N_α is κ -closed for every α of cofinality κ^+ , which we can do thanks to $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \theta$. We then let $E = S_{\kappa^+}^\theta$. We also require that the models M in \mathcal{S} be such that $M \prec \langle H(\theta), \in, \vec{N}, \vec{A} \rangle$. This way we guarantee that the proof of [Claim 3.13](#) goes through in the present situation.

7. GETTING κ -StrPFA TOGETHER WITH 2^κ LARGE

In this section we generalise [Theorem 6.1](#) by proving that κ -StrPFA is consistent with arbitrarily large values of 2^κ . The theorem is the following.

Theorem 7.1. *Assume GCH, and let $\kappa < \kappa^+ < \theta$ be regular cardinals. Then there is a κ -lattice and κ -strongly proper forcing \mathbb{P} which forces $2^\kappa = \theta$ together with κ -StrPFA.*

Remark 7.2. κ -StrPFA is the first forcing axiom we know of the form $\text{FA}_{\kappa^+}(\Gamma)$, here $\Gamma = \{\mathbb{P} \mid \mathbb{P} \text{ is } \kappa\text{-lattice and } \kappa\text{-strongly proper}\}$, such that $\text{FA}_{\kappa^+}(\Gamma)$ is consistent with 2^κ arbitrarily large whereas $\text{FA}_{\kappa^{++}}(\Gamma)$ is false. To see that $\text{FA}_{\kappa^{++}}(\Gamma)$ is false it suffices to consider the poset \mathbb{P} of $< \kappa$ -sized \in -chains of κ -closed elementary submodels $N \prec H(\kappa^{++})$ such that $|N| = \kappa$. \mathbb{P} is κ -lattice and κ -strongly proper, and an application of $\text{FA}_{\kappa^{++}}(\{\mathbb{P}\})$ would cover κ^{++} by a κ^+ -chain of κ -sized sets.

In order to prove [Theorem 7.1](#) it will be convenient to actually prove a slightly stronger result. Given a cardinal κ , a ground model V_0 , a forcing $\mathbb{P} \in V_0$, and a V_0 -generic filter $H \subseteq \mathbb{P}$ such that $V = V_0[H]$, let us call a forcing notion $\mathbb{Q} \in V$ κ - V_0 - H -strongly proper in the case that for every large enough cardinal θ and every $M \in H(\theta)^{V_0}$, if $M \prec H(\theta)^{V_0}$, $|M|^{V_0} = \kappa$, M is κ -closed in V_0 , $\mathbb{P} \in M$, and $\mathbb{Q} \in M[H]$, then given any $q \in \mathbb{Q} \in M[H]$ there is an extension of q in \mathbb{Q} which is strongly $M[H]$ -generic for \mathbb{Q} .

We note that this is a more general notion than that of κ -strong properness, so that the FA_{κ^+} for the class of κ -lattice posets with this property implies κ -StrPFA.

Throughout the following proof of [Theorem 7.1](#), given a forcing notion \mathbb{Q} , $\dot{G}_{\mathbb{Q}}$ will denote the canonical \mathbb{Q} -name for the generic object.

Proof. We start out by letting $\theta_0 = \kappa^{++}$ and fixing, after forcing with $\text{Col}(\kappa^+, < \theta_0)$ if necessary, a $\diamond(S_{\kappa^+}^{\theta_0})$ -sequence $\vec{A} = \langle A_\alpha \mid \alpha \in S_{\kappa^+}^{\theta_0} \rangle$. Let us call this universe V . Our goal will be to build a κ -lattice and κ -strongly proper poset \mathbb{P} forcing $2^\kappa = \theta$ together with $\text{FA}_{\kappa^+}(\Gamma_{G_{\mathbb{P}}}^\kappa)$, where $\Gamma_{G_{\mathbb{P}}}^\kappa$ denotes the class of κ -lattice forcing notions which are κ - V - $\dot{G}_{\mathbb{P}}$ -strongly proper. By the above observation, \mathbb{P} will then force κ -StrPFA.

As in the proof of [Theorem 6.1](#), we fix a filtration $\vec{N} = \langle N_\alpha \mid \alpha < \theta_0 \rangle$ of $H(\theta_0)$ into transitive models such that N_α is κ -closed for each α with $\text{cf}(\alpha) = \kappa^+$ and let $E = S_{\kappa^+}^{\theta_0}$. We consider a sequence $\langle \mathbb{P}_\beta \mid \beta \in E \cup \{\theta_0\} \rangle$, built very much as in the construction in [section 6](#), except that at each stage α we look at whether A_α codes, not a \mathbb{P}_α -name for a relevant forcing, but a $\mathbb{P}_\alpha \times \text{Add}(\kappa, \kappa^+)$ -name for a forcing

which is κ -lattice and κ - V - $\dot{G}_{\mathbb{P}_\alpha \times \text{Add}(\kappa, \kappa^+)}$ -strongly proper (and if so, then working parts at α are conditions in this forcing).

Our forcing \mathbb{P} witnessing [Theorem 7.1](#) will now be $\mathbb{P}_{\theta_0} \times \text{Add}(\kappa, \theta)$. It is clear that \mathbb{P} forces $2^\kappa = \theta$ and has the κ^{++} -c.c. The proof that \mathbb{P} forces $\text{FA}_{\kappa^+}(\Gamma_{G_{\mathbb{P}}^\kappa})$ is along the lines of the corresponding proof for [Theorem 6.1](#). Specifically, suppose \dot{Q}^* is a \mathbb{P} -name for a κ -lattice κ - V - $\dot{G}_{\mathbb{P}}$ -strongly proper forcing, and for $\alpha < \kappa^+$, \dot{D}_α is a \mathbb{P} -name for a dense subset of \dot{Q}^* .

Claim 7.3. *There is a \mathbb{P} -name, \dot{Q} , for a κ -lattice κ - V - $\dot{G}_{\mathbb{P}}$ -strongly proper suborder of \dot{Q}^* of size κ^{++} such that $\dot{D}_\alpha \cap \dot{Q}$ is dense in \dot{Q} for each α .*

Proof. For this, let G_0 be V -generic for \mathbb{P}_{θ_0} and let us work in $W = V[G_0]$. Let \dot{R} be an $\text{Add}(\kappa, \theta)$ -name for \dot{Q}^* . As in the proof of [Lemma 4.1](#), let $N \in V$ be a κ^+ -closed (in V) elementary submodel of size $2^\kappa = \kappa^{++}$ containing everything relevant (including $\text{Add}(\kappa, \theta)$ -names for \dot{D}_α for each $\alpha < \kappa^+$). In particular, $\mathbb{P}_{\theta_0} \subseteq N$ and $N[G_0]$ is therefore κ^+ -closed in W . Let G be W -generic for $\text{Add}(\kappa, \theta)$. We claim that it will suffice to take a name, \dot{Q} , for $\mathbb{Q} = \dot{R}^G \cap N[G_0][G]$. We also let H be a V -generic filter for \mathbb{P} such that $W[G] = V[H]$, and we will use \mathbb{Q}^* to denote $(\dot{Q}^*)^H = \dot{R}^G$.

Using the κ -closedness of $\text{Add}(\kappa, \theta)$ and the κ^+ -closedness of $N[G_0]$ in W it is easy to see that \mathbb{Q} is κ -lattice. To see that it is κ - V - H -strongly proper, let $M \in V$, $M \prec H(\lambda)^V$, for large enough $\lambda \in N$, be κ -closed and of size κ in V , and such that $\mathbb{Q} \in M[H]$. Given $q \in \mathbb{Q} \cap M[H]$, we need to produce a strongly M -generic condition for \mathbb{Q} extending q . As in the proof of [Lemma 4.1](#), we use the closedness of N in W under κ -sequences and get that $M \cap N \in N$ is, in V , κ -closed and of size κ . We then finish as in that proof, noting that any strongly $(M \cap N)[G_0][G]$ -generic condition for \mathbb{Q}^* in $N[G_0][G]$ is a strongly $M[H]$ -condition for \mathbb{Q} . \square

By κ^{++} -c.c. of \mathbb{P} , we may identify \dot{Q} with a $\mathbb{P}_{\theta_0} \times \text{Add}(\kappa, \kappa^{++})$ -name, which we may code by a subset of κ^{++} . Now we use our diamond \vec{A} to capture \dot{Q} as in the proof of [Theorem 6.1](#).⁸ \square

Remark 7.4. [Proposition 2.3](#), i.e., the fact that κ -sequences of ordinals in generic extensions by κ -lattice κ -strongly proper forcings belong to κ -Cohen extensions, is not needed in the proof of [Theorem 7.1](#). This is thanks to the fact that at a stage $\alpha \in E$ in the construction, the models M for which we need to prove strong properness of the relevant forcing come in fact from V .

It is worth pointing out—and follows from a well-known result of Paul Larson—that if MM^{++} holds and we let $\kappa = \omega_2$, then the construction in [Theorem 6.1](#) (and [Theorem 7.1](#)) preserves MM^{++} and so forces ω_2 -StrPFA “on top” of this forcing axiom. And the same thing is of course also true for natural weaker forcing axioms like MM, PFA, and so on.

8. RELAXING STRONGNESS OR G.L.B.’S?

Let Γ_κ be the class of κ -lattice κ -strongly proper posets. As we have seen, while κ -StrPFA, i.e. $\text{FA}_{\kappa^+}(\Gamma_\kappa)$, is consistent with ZFC, it is too weak to decide the size of 2^κ . In fact, this forcing axiom does not seem to have many applications. It does imply certain weak failures of Club Guessing at κ^+ (as pointed out in the introduction), as well as $\mathfrak{d}(\kappa) > \kappa^+$ and the covering number of natural meagre ideals being greater than κ^+ , but we do not know of any other quotable consequences. In this

⁸When the capturing happens at a stage $\alpha \in E$, we have that A_α codes a $\mathbb{P}_\alpha \times \text{Add}(\kappa, \alpha)$ -name, which we can of course identify with a $\mathbb{P}_\alpha \times \text{Add}(\kappa, \kappa^+)$ -name.

final section we address the prospect of (mildly) relaxing some of the constraints in the definition of Γ_κ so as to obtain more powerful forcing axioms.

For the rest of this section, let us fix a regular cardinal $\kappa \geq \omega$ such that $\kappa^{<\kappa} = \kappa$.

Given a model M and a set $X \in M$, let us call $S \subseteq [X]^\kappa$ an M -stationary subset of $[X]^\kappa$ if for every function $F: [X]^{<\omega} \rightarrow X$ with $F \in M$ there is some $N \in S \cap M$ such that $F''[N]^{<\omega} \subseteq N$. This is the natural extension in the $[X]^\kappa$ context of the notion, due to Moore, of M -stationarity for collections of countable sets (see [3]). Also, let us define the κ -Ellentuck topology on $[X]^\kappa$ by declaring basic open sets to be of the form $[s, Y] = \{Z \in [Y]^\kappa \mid s \subseteq Z\}$ for $Y \in [X]^\kappa$ and $s \in [Y]^{<\kappa}$. We will next generalise Moore's Mapping Reflection Principle (MRP) to the present context.

Definition 8.1. κ -MRP is the following statement: Let X be a set, let θ be a cardinal such that $X \in H(\theta)$, and let Σ be a function defined on a club of $[H(\theta)]^\kappa$ and such that for every $M \in \text{dom}(\Sigma)$, $\Sigma(M)$ is both an M -stationary subset of $[X]^\kappa$ and an open subset of $[X]^\kappa$ in the κ -Ellentuck topology. Then there is a \subseteq -continuous \in -chain $\langle M_\alpha \mid \alpha < \kappa^+ \rangle$ of elementary submodels of $H(\theta)$ of size κ such that for each $\alpha < \kappa^+$ of cofinality κ , M_α is κ -closed and there is some $\bar{\alpha} < \alpha$ such that $M_\beta \cap X \in \Sigma(M_\alpha)$ for all $\beta \in [\bar{\alpha}, \alpha)$.

Thus, MRP is just ω -MRP. The following fact can be proved by a generalisation of the argument showing that MRP implies the existence of a well-order of $\mathcal{P}(\omega_1)$ of length ω_2 Σ_1 -definable over $H(\omega_2)$ from any given ladder system on ω_1 and any given ω_1 -sequence of pairwise disjoint stationary subsets of ω_1 .

Fact 8.2. κ -MRP implies that $2^{\kappa^+} = \kappa^{++}$. In fact, a stronger statement is true. Given a club-sequence $\vec{C} = \langle C_\alpha \mid \alpha \in S_\kappa^{\kappa^+} \rangle$ and a sequence $\vec{S} = \langle S_\xi \mid \xi < \kappa^+ \rangle$ of pairwise disjoint stationary subsets of $S_\kappa^{\kappa^+}$, κ -MRP implies that there is a well-order of $\mathcal{P}(\kappa^+)$ of length κ^{++} which is Σ_1 -definable over $H(\kappa^{++})$ from \vec{C} and \vec{S} as parameters.

Definition 8.3. A forcing \mathbb{P} is κ -MRP-strongly proper if for every large enough θ , every κ -closed $M \prec H(\theta)$ of size κ such that $\mathbb{P} \in M$, and every $p \in \mathbb{P} \cap M$ there is $q \leq p$ such that for every $q' \leq_{\mathbb{P}} q$,

$$\mathcal{X}_{q'} = \{X \in [\mathbb{P} \cap M]^\kappa \mid \exists \pi_X(q') \in \mathbb{P} \cap X \forall r \leq_{\mathbb{P}} \pi_X(q'), r \in X \rightarrow r \parallel q'\}$$

is an M -stationary subset of $[\mathbb{P}]^\kappa$.⁹

There is a natural forcing which, for a given open and stationary mapping Σ as in the original statement of MRP, adds by finite approximations a reflecting sequence for Σ . An immediate generalisation of the proof that PFA implies MRP using such forcings yields the following.¹⁰

Fact 8.4. $\text{FA}_{\kappa^+}(\{\mathbb{P} \mid \mathbb{P} \text{ is } \kappa\text{-lattice and } \kappa\text{-MRP-strongly proper}\})$ implies κ -MRP.

The bad news is that, as [Theorem 8.5](#) shows, this forcing axiom is inconsistent when κ is uncountable.

Theorem 8.5. Suppose $\kappa \geq \omega_1$ is such that $\kappa^{<\kappa} = \kappa$. Then

$$\text{FA}_{\kappa^+}(\{\mathbb{P} \mid \mathbb{P} \text{ is } \kappa\text{-lattice and } \kappa\text{-MRP-strongly proper}\})$$

is false.

⁹The notation $r \parallel q'$ means that r is compatible with q' in \mathbb{P} , that is, they have a joint extension.

¹⁰The relevant forcing this time is a natural one for adding a suitable reflecting sequence by $<\kappa$ -sized approximations.

When κ is a successor cardinal, one can prove this inconsistency using the following theorem of Shelah (see [6], Appendix Chap. 3).

Theorem 8.6 (Shelah). *Let $\kappa \geq \omega_1$ be a regular cardinal and let $\langle C_\alpha \mid \alpha \in S_\kappa^{\kappa^+} \rangle$ be a club-sequence with $\text{ot}(C_\alpha) = \kappa$ for all $\alpha \in S_\kappa^{\kappa^+}$. Then there is a sequence $\langle f_\alpha \mid \alpha \in S_\kappa^{\kappa^+} \rangle$ of colourings, with $f_\alpha: C_\alpha \rightarrow \{0, 1\}$ for all α , for which there is no function $G: \kappa^+ \rightarrow 2$ such that for all $\alpha \in S_\kappa^{\kappa^+}$, $G(\xi) = f_\alpha(\xi)$ for club-many $\xi \in C_\alpha$.*

The strategy in this case is to consider $\langle C_\alpha \mid \alpha \in S_\kappa^{\kappa^+} \rangle$ and $\langle f_\alpha \mid \alpha \in S_\kappa^{\kappa^+} \rangle$ as in Theorem 8.6 and to apply the forcing axiom to a natural forcing for adding by $< \kappa$ -sized approximations a regressive function p on $S_\kappa^{\kappa^+}$ such that

- (1) for all $\alpha \in \text{dom}(p)$, $p(\alpha) < \alpha$, and
- (2) for all $\alpha_0 < \alpha_1$, if $\xi \in (C_{\alpha_0} \setminus p(\alpha_0)) \cap (C_{\alpha_1} \setminus p(\alpha_1))$, then $f_{\alpha_0}(\xi) = f_{\alpha_1}(\xi)$.

We then have that $\langle f_\alpha \mid \alpha \in S_\kappa^{\kappa^+} \rangle$ can be uniformised, in fact modulo co-bounded sets, which is a contradiction.

Remark 8.7. \mathbb{P} is also κ^+ -c.c., so this shows the failure of

$$\text{FA}_{\kappa^+}(\{\mathbb{P} \mid \mathbb{P} \text{ is } \kappa\text{-lattice, } \kappa^+\text{-c.c., and } \kappa\text{-MRP-strongly proper}\})$$

when κ is a successor cardinal such that $\kappa^{<\kappa} = \kappa$.

On the other hand, Theorem 8.6 does not seem to be available when κ is inaccessible. We will now give a proof of Theorem 8.5 covering all cases. This proof uses the following result of Shelah.

Theorem 8.8 (Shelah, Claim 3.3 in [5]). *For every uncountable regular cardinal κ there is a club-sequence $\langle C_\alpha \mid \alpha \in S_\kappa^{\kappa^+} \rangle$ such that for every club D of κ^+ there is some $\alpha \in D$ with $\{\zeta < \kappa \mid C_\alpha(\zeta + 1) \in D\}$ stationary (where $\langle C_\alpha(\zeta) \mid \zeta < \kappa \rangle$ is the strictly increasing enumeration of C_α).*

We are now ready to give the proof of Theorem 8.5 in the general case.

Proof. Given $\vec{C} = \langle C_\alpha \mid \alpha \in S_\kappa^{\kappa^+} \rangle$ as in Theorem 8.8, let \mathbb{P} be the following forcing: Conditions in \mathbb{P} are pairs $\langle \mathcal{I}, b \rangle$ such that:

- (1) \mathcal{I} is a collection of $< \kappa$ -many pairwise disjoint intervals of the form $[\alpha, \beta]$ with $\alpha \leq \beta < \kappa^+$,
- (2) b is a regressive function with $\text{dom}(b) \subseteq \{\min(I) \mid I \in \mathcal{I}\} \cap S_\kappa^{\kappa^+}$,
- (3) for each $\alpha \in \text{dom}(b)$, $\{\min(I) \mid I \in \mathcal{I}\} \cap \{C_\alpha(\zeta + 1) \mid \zeta < \kappa\} \cap (b(\alpha), \alpha) = \emptyset$, and
- (4) for each $\alpha \in \text{dom}(b)$ and each $I \in \mathcal{I}$, if $b(\alpha) < \min(I) < \alpha$ and $I' \in \mathcal{I}$ is such that $\min(I) < \min(I') < \alpha$, then $\min(I) < C_\alpha(\zeta) < \min(I')$ for some ζ .

$\langle \mathcal{I}_1, b_1 \rangle \leq_{\mathbb{P}} \langle \mathcal{I}_0, b_0 \rangle$ if

- (1) for every $I \in \mathcal{I}_0$ there is $I' \in \mathcal{I}_1$ with $\min(I') = \min(I)$ and $\max(I) \leq \max(I')$ and
- (2) $b_0 \subseteq b_1$.

Then \mathbb{P} belongs to the relevant class and adds a club of κ^+ violating the club-guessing property of \vec{C} . \square

Question 8.9. Suppose κ is an inaccessible cardinal. Does it necessarily follow that $\text{FA}_{\kappa^+}(\{\mathbb{P} \mid \mathbb{P} \text{ is } \kappa\text{-lattice, } \kappa^+\text{-c.c., and } \kappa\text{-MRP-strongly proper}\})$ fails?

Next we will show that the restriction to κ -strongly proper forcing which, in addition, are κ -lattice is not just a technical artefact of our consistency proofs but is in fact a necessary restriction. This result is essentially due to Shelah (see Appendix Chap. 3, 3.4 in [6]). We include the proof for the reader's convenience.

Theorem 8.10 (Shelah). *Let κ be a successor cardinal. Then*

$$\text{FA}_{\kappa^+}(\{\mathbb{P} \mid \mathbb{P} \text{ is } \kappa\text{-directed closed, } \kappa^+\text{-c.c., and } \kappa\text{-strongly proper}\})$$

is false.

Proof. This is similar to the proof on [Theorem 8.5](#) for the case when κ is a successor cardinal. Let $\langle C_\alpha \mid \alpha \in S_\kappa^+ \rangle$ be a club-sequence with $\text{ot}(C_\alpha) = \kappa$ for all $\alpha \in S_\kappa^+$ and let $\langle f_\alpha \mid \alpha \in S_\kappa^+ \rangle$ be a sequence of colourings, where $f_\alpha: C_\alpha \rightarrow \{0, 1\}$ for all α , which cannot be club-uniformised in the sense of [Theorem 8.6](#). As in the proof of [Theorem 8.5](#) in the successor cardinal case, we will produce a forcing notion \mathbb{P} adding a uniformizing function and belonging to the relevant class. An application of the forcing axiom to \mathbb{P} yields then a contradiction.

Conditions in \mathbb{P} are pairs $p = \langle a_p, \vec{d}_p \rangle$, where

- (1) $a_p \in [S_\kappa^+]^{<\kappa}$;
- (2) $\vec{d}_p = \langle d_p^\alpha \mid \alpha \in a_p \rangle$ is such that, for some successor ordinal $i_p + 1 < \kappa$, $d_p^\alpha: i_p + 1 \rightarrow C_\alpha$ is a strictly increasing and continuous function;
- (3) for all $\alpha_0, \alpha_1 \in a_p$ and for all $\xi \in \text{range}(d_p^{\alpha_0}) \cap \text{range}(d_p^{\alpha_1})$, $f_{\alpha_0}(\xi) = f_{\alpha_1}(\xi)$.

The extension relation \leq on \mathbb{P} is defined by letting $q \leq p$ exactly when

- (1) $a_p \subseteq a_q$,
- (2) for every $\alpha \in a_p$, d_p^α is an initial segment of d_q^α , and
- (3) if $i_p < i_q$, then $(C_{\alpha_0} \cap C_{\alpha_1}) \setminus \min\{d_q^{\alpha_0}(i_q), d_q^{\alpha_1}(i_q)\} = \emptyset$ for all $\alpha_0 \neq \alpha_1 \in a_p$.

Using $\kappa^{<\kappa} = \kappa$ it is straightforward to verify that \mathbb{P} is κ -directed closed, κ^+ -c.c., and κ -strongly proper. Hence, an application of the forcing axiom to \mathbb{P} yields a club-uniformising function for $\langle f_\alpha \mid \alpha \in S_\kappa^+ \rangle$, which is a contradiction. On the other hand, it is not difficult to see that \mathbb{P} is not κ -lattice; in fact, one can easily find compatible conditions $p_0, p_1 \in \mathbb{P}$ which do not have a greatest lower bound. \square

The following is now a natural question.

Question 8.11. Suppose κ is an inaccessible cardinal. Does it necessarily follow that $\text{FA}_{\kappa^+}(\{\mathbb{P} \mid \mathbb{P} \text{ is } \kappa\text{-directed closed and } \kappa\text{-strongly proper}\})$ fails?

We will finish with the following question.

Question 8.12. Is it consistent that there is any uncountable regular κ for which κ -MRP holds? More generally, and in view of [Fact 8.2](#), is there any Π_2 sentence σ with the following properties?

- (1) ZFC proves that if $\kappa \geq \omega_1$ is a regular cardinal and $H(\kappa^+) \models \sigma$ holds, then $2^\kappa = \kappa^+$.
- (2) Some reasonable extension of ZFC proves that one can force the existence of a regular cardinal $\kappa \geq \omega_1$ such that $H(\kappa^+) \models \sigma$.

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