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On simple partitions of $[\kappa]^\kappa$

by

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Abstract. For every uncountable regular cardinal κ, every κ-Borel partition of the space of all members of $[\kappa]^\kappa$ whose enumerating function does not have fixed points has a homogeneous club.

1. Notation. Let $\langle L, <\rangle$ be a linearly ordered set. Given x, $y \in L$, let $(x,y) = \{z \in L : x < z < y\}$. We also define $[x,y)$, $(\leftarrow, y]$, $[x,y]$ and so on in the natural way. An open interval of L is any subset of L of the form (x,y), $(\leftarrow, y)$ or $(x, \rightarrow)$ for some $x, y \in L$. The collection of all unions of open intervals of L is a topology on L if L has at least two points. It is called the *order topology* on L.

We shall be interested in spaces constructed in a canonical way from some ordinal endowed with the order topology derived from the ordinal ordering. Given two ordinals α and β, the *product topology on* α^β is obtained by giving α the order topology and α^β the corresponding product topology. Notice that given any topological space X and any two ordinals β and γ, if β is infinite and $1 \leq \gamma < |\beta|^+$, then X^β and $(X^\beta)^\gamma$ are homeomorphic spaces (when regarded as carrying the corresponding product topologies).

For any set A of ordinals let $\widetilde{A}$ denote the strictly increasing enumeration of the elements of A. Also, whenever α and ξ are ordinals, κ is a cardinal and $X \subseteq \alpha$, X^κ_ξ denotes either the set of all $A \in [\alpha]^\kappa$ such that $\widetilde{A}(\xi) \in X$ or the set of all $A \in \alpha^\kappa$ such that $A(\xi) \in X$. If $\alpha = \kappa$, then we may drop the superscript κ.

Given an ordinal α and a cardinal κ such that $\kappa \leq |\alpha|$, the topology on $[\alpha]^\kappa$ obtained by identifying each $A \in [\alpha]^\kappa$ with $\widetilde{A}$ inside $\alpha^{<\kappa^+}$, where α^β is given the product topology for each $\beta < \kappa^+$ and $\alpha^{<\kappa^+}$ is given the sum topology, will be called the *product topology on* $[\alpha]^\kappa$. With this topology

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$[\alpha]^\kappa$ is homeomorphic, in a canonical way, to the subspace of $\alpha^{<\kappa^+}$ consisting of all strictly increasing functions of size κ. Notice that the set of all $\bigcap_{j<n}[\alpha_j,\beta_j)^\kappa_{\xi_j}$, where $1 \le n < \omega$ and for all $j < n$, $\alpha_j < \beta_j < \alpha$, $\xi_j < \kappa^+$ ($\xi_j < \kappa$ if $\kappa = \alpha$) and α_j is either 0 or a successor ordinal, is a basis for the product topology on $[\alpha]^\kappa$.

Also, if we give α the discrete topology, α^β the corresponding product topology for each $\beta < \kappa^+$, and $\alpha^{<\kappa^+}$ the sum topology, then the topology on $[\alpha]^\kappa$ obtained by identifying each $A \in [\alpha]^\kappa$ with $\widetilde{A}$ inside α^{κ^+} will be called the *product topology on* $[\alpha]^\kappa$ *corresponding to the discrete topology* and has as a basis the collection of all $\bigcap_{j<n}\{\alpha_j\}^\kappa_{\xi_j}$, where $1 \le n < \omega$ and $\xi_j < \kappa^+$, $\alpha_j < \alpha$ for all j.

Of course both topologies above coincide on $[\omega]^{\aleph_0}$. Unless otherwise specified, if α and β are ordinals, α^β (and also $[\alpha]^\beta$ if β is a cardinal) will be assumed to carry the product topology.

Recall that a set $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ is said to be *Ramsey* if there is some $X \in [\omega]^{\aleph_0}$ such that either $[X]^{\aleph_0} \subseteq \mathcal{A}$ or $[X]^{\aleph_0} \cap \mathcal{A} = \emptyset$. Of course it is a theorem of ZFC that there is a non-Ramsey subset of $[\omega]^{\aleph_0}$. However, it is also well known that all sufficiently simple subsets of $[\omega]^{\aleph_0}$ are (perhaps under some additional hypothesis) Ramsey. For example, Galvin and Prikry ([G-P]) proved that all Borel sets are Ramsey, and then Silver ([S]) extended this result to all analytic sets and also proved that even all $\utilde{\Sigma}^1_2$ sets are Ramsey if $\omega_1^{L[a]}$ is countable for every real a. We shall be interested in proving or disproving Ramsey-type regularity properties of subsets of $[\kappa]^\kappa$ according to their complexity. It seems reasonable to use, as a measure of the complexity of a subset A of a topological space, the first place (when available) in some given Borel hierarchy at which A occurs.

Definition 1.1. Let X be a topological space and let μ be an infinite cardinal. $\mathcal{B}_\mu(X)$, the algebra of μ-*Borel subsets* of X, is the μ-subalgebra of $\mathcal{P}(X)$ generated by the open subsets of X. That is, $\mathcal{B}_\mu(X)$ is the $\subseteq$-minimal collection $\mathcal{S}$ of subsets of X such that

(1) every open subset of X belongs to $\mathcal{S}$,

(2) if $A \in \mathcal{S}$, then $X \backslash A \in \mathcal{S}$,

(3) if $\lambda < \mu$ and $\langle A_i : i < \lambda \rangle$ is a sequence of elements of $\mathcal{S}$, then $\bigcup_{i<\lambda} A_i \in \mathcal{S}$.

Definition 1.2. Let X be a topological space and let μ be an infinite cardinal. We define the collections $\utilde{\Sigma}^0_{\xi,\mu}(X)$, $\utilde{\Pi}^0_{\xi,\mu}(X)$ and $\utilde{\Delta}^0_{\xi,\mu}(X)$ of subsets of X in the following way.

(a) $A \in \utilde{\Sigma}^0_{1,\mu}(X)$ iff A is an open subset of X,

and, for every ordinal $\xi \ge 1$,

(b) $A \in \underset{\sim}{\Pi}^0_{\xi,\mu}(X)$ iff $X \setminus A \in \underset{\sim}{\Sigma}^0_{\xi,\mu}(X)$,

(c) if $\xi > 1$, then $A \in \underset{\sim}{\Sigma}^0_{\xi,\mu}(X)$ if and only if there is $\lambda < \mu$ and a sequence $\langle A_i : i < \lambda\rangle$ of elements of $\bigcup_{\zeta<\xi} \underset{\sim}{\Pi}^0_{\zeta,\mu}(X)$ such that $A = \bigcup_{i<\lambda} A_i$,

(d) $\underset{\sim}{\Delta}^0_{\xi,\mu}(X) = \underset{\sim}{\Sigma}^0_{\xi,\mu}(X) \cap \underset{\sim}{\Pi}^0_{\xi,\mu}(X)$.

Of course, for every topological space X and $A \subseteq X$, A is a Borel subset of X iff A is an ω_1-Borel subset of X, and for every $1 \le \xi < \omega_1$, A belongs to $\underset{\sim}{\Sigma}^0_\xi(X)$ (resp. $\underset{\sim}{\Pi}^0_\xi(X)$, $\underset{\sim}{\Delta}^0_\xi(X)$) iff A belongs to $\underset{\sim}{\Sigma}^0_{\xi,\omega_1}(X)$ (resp. $\underset{\sim}{\Pi}^0_{\xi,\omega_1}(X)$, $\underset{\sim}{\Delta}^0_{\xi,\omega_1}(X)$).

Remark 1.1. Let X and Y be topological spaces, let $\mu < \mu'$ be infinite cardinals and let $1 \le \xi < \xi'$ be ordinals.

(i) $\underset{\sim}{\Sigma}^0_{\xi,\mu}(X) \subseteq \underset{\sim}{\Sigma}^0_{\xi,\mu'}(X)$, and similarly for $\underset{\sim}{\Pi}$ and $\underset{\sim}{\Delta}$.

(ii) $\underset{\sim}{\Sigma}^0_{\xi,\mu}(X) \subseteq \underset{\sim}{\Pi}^0_{\xi',\mu}$ and $\underset{\sim}{\Pi}^0_{\xi,\mu}(X) \subseteq \underset{\sim}{\Sigma}^0_{\xi',\mu}$.

(iii) If $\xi \ge 2$, then $\underset{\sim}{\Sigma}^0_{\xi,\mu}(X) \cup \underset{\sim}{\Pi}^0_{\xi,\mu}(X) \subseteq \underset{\sim}{\Delta}^0_{\xi',\mu}$.

(iv) If $f : X \to Y$ is a continuous mapping and A belongs to $\underset{\sim}{\Sigma}^0_{\xi,\mu}(Y)$, then $f^{-1}(A)$ belongs to $\underset{\sim}{\Sigma}^0_{\xi,\mu}(X)$, and similarly for $\underset{\sim}{\Pi}$ and $\underset{\sim}{\Delta}$.

(v) $\underset{\sim}{\Sigma}^0_{\xi,\mu}(X)$ (resp. $\underset{\sim}{\Pi}^0_{\xi,\mu}(X)$) is closed under unions (resp. intersections) of families of less than $\mathrm{cf}(\mu)$ elements. $\underset{\sim}{\Sigma}^0_{n,\mu}(X)$ and $\underset{\sim}{\Pi}^0_{n,\mu}(X)$ are closed under finite unions and finite intersections for $n \le 2$, and if every closed set of X is in $\underset{\sim}{\Pi}^0_{2,\mu}(X)$, then $\underset{\sim}{\Sigma}^0_{\xi,\mu}(X)$ and $\underset{\sim}{\Pi}^0_{\xi,\mu}(X)$ are also closed under finite unions and finite intersections.

The extra hypothesis in (v) above does not always hold. For example, if we let X be $\omega_1 + 1$ with the order topology, then $A = \{\omega_1\}$ is a closed subset of X but it is not in $\underset{\sim}{\Pi}^0_{2,\omega_1}(X)$. This example was given by the referee of this paper.

Fact 1.2. *Let X be a topological space and let μ be an infinite regular cardinal. Then* $\mathcal{B}_\mu(X) = \bigcup_{\xi<\mu} \underset{\sim}{\Sigma}^0_{\xi,\mu}(X) = \bigcup_{\xi<\mu} \underset{\sim}{\Pi}^0_{\xi,\mu}(X)$.

Remark 1.3. Note that given a cardinal κ, $[\kappa]^\kappa$ is homeomorphic to a $\underset{\sim}{\Pi}^0_{2,\kappa^+}$ subset of κ^κ, both $[\kappa]^\kappa$ and κ^κ viewed as carrying the corresponding product topologies. To see this, note that the set of strictly increasing functions from κ into κ can be written as $\bigcap_{\xi<\xi'<\kappa} \bigcup_{\gamma<\kappa} ([0,\gamma+1)_\xi \cap (\gamma,\kappa)_{\xi'})$.

If we want to extend our classification of subsets of a topological space beyond a given Borel hierarchy, a natural thing to do is to consider some generalization of the projective hierarchy of Polish spaces. The following looks like a natural frame for these generalizations.

Definition 1.3. Let Γ be a class of topological spaces and let X be a topological space. We say that Γ is *X-cartesian closed* if $X \in \Gamma$ and $Y \times X \in \Gamma$ for each $Y \in \Gamma$.

Given a family $\{X_i : i < \alpha\}$ of sets and $i_0 < \alpha$, π_{i_0} denotes the canonical projection of $\prod_{i<\alpha} X_i$ onto X_{i_0}.

Definition 1.4. Let X be a topological space. We define, by induction on $n < \omega$, the classes $\utilde{\Sigma}^1_n(Y, X)$, $\utilde{\Pi}^1_n(Y, X)$ and $\utilde{\Delta}^1_n(Y, X)$ for all topological spaces Y.

Suppose Y is a topological space and $A \subseteq Y$. Then

(a) $A \in \utilde{\Sigma}^1_0(Y, X)$ iff A is an open subset of Y,

and, for every $n < \omega$,

(b) $A \in \utilde{\Pi}^1_n(Y, X)$ iff $Y \setminus A \in \utilde{\Sigma}^1_n(Y, X)$,
(c) $A \in \utilde{\Sigma}^1_{n+1}(Y, X)$ iff $A = \pi_0$“B for some $B \in \utilde{\Pi}^1_n(Y \times X, X)$,
(d) $\utilde{\Delta}^1_n(Y, X) = \utilde{\Sigma}^1_n(Y, X) \cap \utilde{\Pi}^1_n(Y, X)$.

$A \subseteq Y$ is a *projective subset of* Y *relative to* X if $A \in \bigcup_n [\utilde{\Sigma}^1_n(Y, X) \cup \utilde{\Pi}^1_n(Y, X)]$.

If Γ is an X-cartesian closed class of topological spaces, the *projective hierarchy on* Γ *relative to* X consists of all projective subsets of Y relative to X for all $Y \in \Gamma$. Also, for every natural number n we define $\utilde{\Sigma}^1_n(\Gamma, X) = \bigcup\{\utilde{\Sigma}^1_n(Y, X) : Y \in \Gamma\}$ and $\utilde{\Pi}^1_n(\Gamma, X) = \bigcup\{\utilde{\Pi}^1_n(Y, X) : Y \in \Gamma\}$.

There are several general facts on the usual projective hierarchy for Polish spaces whose proof can be easily adapted to work in the context of the present abstract notion of projective hierarchy. Here are some examples.

Fact 1.4. *For every topological space* X, *every* X*-cartesian closed class* Γ *of topological spaces and every natural number* n, $\utilde{\Sigma}^1_n(\Gamma, X)$ *and* $\utilde{\Pi}^1_n(\Gamma, X)$ *are closed under continuous preimages*, *in the sense that whenever* $Y, Z \in \Gamma$, $f : Y \to Z$ *is a continuous function and* $A \in \utilde{\Sigma}^1_n(Z, X)$ (*resp.* $\in \utilde{\Pi}^1_n(Z, X)$), *then* $f^{-1}(A) \in \utilde{\Sigma}^1_n(Y, X)$ (*resp.* $\in \utilde{\Pi}^1_n(Y, X)$).

Fact 1.5. *Let* X *and* Y *be topological spaces and suppose there are* $\mathcal{B}$ *and* $\mathcal{P}$ *of the same size such that* $\mathcal{B}$ *is a basis for* Y *consisting of clopen sets and* $\mathcal{P}$ *is a partition of* X *into nonempty clopen sets. Then* $\utilde{\Sigma}^1_0(Y, X) \cup \utilde{\Pi}^1_0(Y, X) \subseteq \utilde{\Delta}^1_1(Y, X)$.

Corollary 1.6. *Suppose* X *and* Y *are topological spaces and suppose there are* $\mathcal{B}$, $\mathcal{C}$ *and* $\mathcal{P}$ *all of the same size such that* $\mathcal{B}$ *and* $\mathcal{C}$ *are*, *respectively*, *bases of* X *and* Y *consisting of clopen sets and* $\mathcal{P}$ *is a partition of* X *into nonempty clopen sets. Then* $\utilde{\Sigma}^1_n(Y, X) \cup \utilde{\Pi}^1_n(Y, X) \subseteq \utilde{\Delta}^1_{n+1}(Y, X)$ *for every* $n < \omega$.

Fact 1.7. *Let* κ *be an infinite cardinal*, *let* Y *be a topological space and suppose* κ *is endowed with a topology so that there is a closed set* $J \subseteq \kappa$ *of size* κ *which is a union of clopen singletons. Then*, *giving* κ^κ *the correspond-*

ing product topology, $\underset{\sim}{\Sigma}^1_n(Y,\kappa^\kappa)$ *and* $\underset{\sim}{\Pi}^1_n(Y,\kappa^\kappa)$ *are closed under unions and intersections of families of at most* κ *elements for all* $n\ge 1$.

REMARK 1.8. Suppose Y, κ and $X=\kappa^\kappa$ are such that the conclusions of Facts 1.5 and 1.7 hold for X, Y and κ. Then $\mathcal{B}_{\kappa^+}(Y)\subseteq \underset{\sim}{\Sigma}^1_1(Y,X)$.

On the other hand, as the referee has shown with the example I am about to quote, there are elementary properties of the projective hierarchy for Polish spaces which do not hold in general for the projective hierarchy on the class of all $[\kappa]^\kappa\times \overset{j}{\ldots}\times[\kappa]^\kappa$ relative to $[\kappa]^\kappa$ ($[\kappa]^\kappa$ given the product topology).

FACT 1.9 (Referee). *Let* $\kappa\ge\omega_1$ *be a regular cardinal and let* $X=Y=[\kappa]^\kappa$ *with the product topology. Then the set of all those members of* Y *whose first element is a successor ordinal is an open subset of* Y, *yet* $A\notin \underset{\sim}{\Sigma}^1_1(Y,X)$.

2. Simple partitions. We are interested in finding large homogeneous sets of partitions of $[\kappa]^\kappa$, where κ is an uncountable regular cardinal. As Remark 2.1 below shows, the product topology on $[\kappa]^\kappa$ corresponding to the discrete topology is too large in general, in the sense that there are often partitions of $[\kappa]^\kappa$ both open (and in fact of a very simple form) and without large homogeneous sets.

REMARK 2.1. Given a cardinal κ, every partition $\chi:[\kappa]^2\to 2$ translates naturally into a partition of $[\kappa]^\kappa$. For such a χ let $\mathcal{A}$ be the union of all $\{\alpha\}_0\cap\{\beta\}_1$ such that $\alpha<\beta<\kappa$ and $\chi(\{\alpha,\beta\})=1$. Let $A\in[\kappa]^\kappa$. If $[A]^\kappa\subseteq\mathcal{A}$, then $\chi``[A]^2=\{1\}$, and if $[A]^\kappa\cap\mathcal{A}=\emptyset$, then $\chi``[A]^2=\{0\}$. This shows that if $\kappa\not\to(\kappa)^2_2$, then there is a set $\mathcal{A}=\bigcup_{i<\kappa}(\{\alpha_i\}_0\cap\{\beta_i\}_1)$ such that $[A]^\kappa\cap\mathcal{A}\ne\emptyset$ and $[A]^\kappa\setminus\mathcal{A}\ne\emptyset$ for every $A\in[\kappa]^\kappa$.

REMARK 2.2. If $[\kappa]^\kappa$ is given the product topology instead of the product topology corresponding to the discrete topology, then the partition in Remark 2.1 is still clearly κ^+-Borel and in fact belongs to $\underset{\sim}{\Sigma}^0_{2,\kappa^+}([\kappa]^\kappa)$. This shows that the natural generalization (even with respect to the product topology) of the Galvin–Prikry result does not apply to $[\kappa]^\kappa$ if $\kappa\not\to(\kappa)^2_2$.

The following negative result was pointed out by the referee in response to a false version of Theorem 2.5.

FACT 2.3 (Referee). *Suppose* κ *is an uncountable regular cardinal. Then there is* $\mathcal{A}\subseteq[\kappa]^\kappa$, $\mathcal{A}$ *open in the product topology*, *such that* $[S]^\kappa\cap\mathcal{A}\ne\emptyset$ *and* $[S]^\kappa\setminus\mathcal{A}\ne\emptyset$ *for every stationary* $S\subseteq\kappa$.

Proof. Let $\mathcal{A}$ consist of all those $A\in[\kappa]^\kappa$ such that $\widetilde{A}(\xi)=\xi$ for some ξ. Then $\mathcal{A}$ is open since it can be written as $\bigcup_{\xi<\kappa}[0,\xi+1)_\xi$. However, it is clear that if $S\subseteq\kappa$ is stationary, then $[S]^\kappa\cap\mathcal{A}\ne\emptyset$ and $[S]^\kappa\setminus\mathcal{A}\ne\emptyset$: on the one hand, $S \cap C \in \mathcal{A}$, where $C = \{\xi < \kappa : \mathrm{ot}(S \cap \xi) = \xi\}$, and on the other hand, every $A \in [\kappa]^\kappa$ can be easily refined to a set B in $[\kappa]^\kappa \setminus \mathcal{A}$. ∎

The proof of Fact 2.3 suggests restricting ourselves to partitions of the closed subspace of $[\kappa]^\kappa$ consisting of all those members of $[\kappa]^\kappa$ whose enumerating function does not have fixed points. We will denote this space by $\mathcal{X}_\kappa$. For this space we do have positive homogeneity results. Our main result (Theorem 2.5) is that all open partitions $\mathcal{A}$ of $\mathcal{X}_\kappa$ have a homogeneous club modulo $\mathcal{X}_\kappa$ (meaning that there is a club $C \subseteq \kappa$ such that either $[C]^\kappa \cap \mathcal{X}_\kappa \subseteq \mathcal{A}$ or $[C]^\kappa \cap \mathcal{X}_\kappa \cap \mathcal{A} = \emptyset$). From this it trivially follows (Corollary 2.6) that in contrast to Remark 2.2 ([1]), all κ-Borel partitions of $\mathcal{X}_\kappa$ have large (actually closed and unbounded) homogeneous sets modulo $\mathcal{X}_\kappa$.

For every natural number $n \geq 1$, every cardinal κ and every sequence $\mathcal{I} = \langle\langle(\alpha_{i,j}, \beta_{i,j}) : j < n\rangle : i < \kappa\rangle$ of n-sequences of open intervals of ordinals in κ, let $c^{\mathcal{I}} : [\kappa]^n \to 2$ be the partition given by $c^{\mathcal{I}}(\{\gamma_0, \ldots, \gamma_{n-1}\}) = 1$ iff $\gamma_0 < \ldots < \gamma_{n-1}$ and, for some $i < \kappa$, $\alpha_{i,j} < \gamma_j < \beta_{i,j}$ for every $j < n$.

Part (ii) of the following lemma and its proof will be crucial in the proof of Theorem 2.5.

LEMMA 2.4. *Let κ be an uncountable regular cardinal. Then for every n with $1 \leq n < \omega$ and every sequence $\mathcal{I} = \langle\langle(\alpha_{i,j}, \beta_{i,j}) : j < n\rangle : i < \kappa\rangle$ of n-sequences of open intervals of ordinals in κ there is a club $C \subseteq \kappa$ such that either $c^{\mathcal{I}}``[C]^n = \{0\}$ or $c^{\mathcal{I}}``[C]^n = \{1\}$.*

Proof. This is proved by induction on n. Take first $n = 1$. If there is no club $C \subseteq \kappa$ such that $c^{\mathcal{I}}``[C]^1 = \{1\}$, i.e., such that $C \subseteq \bigcup_{i<\kappa}(\alpha_{i,0}, \beta_{i,0})$, then $A = \kappa \setminus \bigcup_{i<\kappa}(\alpha_{i,0}, \beta_{i,0})$ is stationary. In particular A is an unbounded subset of κ, and being the complement of an open subset of κ, it is closed. Hence, A is a club of κ such that $c^{\mathcal{I}}``[A]^1 = \{0\}$. Now suppose that $n > 1$ and that the lemma holds for $n - 1$.

Suppose first that there is some stationary set $S \subseteq \kappa$ such that for every $\gamma \in S$ there is some $A_\gamma \in [C \setminus (\gamma + 1)]^\kappa$ so that $c^{\mathcal{I}}(\{\gamma\} \cup s) = 0$ for all $s \in [A_\gamma]^{n-1}$. For every $\gamma \in S$ let C_γ be the club of all limit points of A_γ (and let $C_\gamma = \kappa$ if $\gamma \in \kappa \setminus S'$). Let $T = S \cap \Delta_{\gamma<\kappa} C_\gamma$ and pick $\gamma_0 < \ldots < \gamma_{n-1}$ in T. For every j $(0 < j \leq n-1)$, $\gamma_j = \sup_{k<\mathrm{cf}(\gamma_j)} \gamma_j^k$, where for all $k_j < \mathrm{cf}(\gamma_j)$ $(0 < j \leq n-1)$, $c^{\mathcal{I}}(\{\gamma_0, \gamma_1^{k_1}, \ldots, \gamma_{n-1}^{k_{n-1}}\}) = 0$. But then, since all intervals of ordinals occurring in the members of $\mathcal{I}$ are open, it follows that $c^{\mathcal{I}}(\{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}) = 0$. Finally, let C be the club of limit points of T. Again, since T is 0-homogeneous for $c^{\mathcal{I}}$ and since the intervals

([1]) The argument for Remark 2.1 also shows that if κ is a regular cardinal such that $\kappa \nrightarrow (\kappa)^2_2$, then there is a set $\mathcal{A} = \bigcup_{i<\kappa}(\{\alpha_i\}_0 \cap \{\beta_i\}_1)$ such that $[A]^\kappa \cap \mathcal{X}_\kappa \cap \mathcal{A} \neq \emptyset$ and $([A]^\kappa \cap \mathcal{X}_\kappa) \setminus \mathcal{A} \neq \emptyset$ for every $A \in [\kappa]^\kappa$.

of ordinals occurring in the members of $\mathcal{I}$ are open, it follows that C itself is 0-homogeneous for $c^{\mathcal{I}}$.

Now suppose there is no stationary set $S \subseteq \kappa$ as above. Notice that for every $\gamma < \kappa$, the partition $c^{\mathcal{I},\gamma}$ of $[\kappa \setminus (\gamma+1)]^{n-1}$ given by $c^{\mathcal{I},\gamma}(s) = c^{\mathcal{I}}(\{\gamma\} \cup s)$ is $c^{\mathcal{I}'} \restriction [\kappa \setminus (\gamma+1)]^{n-1}$ for some sequence $\mathcal{I}'$ (2) of $n-1$-sequences of open intervals of ordinals in κ. But then, by our assumption and by the induction hypothesis, there is a club $C \subseteq \kappa$ such that for every $\gamma \in C \cap S$ there is a club $C_\gamma \subseteq \kappa$ so that $c^{\mathcal{I}}(\{\gamma\} \cup s) = 1$ for all $s \in [C_\gamma]^{n-1}$. Let $C' = C \cap \Delta_{\gamma<\kappa} C_\gamma$ and pick $\gamma_0 < \ldots < \gamma_{n-1}$ in C'. Since $\gamma_1, \ldots, \gamma_{n-1} \in C_{\gamma_0}$, $c^{\mathcal{I}}(\{\gamma_0, \ldots, \gamma_{n-1}\}) = 1$. ■

THEOREM 2.5. *Let κ be an uncountable regular cardinal. Then for every $\mathcal{A} \subseteq [\kappa]^\kappa$ which is open in the product topology there is a club $C \subseteq \kappa$ such that either $[C]^\kappa \cap \mathcal{X}_\kappa \subseteq \mathcal{A}$ or else $[C]^\kappa \cap \mathcal{X}_\kappa \cap \mathcal{A} = \emptyset$.*

Proof. First note that if $\mathcal{A}$ is an open subset of $[\kappa]^\kappa$, then there is an open $\mathcal{A}' \subseteq \mathcal{A}$ such that $\mathcal{A} \cap \mathcal{X}_\kappa \subseteq \mathcal{A}'$ and $\mathcal{A}'$ is a union of intersections of the form $\bigcap_{j<n} (\alpha_j, \beta_j)_{\xi_j}$, where $n < \omega$ and $\xi_j \le \alpha_j < \beta_j < \kappa$ for all j. Hence it suffices to prove that if $\mathcal{A}$ is of the form $\bigcup_{i<\kappa} \bigcap_{j<n} (\alpha_{i,j}, \beta_{i,j})_{\xi_{i,j}}$, where $n \ge 1$ is some fixed natural number, $\xi_{i,j} \le \alpha_{i,j} < \beta_{i,j}$ for each i and j, and $\xi_{i,j} < \xi_{i,j'}$ for all $j < j' < n$, then there is a club $C \subseteq \kappa$ such that either $[C]^\kappa \subseteq \mathcal{A}$ or else $[C]^\kappa \cap \mathcal{X}_\kappa \cap \mathcal{A} = \emptyset$.

For every $\xi < \kappa$ let $\mathcal{I}_{\langle\xi\rangle} = \langle\langle(\alpha_{i,j}, \beta_{i,j}) : j < n\rangle : i < \kappa,\ \xi_{i,0} = \xi\rangle$. Consider first the case in which for every $\xi < \kappa$ there is some $X_\xi \in [\kappa]^\kappa$ such that $c^{\mathcal{I}_{\langle\xi\rangle}}$ "$[X_\xi]^n = \{0\}$. As in the proof of Lemma 2.4, for every $\xi < \kappa$, let C_ξ be the club of limit points of X_ξ. Since the intervals $(\alpha_{i,j}, \beta_{i,j})$ are open, it follows that $[\Delta_{\xi<\kappa} C_\xi]^\kappa \cap \mathcal{X}_\kappa \cap \mathcal{A} = \emptyset$. To see this, suppose $A \in [\Delta_{\xi<\kappa} C_\xi]^\kappa$ is in $\mathcal{X}_\kappa \cap \mathcal{A}$. Then there is some i such that $\xi_{i,j} \le \alpha_{i,j} < \widetilde{A}(\xi_{i,j}) < \beta_{i,j}$ for all $j < n_i$. But then $\widetilde{A}(\xi_{i,j})$ is a limit point of ordinals in $X_{\xi_{i,0}}$ for all j. Since the intervals $(\alpha_{i,j}, \beta_{i,j})$ are open, there are some $\gamma_0 < \ldots < \gamma_{n-1}$ in $X_{\xi_{i,0}}$ such that $\alpha_{i,j} < \gamma_j < \beta_{i,j}$ for all $j < n_i$, which of course is a contradiction.

In the other case, by Lemma 2.4 there is some $\xi_0 < \kappa$ and some club $C_0 \subseteq \kappa$ such that $c^{\mathcal{I}_{\langle\xi_0\rangle}}$ "$[C_0]^n = \{1\}$. Let also $T_0 = \{\langle\xi_0\rangle\}$. We will check that in this case $[C]^\kappa \subseteq \mathcal{A}$ for some club $C \subseteq \kappa$.

For every $\gamma \in C_0$ and every $\xi < \kappa$ let $\mathcal{I}_{\langle\xi_0,\gamma,\xi\rangle} = \langle\langle(\alpha_{i,j}, \beta_{i,j}) : 1 \le j < n\rangle : i < \kappa,\ \xi_{i,0} = \xi_0,\ \alpha_{i,0} < \gamma < \beta_{i,0}$ and $\xi_{i,1} = \xi\rangle$. Again, given such a γ, consider first the case in which for every $\xi < \kappa$ there is some $X^\gamma_\xi \in [C_0]^\kappa$ such that $c^{\mathcal{I}_{\langle\xi_0,\gamma,\xi\rangle}}$ "$[X^\gamma_\xi]^{n-1} = \{0\}$. Then we say that γ is a $\langle\xi_0\rangle$-*bad ordinal.* Also, for every ξ let D^γ_ξ be the club of all limit points of X^γ_ξ and let $D^\gamma = \Delta_{\xi<\kappa} D^\gamma_\xi$. In the other case there is some $\xi_{\langle\xi_0,\gamma\rangle} < \kappa$ and, by Lemma 2.4, some club

(2) Specifically, $\mathcal{I}' = \langle\langle(\alpha_{i,j}, \beta_{i,j}) : 1 \le j < n\rangle : i < \kappa,\ \alpha_{i,0} < \gamma < \beta_{i,0}\rangle$.

$D^\gamma \subseteq \kappa$ such that $c^{\mathcal{I}_\varphi}$ "$[C_0 \cap D^\gamma]^{n-1} = \{1\}$, where $\varphi = \langle \xi_0, \gamma, \xi_{\langle \xi_0, \gamma\rangle}\rangle$. If this happens, then we say that γ is a $\langle \xi_0\rangle$-*good ordinal*.

Suppose that the set $S \subseteq C_0$ of $\langle \xi_0\rangle$-bad ordinals were stationary. Let $T = S \cap \Delta_{\gamma \in S} D^\gamma$. Then, for all $\gamma_0 < \gamma_1 < \ldots < \gamma_{n-1}$ in T, $\gamma_1, \ldots, \gamma_{n-1} \in \Delta_{\xi<\kappa} D^{\gamma_0}_\xi$. Now, by an argument as in the second paragraph of this proof, it follows that $c^{\mathcal{I}_{\langle \xi_0\rangle}}(\{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}) = 0$, which is a contradiction.

Hence, there is a club $C \subseteq \kappa$ such that all $\gamma \in C$ are $\langle \xi_0\rangle$-good ordinals. Let $C_1 = C \cap \Delta_{\gamma \in C \cap S} D^\gamma$. Then, for all $\gamma_0 < \gamma_1 < \ldots < \gamma_{n-1}$ in C_1, $\gamma_1, \ldots, \gamma_{n-1} \in D^{\gamma_0}$, and so $c^{\mathcal{I}_\varphi}(\{\gamma_1, \ldots, \gamma_{n-1}\}) = 1$ for $\varphi = \langle \xi_0, \gamma_0, \xi_{\langle \xi_0, \gamma_0\rangle}\rangle$. Let $T_1 = \{\langle \xi_0, \gamma, \xi_{\langle \xi_0,\gamma\rangle}\rangle : \gamma \in C_1\}$.

Now suppose $1 < j < n$ and suppose that for every $k < j$, C_k and T_k have been defined, C_k is a club of κ and T_k is a set of sequences

$$\varphi = \langle \xi_0, \gamma^\varphi_0, \xi^\varphi_1, \gamma^\varphi_1, \ldots, \xi^\varphi_{k-1}, \gamma^\varphi_{k-1}, \xi^\varphi_k\rangle,$$

where $\gamma^\varphi_0 < \gamma^\varphi_1 < \ldots < \gamma^\varphi_{k-1}$ range over C_k, $\xi^\varphi_l < \gamma^\varphi_l$ for all $l < k$ and φ is such that $c^{\mathcal{I}_\varphi}(\{\gamma_k, \ldots, \gamma_{n-1}\}) = 1$ for all $\gamma_k < \ldots < \gamma_{n-1}$ in $C_k \setminus (\gamma^\varphi_{k-1}+1)$, where $\mathcal{I}_\varphi = \langle\langle(\alpha_{i,j}, \beta_{i,j}) : k \le j < n\rangle : i < \kappa$, $\xi_{i,l} = \xi^\varphi_l$ and $\alpha_{i,l} < \gamma^\varphi_l < \beta_{i,l}$ for every $l < k$, and $\xi_{i,k} = \xi^\varphi_k\rangle$. Suppose also that for every $\varphi \in T_k$, φ is the only $\psi \in T_k$ such that $\psi\restriction 2k = \varphi\restriction 2k$.

For every $\varphi \in T_{j-1}$, every $\gamma \in C_{j-1}$ and every $\xi < \kappa$ define $\mathcal{I}_{\varphi,\langle\gamma,\xi\rangle}$ (whenever φ and ψ are two sequences, φ, ψ will denote their concatenation) in the obvious way. Similarly to the $j = 1$ case, say that $\gamma \in C_{j-1}$ is a φ-*bad ordinal* if for every $\xi < \kappa$ there is some $X^\gamma_\xi \in [C_{j-1}]^\kappa$ such that $c^{\mathcal{I}_{\varphi,\langle\gamma,\xi\rangle}}$ "$[X^\gamma_\xi]^{n-j} = \{0\}$. Otherwise call it a φ-*good ordinal*. If $\langle X^\gamma_\xi : \xi < \kappa\rangle$ witnesses that γ is a φ-bad ordinal, then for every ξ let D^γ_ξ be the club of all limit points of X^γ_ξ and let $D^\gamma = \Delta_{\xi<\kappa} D^\gamma_\xi$. If γ is φ-good, by Lemma 2.4 there is some $\xi_{\varphi,\langle\gamma\rangle} < \kappa$ and some club $D^\gamma_\varphi \subseteq \kappa$ such that $c^{\mathcal{I}_{\varphi'}}$ "$[D^\gamma_\varphi]^{n-j} = \{1\}$, where $\varphi' = \varphi, \langle\gamma, \xi_{\varphi,\langle\gamma\rangle}\rangle$.

Suppose that for some $\varphi \in T_{j-1}$ the set $S \subseteq C_{j-1}$ of φ-bad ordinals were stationary. Let $T = S \cap \Delta_{\gamma\in S} D^\gamma$. Then for all $\gamma_{j-1} < \gamma_j < \ldots < \gamma_{n-1}$ in T, $\gamma_j, \ldots, \gamma_{n-1} \in \Delta_{\xi<\kappa} D^{\gamma_{j-1}}_\xi$, and arguing as in the $j = 1$ case it would follow that $c^{\mathcal{I}_\varphi}(\{\gamma_{j-1}, \gamma_j, \ldots, \gamma_{n-1}\}) = 0$, which is a contradiction.

It follows that for every $\varphi \in T_{j-1}$ there is a club $C^\varphi \subseteq \kappa$ such that all $\gamma \in C^\varphi$ are φ-good ordinals. Let $D^\varphi = C^\varphi \cap \Delta_{\gamma\in C^\varphi} D^\gamma_\varphi$. For every $\gamma < \kappa$ let D'_γ be the intersection of all D^φ for $\varphi \in T_{j-1}$ such that $\gamma^\varphi_{j-2} < \gamma$. Finally, let C_j be the club of limit ordinals in $\Delta_{\gamma<\kappa} D'_\gamma$ and let T_j be the set of all $\varphi, \langle\gamma, \xi_{\varphi,\langle\gamma\rangle}\rangle$, where $\varphi \in T_{j-1}$ and $\gamma^\varphi_0 < \ldots < \gamma^\varphi_{j-2} < \gamma$ range over C_j. Now fix such an element $\varphi, \langle\gamma, \xi_{\varphi,\langle\gamma\rangle}\rangle$ of T_j and pick $\gamma_j < \ldots < \gamma_{n-1}$ in C_j with $\gamma_j > \gamma$. Since $\gamma_j, \ldots, \gamma_{n-1}$ belong to D'_γ, they also belong to D^φ. As $\gamma \in C^\varphi$, it follows that these ordinals are in D^γ_φ, and so $c^{\mathcal{I}_{\varphi'}}(\{\gamma_j, \ldots, \gamma_{n-1}\}) = 1$ for $\varphi' = \varphi, \langle\gamma, \xi_{\varphi,\langle\gamma\rangle}\rangle$.

Let us finally check that $[C_{n-1}]^\kappa \subseteq \mathcal{A}$:

Pick $X \in [C_{n-1}]^\kappa$ and let $\gamma_0 = \widetilde{X}(\xi_0)$. Let ξ_1 be the unique ξ such that $\langle \xi_0, \gamma_0, \xi \rangle \in T_1$ and let $\gamma_1 = \widetilde{X}(\xi_1)$. Proceeding in this way, we find $\varphi \in T_{n-1}$, $\varphi = \langle \xi_0, \gamma_0, \ldots, \xi_{n-2}, \gamma_{n-2}, \xi_{n-1} \rangle$ such that $\widetilde{X}(\xi_j) = \gamma_j$ for all $j < n-1$. Let also $\gamma_{n-1} = \widetilde{X}(\xi_{n-1})$. Then $c^{\mathcal{I}_\varphi}(\{\gamma_{n-1}\}) = 1$, which means that there is some $i < \kappa$ such that for every $j < n$, $\xi_{i,j} = \xi_j$ and $\alpha_{i,j} < \gamma_j = \widetilde{X}(\xi_j) < \beta_{i,j}$. Hence, $X \in \mathcal{A}$. ∎

Corollary 2.6. *If κ is an uncountable regular cardinal and $\mathcal{A}$ is a κ-Borel subset of $[\kappa]^\kappa$ (relative to the product topology on $[\kappa]^\kappa$), then there is a club $C \subseteq \kappa$ such that either $[C]^\kappa \cap \mathcal{X}_\kappa \subseteq \mathcal{A}$ or else $[C]^\kappa \cap \mathcal{X}_\kappa \cap \mathcal{A} = \emptyset$.*

The next result shows that Theorem 2.5 is in some sense optimal: if we allow unions not only of finite intersections, but of countable intersections of elements in the natural basis for the product topology, then we can easily build partitions of $\mathcal{X}_\kappa$ without homogeneous sets.

Fact 2.7. *Let κ be an uncountable regular cardinal. Then there is a collection $\{\mathcal{A}_\gamma : \gamma < \kappa\}$ of pairwise disjoint subsets of $[\kappa \cap \mathrm{cf}(\omega)]^\kappa$ such that*

(i) *for every $\gamma < \kappa$, $\mathcal{A}_\gamma = \bigcup_{i<\kappa} \bigcap_{j<\omega} (\alpha^\gamma_{i,j}, \beta^\gamma_{i,j})_{\xi^\gamma_{i,j}}$ for some $\alpha^\gamma_{i,j}, \beta^\gamma_{i,j}, \xi^\gamma_{i,j} < \kappa$,*

(ii) *for every $A \in [\kappa]^\kappa$ and every $\gamma < \kappa$, $[A]^\kappa \cap \mathcal{X}_\kappa \cap \mathcal{A}_\gamma \neq \emptyset$.*

Proof. Let $\langle S_\gamma : \gamma < \kappa \rangle$ be a partition of $\kappa \cap \mathrm{cf}(\omega)$ into stationary subsets. Fix also, for every $\alpha < \kappa$ of countable cofinality, an increasing ω-sequence e_α converging to α. For every γ let $\mathcal{A}_\gamma = \bigcup_{\alpha \in S_\gamma} \bigcap_{j<\omega} (e_\alpha(j), \alpha)_j$. If $\gamma < \kappa$ and $A \in \mathcal{A}_\gamma$, then $\sup_{j<\omega} \widetilde{A}(j) \in S_\gamma$, and so $\mathcal{A}_\gamma \cap \mathcal{A}_{\gamma'} = \emptyset$ if $\gamma \neq \gamma'$. Now take $A \in [\kappa]^\kappa$ and $\gamma < \kappa$. Since the set of all ordinals in κ which are limit points of A is a club of κ, there is such an ordinal α in S_γ with $\alpha > \omega$. Take a strictly increasing sequence $(\alpha_j)_{j<\omega}$ of elements in A such that $\omega < \alpha_0$ and $e_\alpha(j) < \alpha_j < \alpha$ for all j. Then we may take $B \subseteq A \setminus \alpha$ of size κ such that $\{\alpha_j : j < \omega\} \cup B \in \mathcal{X}_\kappa$. But also $\{\alpha_j : j < \omega\} \cup B \in \mathcal{A}_\gamma$. ∎

Remark 2.8. All $\mathcal{A}_\gamma$ in Fact 2.7 are also in $\utilde{\Sigma}^0_{3,\kappa^+}([\kappa]^\kappa)$. Hence, Corollary 2.6 is also optimal.

The following fact shows that there cannot be any version of Theorem 2.5 for open partitions of the open space $[\kappa]^\kappa \setminus \mathcal{X}_\kappa$; more precisely, there are always open $\mathcal{A} \subseteq [\kappa]^\kappa$ such that both $([C]^\kappa \setminus \mathcal{X}_\kappa) \cap \mathcal{A}$ and $([C]^\kappa \setminus \mathcal{X}_\kappa) \setminus \mathcal{A}$ are nonempty for every club $C \subseteq \kappa$.

Fact 2.9. *Let $\kappa \geq \omega_1$ be a regular cardinal. Then there is $\mathcal{A} \subseteq [\kappa]^\kappa$, open in the product topology, such that for every club $C \subseteq \kappa$ there are stationary $S, T \subseteq C$ (hence in particular S and T belong to $[\kappa]^\kappa \setminus \mathcal{X}_\kappa$) such that $S \in \mathcal{A}$ and $T \notin \mathcal{A}$.*

Proof. Fix a stationary and co-stationary set $\overline{S} \subseteq \kappa$ and define $\mathcal{A} = \bigcup\{[0, \xi+1)_\xi : \xi \in \overline{S}\}$. Note that a member A of $[\kappa]^\kappa$ belongs to $\mathcal{A}$ if and only if there is some $\xi \in A \cap \overline{S}$ such that $\mathrm{ot}(A \cap \xi) = \xi$. Now the conclusion follows trivially. ∎

Given a cardinal κ, a natural number $n \geq 1$ and a linear order on $[\kappa]^\kappa$, the *product topology on* $[[\kappa]^\kappa]^n$ is the topology on $[[\kappa]^\kappa]^n$ obtained by giving $([\kappa]^\kappa)^n$ the product topology and identifying each element of $[[\kappa]^\kappa]^n$ with its strictly $\leq$-increasing enumeration. Notice that this definition is independent of the particular choice of $\leq$.

Fact 2.10 shows that Theorem 2.5, which can be regarded as a result concerning $[[\kappa]^\kappa]^n$ for $n = 1$, cannot be extended to greater n.

Fact 2.10. *Let κ be an infinite cardinal and let $n \geq 2$ be a natural number. Then there is an open subset $\mathcal{A}$ of $[[\kappa]^\kappa]^n$ such that for every $X \in [\kappa]^\kappa$ there are $Y_j, Z_j \in [X]^\kappa$ for $1 \leq j < n$ so that $\{X, Y_1, \ldots, Y_{n-1}\} \in \mathcal{A}$ and $\{X, Z_1, \ldots, Z_{n-1}\} \in [[\kappa]^\kappa]^n \setminus \mathcal{A}$.*

Proof. We can clearly assume that $n = 2$. Let $\leq_{\mathrm{lex}}$ be the lexicographic order on $[\kappa]^\kappa$, i.e., $X <_{\mathrm{lex}} Y$ iff $\widetilde{X}(\delta_0) < \widetilde{Y}(\delta_0)$, where δ_0 is the first δ such that $\widetilde{X}(\delta) \neq \widetilde{Y}(\delta)$. For every $\gamma < \kappa$ let $\mathcal{A}_\gamma$ be the set of all $\{X, Y\} \in [[\kappa]^\kappa]^2$ such that $X <_{\mathrm{lex}} Y$, $\widetilde{X}(0) < \gamma$ and $\widetilde{Y}(0) > \gamma$. Then $\mathcal{A}_\gamma$ is an open subset of $[[\kappa]^\kappa]^2$ for every γ. Let $\mathcal{A} = \bigcup_{\gamma<\kappa} \mathcal{A}_\gamma$. Pick $X \in [\kappa]^\kappa$. Let $Y = X \setminus (\widetilde{X}(0)+2)$. Then $X <_{\mathrm{lex}} Y$ and $\{X, Y\} \in \mathcal{A}_{\widetilde{X}(0)+1} \subseteq \mathcal{A}$. Let $Z = X \setminus \{\widetilde{X}(1)\}$. Then $X <_{\mathrm{lex}} Z$ but $\widetilde{X}(0) = \widetilde{Z}(0)$, and so $\{X, Z\} \notin \mathcal{A}$. ∎

Question 2.1. Can one find true (or consistent) Open Coloring Axiom-like statements for $[[\kappa]^\kappa]^2$? For $\kappa = \omega$ we have of course Todorčević's Open Coloring Axiom (see [F], [T], [V]).

Definition 2.1. If κ is a cardinal and $\mathcal{A} \subseteq [\kappa]^\kappa$, then $\mathcal{A}$ is a *completely Ramsey subset* of $[\kappa]^\kappa$ if for every $A \in [\kappa]^\kappa$ there is some $B \in [A]^\kappa$ such that either $[B]^\kappa \subseteq \mathcal{A}$ or $[B]^\kappa \cap \mathcal{A} = \emptyset$.

Fact 2.11. *Suppose κ is a Ramsey cardinal and*

$$\mathcal{A} = \bigcup_{i<\kappa} \bigcap_{j<n_i} [\alpha_{i,j}, \beta_{i,j})_{\xi_{i,j}},$$

where $n_i < \omega$ for all i, $\alpha_{i,j}, \beta_{i,j} < \kappa$ and $\xi_{i,j} < \kappa$ for all $j < n_i$ and all $i < \kappa$, and furthermore, $\{\xi_{i,j} : j < n_i,\ i < \kappa\}$ is bounded in κ. Then $\mathcal{A}$ is a completely Ramsey subset of $[\kappa]^\kappa$.

Proof. We can clearly assume that $(\xi_{i,j} : j < n_i)$ is a one-to-one sequence for each i. Let $\gamma < \kappa$ be a bound for $\{\xi_{i,j} : j < n_i,\ i < \kappa\}$. Pick $A \in [\kappa]^\kappa$ and define $\chi : [A]^{<\omega} \to \gamma^{<\omega} \cup \{\gamma\}$ by letting $\chi(s)$ be, when available, some $t \in \gamma^{<\omega}$ for which there is some $i < \kappa$ such that $t = (\xi_{i,j} : j < n_i)$, $n_i = |s|$, and for each $k < n_i$, if $\xi_{i,j}$ is the kth member of t, then $\alpha_{i,j} \le \widetilde{s}(k) < \beta_{i,j}$; otherwise, let $\chi(s) = \gamma$.

Recall that a cardinal κ is Ramsey if and only if $\kappa \to (\kappa)^{<\omega}_\delta$ for all $\delta < \kappa$ (see [K, Proposition 7.14]). It follows that there is some $B \in [A]^\kappa$ such that for all n, $\chi(s) = \chi(s')$ for all $s, s' \in [B]^n$. Now suppose there is some n such that $\chi``[B]^n = \{t\}$ for some $t = (\xi_j : j < n) \in \gamma^{<\omega}$. Pick any $B' \in [B]^\kappa$ and let $s = \{\widetilde{B}'(\xi_j)\}_{j<n}$. Since $\chi(s) = t$, we know that there is some i such that $t = (\xi_{i,j} : j < n_i)$ and $\alpha_{i,j} \le \widetilde{B}'(\xi_j) = \widetilde{B}'(\xi_{i,j}) < \beta_{i,j}$ for all $j < n_i$. Hence, $B' \in \mathcal{A}$, and since $B' \in [B]^\kappa$ was arbitrary, $[B]^\kappa \subseteq \mathcal{A}$. That $[B]^\kappa \cap \mathcal{A}$ is empty if $\chi``[B]^{<\omega} = \{\gamma\}$ also follows easily. ∎

QUESTION 2.2. Suppose κ is a Ramsey cardinal. Is every open subset of $[\kappa]^\kappa$ completely Ramsey?

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