On a convenient property about $[\gamma]^{\aleph_0}$

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Abstract

Several situations are presented in which there is an ordinal γ such that $\{X \in [\gamma]^{\aleph_0} : X \cap \omega_1 \in S \text{ and } ot(X) \in T\}$ is a stationary subset of $[\gamma]^{\aleph_0}$ for all stationary $S, T \subseteq \omega_1$. A natural strengthening of the existence of an ordinal γ for which the above conclusion holds lies, in terms of consistency strength, between the existence of the sharp of H_{ω_2} and the existence of sharps for all reals. Also, an optimal model separating $BSPFA$ and BMM is produced and it is shown that a strong form of BMM involving only parameters from H_{ω_2} implies that every function from ω_1 into ω_1 is bounded on a club by a canonical function.

1 Introduction

Given a class Γ of partially ordered sets (posets), the *bounded forcing axiom for* Γ ($BFA(\Gamma)$) is the statement that for every $\mathbb{P} \in \Gamma$ and every collection $\{A_i : i < \omega_1\}$ of maximal antichains of $\mathbb{P}$ of size at most $\aleph_1$ there is a filter $G \subseteq \mathbb{P}$ such that $G \cap A_i \neq \emptyset$ for each i. Recall that $BFA(\Gamma)$ can be characterized, in all naturally occurring cases, as a principle of generic absoluteness for Σ_1 formulas with parameters in H_{ω_2}.

Theorem 1.1 *(Bagaria ([B])) Given a class Γ of complete Boolean algebras, $BFA(\Gamma)$ holds if and only if for every $a \in H_{\omega_2}$ and every Σ_1 formula $\varphi(x)$, $H_{\omega_2} \models \varphi(a)$ iff there is some $\mathbb{P}$ in Γ such that $\Vdash_{\mathbb{P}} H_{\omega_2} \models \varphi(\check{a})$.*

Most natural classes Γ of posets, and in particular all classes of posets considered in this paper, are closed under completion.[1] Notice that for these Γ, $BFA(\Gamma)$ is equivalent to the corresponding principle of generic absoluteness in Theorem 1.1.

In this paper I look at combinatorial features that have arisen in the study of *Bounded Martin's Maximum* (BMM), the bounded forcing axiom for the class of those posets that preserve all stationary subsets of ω_1. Notice that BMM is the bounded form of the maximal forcing axiom Martin's Maximum from [Fo-M-S].

[1] That is, if $\mathbb{P} \in \Gamma$ and $\mathbb{B}$ is the (unique up to isomorphism) complete Boolean algebra such that $\mathbb{P}$ can be densely embedded in $\mathbb{B}$, then $\mathbb{B} \in \Gamma$.

Until fairly recently, an important open problem in set theory was whether BMM decides the size of the continuum. A number of partial positive results– usually of the form 'BMM + (some extra hypothesis) implies (a certain statement implying) $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$'– were obtained[2] before S. Todorčević found in April 2002 a proof that BMM alone implies $2^{\aleph_0} = \aleph_2$ ([T2]).

The particular statement implying $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ referred to in the last paragraph is the following.

Definition 1.1 *(Woodin ([Wo], Definition 5.12)) ψ_{AC} is the statement that if S and T are stationary and co-stationary subsets of ω_1, then there are an ordinal δ, a surjection $e : \omega_1 \longrightarrow \delta$ and a club $C \subseteq \omega_1$ such that $S \cap C = \{\nu \in C \,:\, ot(e``\nu) \in T\}$.*

It is well–known (see for example [J], p. 445) that given any ordinal $\delta < \omega_2$ and any surjective function $e : \omega_1 \longrightarrow \delta$, the function $g : \omega_1 \longrightarrow \omega_1$ given by $g(\nu) = ot(e``\nu)$ represents δ in the generic ultrapower derived from forcing with $\mathcal{P}(\omega_1)/NS_{\omega_1}$ (where NS_{ω_1} denotes the nonstationary ideal over ω_1), i.e., $\mathcal{P}(\omega_1)/NS_{\omega_1}$ forces that the set of M–ordinals below the class of g in M is well–ordered in order type δ, where M is the generic ultrapower $(V^{\omega_1} \cap V)/G$. Such a function g is called *the canonical function for δ (derived from e)*. This terminology is justified by the easily checked fact that if $e' : \omega_1 \longrightarrow \delta$ is any other such surjection and g' is defined as g with e' instead of e, then g and g' agree on a club. Throughout this paper, by a canonical function I will mean a canonical function for some ordinal below ω_2. Also, in the context of forcing with $\mathcal{P}(\omega_1)/NS_{\omega_1}$, M and j will denote the corresponding generic ultrapower and generic elementary embedding, respectively.

It is easy to see that ψ_{AC} is equivalent to the assertion that for all stationary and co-stationary $S, T \subseteq \omega_1$ there is some $\delta < \omega_2$ such that $S \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} \delta \in j(T)$ and $\omega_1 \backslash S \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} \delta \notin j(T)$, i.e., letting $\mathbb{B}$ be the regular open completion of $\mathcal{P}(\omega_1)/NS_{\omega_1}$, the class $[S]_{\mathbb{B}}$– under the natural embedding of $\mathcal{P}(\omega_1)/NS_{\omega_1}$ inside $\mathbb{B}$– of S is equal to $[\![\check{\delta} \in j(\check{T})]\!]_{\mathbb{B}}$, the Boolean value of the formula $\delta \in j(T)$. It is also easy to see that ψ_{AC} implies $2^{\aleph_1} = \aleph_2$ and also that $L(\mathcal{P}(\omega_1)) \models ZFC$ ([Wo], Lemma 5.13). ψ_{AC} actually also implies $2^{\aleph_0} = 2^{\aleph_1}$ by a harder argument ([Wo], Theorem 3.51).

Definition 1.2 *The Club Bounding Principle (CBP) is the statement that every function from ω_1 into ω_1 is bounded on a club by a canonical function.*

ψ_{AC} implies CBP ([As-W]), which in turn implies, by a result of Deiser and Donder ([D-Do]), the existence of an inner model with an inaccessible limit of measurable cardinals.

In contrast to this, a particularly annoying fact about BMM is that nothing nontrivial can be said so far about its consistency strength in general.[3] On the

[2] See [As2] for these developments.

[3] Added in proof: The situation has changed quite dramatically since this paper was completed. Ralf Schindler has proved that BMM implies the existence, for every set X, of an inner model containing X and with a strong cardinal ([Sc]).

other hand, the *Bounded Semiproper Forcing Axiom* $BSPFA$, namely $BFA(\Gamma)$ for the class Γ of semiproper posets, is equiconsistent with the existence of a reflecting cardinal (a regular cardinal κ is *reflecting* if H_κ is correct about Σ_2 statements with parameters).[4]

It is easy to see that a cardinal is reflecting in L whenever it is reflecting in the universe, and also that if κ is a Mahlo cardinal, then there are stationarily many λ in κ which are reflecting in V_κ. BMM certainly implies that ω_2 is reflecting in L, but nothing better than that is yet known to hold in general.

However, it should be noted that ψ_{AC}– and thus the existence of an inner model with an inaccessible limit of measurable cardinals– follows from BMM plus the existence of a cardinal with a certain weak Erdős property denoted by $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$ ([As-W]).

The meaning of this notation is that for every first order structure $\mathcal{A} = \langle L_\kappa[A], \in, A\rangle$ there is a sequence $(I_\alpha \,:\, \alpha < \omega_1)$ such that

(i) for all $\alpha < \omega_1$, I_α is a set of indiscernibles for $\mathcal{A}^+$ of order type $\omega\alpha$, where $\mathcal{A}^+ = (\mathcal{A}, \dot{\xi})_{\xi<\omega_1}$, and

(ii) for all α, $\beta < \omega_1$, all formulas $\varphi(v_0, \ldots, v_{n+m-1})$, $\vec{\xi} \in \omega_1^n$, $\vec{\gamma} \in I_\alpha^m$ and $\vec{\gamma}' \in I_\beta^m$, if $\vec{\gamma}$ and $\vec{\gamma}'$ are strictly increasing,

$$\mathcal{A}^+ \models \varphi(\vec{\xi}, \vec{\gamma}) \longleftrightarrow \varphi(\vec{\xi}, \vec{\gamma}').$$

The consistency strength of this partition relation lies strictly below that of an ω_1–Erdős cardinal. This fact was used in [As-W] to produce the first known model of $BSPFA$ in which BMM fails. Also, Schindler has observed that if BMM holds and ω_1 is inaccessible to reals (i.e., $\omega_1^{L[a]} < \omega_1$ for all reals a), then $X^\sharp$ exists for every set of ordinals X and there is an inner model with a strong cardinal. These results indicate that BMM could have large consistency strength by itself, though it is not yet known whether it implies that ω_1 is inaccessible to reals.

In this paper I am mostly interested in a certain combinatorial property for $[\gamma]^{\aleph_0}$– given some ordinal $\gamma \geq \omega_2$– which is relevant to the obtainment of ψ_{AC} in the presence of BMM. As I will shortly show– this argument was given for example also in [As-W], but I reproduce it here for the reader's convenience–, the existence of some γ with that property implies, in the presence of BMM, that ψ_{AC} holds.

Recall that, given a set $\mathcal{X}$, $C \subseteq [\mathcal{X}]^{\aleph_0}$ is a *club of* $[\mathcal{X}]^{\aleph_0}$ if and only if C is an $\subseteq$–unbounded subset of $[\mathcal{X}]^{\aleph_0}$ which is also closed, i.e., the union of every countable $\subseteq$–increasing sequence of elements of C belongs to C. The following well–known fact will be often used.

[4] The bounded form of the Proper Forcing Axiom was shown by Goldstern and Shelah in [Go-S]– where it was introduced– to imply that ω_2 is reflecting in L. On the other hand, if κ is reflecting, then there is a semiproper iteration $\mathbb{P} \subseteq V_\kappa$ forcing $BSPFA$ over V ([Go-S]).

Lemma 1.2 *([Ku]) For every set $\mathcal{X}$ and every club $E \subseteq [\mathcal{X}]^{\aleph_0}$ there is a function $F : [\mathcal{X}]^{<\omega} \longrightarrow \mathcal{X}$ such that all $X \in [\mathcal{X}]^{\aleph_0}$ which are closed under F (i.e., are such that $F``[X]^{<\omega} \subseteq X$) belong to E.*

$A \subseteq [\mathcal{X}]^{\aleph_0}$ is a *stationary subset of* $[\mathcal{X}]^{\aleph_0}$ if A intersects each club of $[\mathcal{X}]^{\aleph_0}$. If $\omega_1 \subseteq \mathcal{X}$, $A \subseteq [\mathcal{X}]^{\aleph_0}$ is a *projective stationary subset of* $[\mathcal{X}]^{\aleph_0}$ ([F-J]) if and only if $\{X \in A : X \cap \omega_1 \in S\}$ is a stationary subset of $[\mathcal{X}]^{\aleph_0}$ for every stationary $S \subseteq \omega_1$.

Given a set $\mathcal{X}$ and $A \subseteq [\mathcal{X}]^{\aleph_0}$, we define the following poset $\mathbb{P}_A$: $p \in \mathbb{P}_A$ if and only if p is a strictly $\subseteq$–increasing and $\subseteq$–continuous (i.e., if $\nu \in dom(p)$ is a limit ordinal, then $p(\nu) = \bigcup_{\nu' < \nu} p(\nu')$) sequence of elements of A whose length is some countable successor ordinal. q extends p if and only if $p \subseteq q$.

The following fact is proved in [F-J].

Lemma 1.3 *Let X be a set and let A be a stationary subset of $[X]^{\aleph_0}$. Then, $\mathbb{P}_A$ forces the existence of a strictly $\subseteq$–increasing and $\subseteq$–continuous sequence $\langle X_\nu : \nu < \omega_1 \rangle$ of elements of A such that $X = \bigcup_{\nu < \omega_1} X_\nu$. Suppose that $\omega_1 \subseteq \mathcal{X}$. Then $\mathbb{P}_A$ preserves stationary subsets of ω_1 if and only if A is a projective stationary subset of $[X]^{\aleph_0}$.*

From Lemma 1.3 we get that the following bounded form of the Projective Stationary Reflection principle ([F-J])[5] is a consequence of BMM.

Definition 1.3 *$BPSR$ is the following statement.*

Suppose γ is an ordinal, $a \in H_{\omega_2}$, $\alpha < \gamma$ and $A \subseteq [\gamma]^{\aleph_0}$ is a projective stationary subset of $[\gamma]^{\aleph_0}$ which is Σ_1 definable with a, α and γ as parameters (i.e., there is some Σ_1 formula $\varphi(x, y, z)$ such that, for every set X, $X \in A$ iff $\models_1 \varphi(X, a, \alpha, \gamma)$[6]). Then there are some $\overline{\alpha} < \delta < \omega_2$ and some strictly $\subseteq$–increasing and $\subseteq$–continuous sequence $\langle X_\nu : \nu < \omega_1 \rangle$ such that $\delta = \bigcup_{\nu < \omega_1} X_\nu$ and, for every $\nu < \omega_1$, $H_{\omega_2} \models \varphi(X_\nu, a, \overline{\alpha}, \delta)$.

Suppose BMM (or, more generally, $BPSR$) holds. In order to verify ψ_{AC} it suffices, by Lemma 1.3, to find some ordinal $\gamma \geq \omega_2$ so that, whenever S and T are stationary and co-stationary subsets of ω_1, $\{X \in [\gamma]^{\aleph_0} : X \cap \omega_1 \in S \text{ iff } ot(X) \in T\}$ is a projective stationary subset of $[\gamma]^{\aleph_0}$. Then, by $BPSR$ there is $\delta < \omega_2$ and a strictly $\subseteq$–increasing and $\subseteq$–continuous sequence $\langle X_\nu : \nu < \omega_1 \rangle$ of countable subsets of δ such that, for each ν, $X_\nu \cap \omega_1 \in S$ if and only if $ot(X_\nu) \in T$ and such that $\delta = \bigcup_\nu X_\nu$. Let $e : \omega_1 \longrightarrow \delta$ be any surjection and let $C \subseteq \omega_1$ be the club of all ordinals ν such that $\nu = (e``\nu) \cap \omega_1$ and $e``\nu = X_\nu$. Then, for every $\nu \in C$, $\nu \in S$ iff $ot(e``\nu) \in T$.

Given an ordinal γ and two subsets Y and Z of ω_1, let

$$\mathcal{X}^\gamma_{Y,Z} = \{X \in [\gamma]^{\aleph_0} : X \cap \omega_1 \in Y \text{ and } ot(X) \in Z\}$$

[5]The Projective Stationary Reflection principle is equivalent to Todorčević's Strong Reflection Principle (see also [F-J] for a proof).

[6]$\models_n$ denotes the definable satisfaction relation for Σ_n formulas.

Now I define the Chang's Conjecture type property I will focus on in most of this paper.

Definition 1.4 *For an ordinal $\gamma \geq \omega_2$, $\mathcal{S}^\gamma$ is the statement that for all stationary subsets S and T of ω_1, $\mathcal{X}^\gamma_{S,T}$ is a stationary subset of $[\gamma]^{\aleph_0}$.*

Also, $\mathcal{S}$ is the statement that there is some ordinal $\gamma \geq \omega_2$ such that $\mathcal{S}^\gamma$.

If there is some $\gamma \geq \omega_1$ such that $\mathcal{S}^\gamma$, then γ is obviously a limit ordinal and it is at least ω_2 (see Remark 1).

The reason I am interested in $\mathcal{S}$ is that, as it easily follows from the paragraph before Definition 1.4, this combinatorial property suffices to prove– in the presence of $BPSR$– ψ_{AC} (actually, this argument is enough to establish the more general version of ψ_{AC} in which the conclusion of Definition 1.1 holds for all stationary S, $T \subseteq \omega_1$ such that T is also co-stationary). In fact, $\mathcal{S}^\gamma$ is equivalent to $\{X \in [\gamma]^{\aleph_0} : X \cap \omega_1 \in S \text{ iff } ot(X) \in T\}$ being a projective stationary subset of $[\gamma]^{\aleph_0}$ for all stationary S, $T \subseteq \omega_1$ such that T is also co-stationary.

Besides its convenience, $\mathcal{S}$ appears to be quite a reasonable principle in itself: Let γ be a fixed uncountable ordinal and consider a countable subset of γ. Let us look for parameters describing X which are, moreover, countable ordinals. One such parameter is– if X is in the club of countable subsets whose intersection with ω_1 is a countable ordinal– $X \cap \omega_1$. Another countable parameter describing X is of course its order type. $X \cap \omega_1$ and $ot(X)$ appear to be the most obvious countable parameters describing a "typical" $X \in [\gamma]^{\aleph_0}$. It is then natural to ask for the largeness of the subsets of $[\gamma]^{\aleph_0}$ obtained by considering all $X \in [\gamma]^{\aleph_0}$ whose describing parameters are combined in several ways. The result of demanding that $\mathcal{X}^\gamma_{S,T}$ be large– where "large" means stationary– for large– i.e., stationary– S, $T \subseteq \omega_1$ is precisely $\mathcal{S}^\gamma$.

So far, $\mathcal{S}$ has been known to hold in a variety of situations. For example, Woodin proved that if κ is a measurable cardinal, then $\mathcal{S}^\kappa$. He also proved $\mathcal{S}^\gamma$ for $\gamma = (2^{2^{\aleph_1}})^+$ in [Wo], Lemma 10.95, from the assumption that NS_{ω_1} is precipitous. There he uses this to argue that, if BMM holds and either there is a measurable cardinal or NS_{ω_1} is precipitous, then ψ_{AC} also holds. Also, P. Larson and Woodin independently proved that if NS_{ω_1} is presaturated– actually, it suffices to assume that NS_{ω_1} is precipitous and that $\mathcal{P}(\omega_1)/NS_{\omega_1}$ forces $j(\omega_1^V) = \omega_2^V$–, then $\mathcal{S}^{\omega_2}$ (see [L] or the proof of [Wo], Theorem 5.14). I proved that if κ is an ω_1–Erdős cardinal, then there are unboundedly many inaccessible cardinals γ below κ such that $\mathcal{S}^\gamma$. Then, P. Welch lowered the large cardinal hypothesis in this result: $\mathcal{S}^\gamma$ holds for unboundedly many inaccessible $\gamma < \kappa$ in case κ is a cardinal with the partition property $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$ ([As-W]). Welch's proof can be adapted to show that $\mathcal{S}^\kappa$ also holds for such a cardinal κ.

The following observation– which was noticed before by P. Larson ([L])– will be crucial at a couple of places in this paper.

Lemma 1.4 *(Absoluteness Lemma) Let $M \subseteq N$ be two transitive models of enough set theory, let γ be an ordinal in M, let α and β be two countable* *ordinals in M such that $\alpha < \beta < \gamma$ and let $F : [\gamma]^{<\omega} \longrightarrow \gamma$ be a function in M. Then, in M there is a countable subset of γ closed under F whose intersection with ω_1^M is α and whose order type is β if and only if there is such a set in N.*

Proof: Letting $g : \omega \longrightarrow \alpha$ and $h : \omega \longrightarrow \beta$ be bijections in M, we have a poset $\mathbb{P}$ in M of finite approximations to a set $X \in [\gamma]^{\aleph_0}$ such that X is closed under F, $X \cap \omega_1^M = \alpha$ and $ot(X) = \beta$:

A condition p in $\mathbb{P}$ is some triple (n_p, x_p, h_p) such that

(a) n_p is a natural number,

(b) $x_p \in [\gamma]^{<\omega}$,

(c) $g``n_p \subseteq x_p \cap \omega_1^M \subseteq \alpha$, and

(d) $h_p : x_p \longrightarrow \beta$ is order preserving and $h``n_p \subseteq range(h_p)$.

$q \leq p$ iff $p = q$ or else

(e) $n_p < n_q$,

(f) $x_p \subseteq x_q$, $h_p \subseteq h_q$, and

(g) $F : [x_p]^{<\omega} \subseteq x_q$.

It is easily seen that $\mathbb{P}$ is well–founded if and only if there is no $X \in [\gamma]^{\aleph_0}$ closed under F such that $X \cap \omega_1^M = \alpha$ and $ot(X) = \beta$. Now, the absoluteness of well–foundedness between transitive models of set theory establishes the lemma. □

The rest of the paper is divided into four sections. In Section 2, I give several consequences of $\mathcal{S}$ showing that it implies the consistency of small large cardinals with ZFC. In Section 3 it is shown that $\mathcal{S}$ holds in various situations. The main result in Section 4 is that a natural strengthened version of $\mathcal{S}$ implies that the sharp of every real exists (on the other hand, from the results of Section 3 it follows that this version of $\mathcal{S}$ holds in $L[H_{\omega_2}]$ provided $H_{\omega_2}^\sharp$ exists). Section 5 is somewhat independent of the rest of the paper. It starts with a result showing that a certain principle of generic absoluteness for H_{ω_2} extending BMM implies CBP. Then, an optimal model separating $BSPFA$ from BMM is given. Finally, a certain strengthening of the negation of CBP is presented and it is shown that it and its negation follow from $\diamondsuit_{\omega_1}$ and BMM, respectively.

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2 Large cardinal consequences of $\mathcal{S}$

I will start by considering local versions of $\mathcal{S}$. The ordinals arising naturally from these weaker versions– i.e., those defined as the first ordinal γ such that the property holds for $[\gamma]^{\aleph_0}$– build a hierarchy.

Definition 2.1 *Let Y and Z be subsets of ω_1.*

(a) $\mathcal{S}_{Y,Z}$ is the statement that there is some ordinal $\gamma \geq \omega_1$ such that $\mathcal{X}^{\gamma}_{Y,Z}$ is a stationary subset of $[\gamma]^{\aleph_0}$.

(b) Assuming $\mathcal{S}_{Y,Z}$, $\gamma(Y,Z)$ is the first ordinal γ witnessing this.

Notice that for every $\gamma \geq \omega_1$ and every nonstationary $Y \subseteq \omega_1$, letting C be any club of ω_1 disjoint from Y, $\{X \in [\gamma]^{\aleph_0} : X \cap \omega_1 \in C\}$ is a club of $[\gamma]^{\aleph_0}$ disjoint from $\mathcal{X}^{\gamma}_{Y,Z}$ for every $Z \subseteq \omega_1$. Therefore, $\mathcal{X}^{\gamma}_{Y,Z}$ can only be stationary for stationary Y and– also trivially– for unbounded $Z \subseteq \omega_1$.

Fact 2.1 *Let S, Z and Z' be subsets of ω_1, S stationary, and suppose that Z' consists of limit points of Z. Assume $\mathcal{S}_{S,Z'}$. Then, $\mathcal{S}_{S,Z}$. Moreover, below $\gamma(S,Z')$ there are unboundedly many ordinals γ witnessing $\mathcal{S}_{S,Z}$.*

Proof: Assume otherwise. Then there is some $\gamma_0 < \gamma(S,Z') := \gamma$ such that $\mathcal{X}^{\xi}_{S,Z}$ is nonstationary for every ξ such that $\gamma_0 < \xi < \gamma$. Letting $\theta > |\gamma|$ be a cardinal, there is a witness $F_\xi : [\xi]^{<\omega} \longrightarrow \xi$ to this in H_θ for every such ξ. Then, since $\mathcal{X}^{\gamma}_{S,Z'}$ is stationary, there is some $N \preccurlyeq H_\theta$ containing γ_0 and F_ξ for each ξ in N between γ_0 and γ, and such that $N \cap \omega_1 \in S$ and $\alpha := ot(N \cap \gamma) \in Z'$. α is a limit point of Z, and so there is some ξ in $N \cap \gamma$ above γ_0 such that $ot(N \cap \xi) \in Z$. But then, $N \cap \xi$ is an element of $\mathcal{X}^{\xi}_{S,Z}$ closed under F_ξ, which is a contradiction. $\square$

Lemma 2.2 *Let S and Z be subsets of ω_1, S stationary, and let γ be an ordinal between ω_1 and ω_2. If $S \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} \gamma \in j(Z)$, then $\mathcal{X}^{\gamma}_{S,Z}$ is stationary and in fact there is a club $E \subseteq [\gamma]^{\aleph_0}$ such that every element of E whose intersection with ω_1 is in S has its order type in Z.*

Proof: Fix any surjective function $e : \omega_1 \longrightarrow \gamma$. The canonical function g for γ derived from e is such that for some club $C \subseteq \omega_1$, if $\nu \in C \cap S$, then $(e``\nu) \cap \omega_1 = \nu$ and $g(\nu) = ot(e``\nu) \in Z$. Therefore we can take $E = \{e``\nu : \nu \in C\}$. $\square$

Now the following fact, showing that some $\mathcal{X}^{\gamma}_{S,Z}$ may be stationary even if Z is nonstationary, holds trivially.

Fact 2.3 *Let S be a stationary subset of ω_1 such that $S \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} j(\check{\omega}_1) = \check{\omega}_2$ (i.e., S forces that the set of M–ordinals below ω_1^M– M being the generic* *ultrapower– is well–ordered in order type ω_2^V). Then, for every stationary $S' \subseteq S$ and every unbounded $Z \subseteq \omega_1$ there are unboundedly many ordinals $\gamma < \omega_2$ for which there is some stationary $S_\gamma \subseteq S'$ and some club $E_\gamma \subseteq [\gamma]^{\aleph_0}$ such that every element of E_γ whose intersection with ω_1 is in S_γ has its order type in Z.*

The conclusion of Fact 2.3– and in fact the apparently weaker version of it where Z is required to be a club– is actually equivalent to $S \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} j(\check{\omega}_1) = \check{\omega}_2$. To see this, assume without loss of generality that $S \Vdash j(\check{\omega}_1) \neq \check{\omega}_2$. Then there is some stationary $S' \subseteq S$ and some function $f : \omega_1 \longrightarrow \omega_1$ representing– modulo S'– an ordinal above all $\alpha < \omega_2^V$ in the generic ultrapower M (i.e., S' forces that the class of f in M is an ordinal of M lying above the class of every canonical function in V). Let Z be a club of ω_1 in V so that $min(Z \backslash (\nu + 1)) > f(\nu)$ for every $\nu \in S'$. Since the identity on ω_1 represents ω_1^V, S' forces that the least element of $j(Z)$ above ω_1^V lies above α for all $\alpha < \omega_2^V$. Hence, for every γ such that $\omega_1 < \gamma < \omega_2$, $\mathcal{X}^\gamma_{S',Z}$ is nonstationary.

There can be ordinals $\gamma > \omega_2$ such that $\mathcal{X}^\gamma_{S,Z}$ is stationary in $[\gamma]^{\aleph_0}$ for some stationary S and some nonstationary Z. For example, suppose we can collapse some cardinal λ to ω_1 while preserving ω_1 and forcing that there is some stationary subset S of ω_1 as in the hypothesis of Fact 2.3. Now, given any stationary $S' \subseteq \omega_1$ and any unbounded subset Z of ω_1, S' and Z in the ground model, if $S \cap S'$ is stationary, then there are unboundedly many $\gamma < \lambda$ such that $\mathcal{X}^\gamma_{S',Z}$ is a stationary subset of $[\gamma]^{\aleph_0}$ in the extension. But then, by the absoluteness Lemma 1.4, the same is true in the ground model. If κ is a so-called almost $<\omega_1$–Erdős cardinal, then by [Do-Ko], the Levy collapse turning κ into ω_2 with countable conditions forces $\Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} j(\check{\omega}_1) = \check{\omega}_2$ (see the comment after the proof of Fact 2.5). Hence, if κ is such a cardinal and S, Z are unbounded subsets of ω_1, S stationary, then there are unboundedly many ordinals $\gamma < \kappa$ such that $\mathcal{X}^\gamma_{S,Z}$ is a stationary subset of $[\gamma]^{\aleph_0}$.

Notice also that there is no ordinal $\gamma \geq \omega_1$ such that $\mathcal{X}^\gamma_{\omega_1,Z}$ is stationary for *all* unbounded $Z \subseteq \omega_1$, since otherwise $\{ot(X) : X \in E\}$ would be a co-bounded subset of ω_1 for every club $E \subseteq [\gamma]^{\aleph_0}$.

The strengthening of ψ_{AC} asserting that the conclusion of Definition 1.1 holds for (S, Z) whenever $S \subseteq \omega_1$ is a stationary and co-stationary subset of ω_1 and $Z \subseteq \omega_1$ is unbounded, is false. In fact, letting Z be, for example, the set of all countable successor ordinals, there obviously is no stationary and co-stationary $S \subseteq \omega_1$ such that the conclusion of Definition 1.1 holds for (S, Z), since being or not being a successor ordinal cannot be changed by forcing. However, I do not know the answer to the following:

Question 2.1 *Can there be a nonstationary $Z \subseteq \omega_1$ such that for every stationary and co-stationary $S \subseteq \omega_1$ there are a δ with $\omega_1 < \delta < \omega_2$, a surjection $e : \omega_1 \longrightarrow \delta$ and a club $C \subseteq \omega_1$ such that $S \cap C = \{\nu \in C : ot(e``\nu) \in Z\}$?*

Remark 1 There is no γ, $\omega_1 < \gamma < \omega_2$, such that $\mathcal{X}^\gamma_{\omega_1,C}$ is stationary for all clubs $C \subseteq \omega_1$– in other words, such that $\{ot(X) : X \in E\}$ is a stationary subset of ω_1 for every club $E \subseteq [\gamma]^{\aleph_0}$. In fact, there is no such γ such that $\bigcup_{X \in E}\{\nu :$

$\omega_1 \cap X \in \nu \leq ot(X)\}$ is a stationary subset of ω_1 for every club $E \subseteq [\gamma]^{\aleph_0}$. The reason is that such a γ can be covered by an $\subseteq$–increasing $\subseteq$–continuous sequence $(X_\nu : \nu < \omega_1)$ of countable subsets of γ such that $X_\nu \cap \omega_1 = \nu$ for all ν in some club $D \subseteq \omega_1$. Suppose $Y = \bigcup_{\nu \in D}(ot(X_\nu)+1)\backslash(\nu+1)$ were stationary. Then, since the mapping sending $\xi \in Y$ to the least ν such that $\nu < \xi \leq ot(X_\nu)$ is regressive, there would be some ν_0 such that $ot(X_{\nu_0})$ is uncountable, which is absurd.

$\mathcal{S}^{\omega_2}$ is quite strong, in the sense that it implies the existence of $0^\sharp$ and more. Consider the following forms of Chang's Conjecture:

Definition 2.2 *(a) $P_1(\omega_1)$ is the statement that for every club $E \subseteq [\omega_2]^{\aleph_0}$, $\{ot(X) : X \in E\}$ includes a club of ω_1.*

(b) $P_2(\omega_1)$ is the statement that for every club $E \subseteq [\omega_2]^{\aleph_0}$, $\{ot(X) : X \in E\}$ is a stationary subset of ω_1.

(c) $P_1(\omega_1)^+$ is the statement that for every club $E \subseteq [\omega_2]^{\aleph_0}$ there is a club $C \subseteq \omega_1$ such that $\{ot(X) : X \in E, X \cap \omega_1 = \nu\}$ includes a club for all $\nu \in C$.

$P_1(\omega_1)^+$ trivially implies $\mathcal{S}^{\omega_2}$. $P_1(\omega_1)$ and $P_2(\omega_1)$ are defined in [Do-Le], Definition 2.17. There it is shown that $P_1(\omega_1)$ and $P_2(\omega_2)$ are equiconsistent with the existence of a so-called nearly $<\omega_1$–Erdős cardinal. This large cardinal notion in particular implies that $0^\sharp$ exists. The following result follows trivially from the fact that the conclusion of $\mathcal{S}^{\omega_2}$ holds with $S = \omega_1$ for all stationary $T \subseteq \omega_1$.

Fact 2.4 *$\mathcal{S}^{\omega_2}$ implies $P_1(\omega_1)$.*

Welch has proved that $P_1(\omega_1)^+$ is also equiconsistent with a nearly $<\omega_1$–Erdős cardinal: the Levy collapse of such a cardinal to ω_2 forces $P_1(\omega_1)^+$. I do not know whether $P_1(\omega_1)^+$ is equivalent to $\mathcal{S}^{\omega_2}$.

P. Larson has proved that CBP implies $P_2(\omega_1)$. In fact, it implies the strong version of $P_2(\omega_1)$ asserting that for every club $E \subseteq [\omega_2]^{\aleph_0}$ there are club–many $\nu < \omega_1$ such that $\{ot(X) : X \in E, X \cap \omega_1 = \nu\}$ is stationary (see [As-W]).

Question 2.2 *Does CBP imply $\mathcal{S}^{\omega_2}$?*

Let me turn briefly to a related remark. Suppose there is some stationary S such that $S \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} j(\check{\omega}_1) = \check{\omega}_2$. Let G be a $\mathcal{P}(\omega_1)/NS_{\omega_1}$–generic filter over V containing S and let M and j be, respectively, the corresponding generic ultrapower and generic elementary embedding. If T is a stationary subset of ω_1 in V, then $j(T)$ is a subset of $\omega_1^M = \omega_2^V$ which of course is stationary in M. One may ask, however, whether $j(T)$ is also stationary in $V[G]$, or whether it intersects at least each club of ω_2 in V. As the following result shows, there are models of $\Vdash j(\check{\omega}_1) = \check{\omega}_2$ in which these questions are answered negatively.

Fact 2.5 *There is a club $D \subseteq \omega_2$ whose minimum is above ω_1 such that*

(a) if for every club $C \subseteq \omega_1$ there is some stationary $S \subseteq \omega_1$ such that $S \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} D \cap j(C) \neq \emptyset$, then $P_2(\omega_1)$ holds,

(b) if for every stationary $T \subseteq \omega_1$ there is some stationary $S \subseteq \omega_1$ such that $S \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} D \cap j(T) \neq \emptyset$, then $P_1(\omega_1)$ holds,

(c) if $\Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} D \cap j(C) \neq \emptyset$ for every club $C \subseteq \omega_1$, then $\{ot(X) : X \in E,\, X \cap \omega_1 \in S\}$ is stationary for every stationary $S \subseteq \omega_1$ and every club $E \subseteq [\omega_2]^{\aleph_0}$, and

(d) if $\Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} D \cap j(T) \neq \emptyset$ for every stationary $T \subseteq \omega_1$, then $\mathcal{S}^{\omega_2}$ holds.

Proof: Let $G : [\omega_2]^{<\omega} \longrightarrow \omega_2$ be a function such that every $X \in [\omega_2]^{\aleph_0}$ closed under G is the intersection of some $N \preccurlyeq (H_{\omega_3}, \in, <)$ with ω_2, where $<$ is a well–order of H_{ω_3}. Let D be the club of all $\alpha < \omega_2$ above ω_1 which are closed under G. Assume $P_2(\omega_2)$ fails. Let (F_0, C_0) be the $<$–least pair (F, C) such that F is a function from $[\omega_2]^{<\omega}$ into ω_2 and $C \subseteq \omega_1$ is a club disjoint from $\{ot(X) : X \in [\omega_2]^{\aleph_0}$ is closed under $F\}$. Suppose there is some stationary $S \subseteq \omega_1$ and some γ in D such that $S \Vdash \gamma \in j(C_0)$. Since γ is closed under G, there is some $X \in [\gamma]^{\aleph_0}$ which is closed under G and such that $X \cap \omega_1 \in S$ and $ot(X) \in C_0$. Then, there is some $N \preccurlyeq (H_{\omega_3}, \in, <)$ such that $N \cap \omega_2 = X$. Since F_0 is in N, X is closed under F_0, contradicting the fact that $ot(X)$ is in C_0. Similarly one proves that D witnesses (b), (c) and (d) as well. □

The assumption that $\mathcal{P}(\omega_1)/NS_{\omega_1}$ forces that $j(\check{\omega}_1) = \check{\omega}_2$ is equiconsistent with the existence of an almost $<\omega_1$–Erdős cardinal ([Do-Ko], Theorems C and D), whose consistency strength is strictly weaker than that of a nearly $<\omega_1$–Erdős cardinal. This shows the consistency of the existence of clubs $C \subseteq \omega_1$, $D \subseteq \omega_2$ such that $\Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}}$ "$j(\check{\omega}_1) = \check{\omega}_2$ and $j(C) \cap D = \emptyset$".

Welch has remarked that if $\kappa \geq \omega_2$ is a successor cardinal and $\mathcal{S}^{\kappa^+}$ holds, then $x^\dagger$ exists for every real x, and that this already appears in [Do-Ko], where the more general fact is proved that the existence of $x^\dagger$ for every real x follows from the existence of such a cardinal κ such that the weak Chang's Conjecture at κ holds, that is, such that for any first order structure $\mathcal{M}$ of countable language and with universe κ^+ there is some ordinal $\gamma < \kappa$ with the property that $\{ot(X) : X \preccurlyeq \mathcal{M},\, X \cap \kappa = \gamma,\, |X| = |\gamma|\}$ is unbounded in $|\gamma|^+$.

From $\mathcal{S}$ alone– actually a weak form of $\mathcal{S}$ suffices– we can infer that ω_1 is weakly compact in $L[a]$ for every real a.

Fact 2.6 *Suppose there is an ordinal $\gamma > \omega_1$ such that $\mathcal{X}^{\gamma}_{\omega_1, C}$ is a stationary subset of $[\gamma]^{\aleph_0}$ for every club $C \subseteq \omega_1$. More generally, suppose that for every club $E \subseteq [\gamma]^{\aleph_0}$, $\bigcup_{X \in E}\{\nu : \omega_1 \cap X \in \nu \leq ot(X)\}$ is a stationary subset of ω_1. Then, ω_1 is a weakly compact limit of weakly compact cardinals in $L[a]$ for every real a.*

Proof: A previous step will be to prove that ω_1 is inaccessible to reals.

Claim 2.7 *ω_1 is inaccessible in $L[r]$ for every real r.*

Proof: Otherwise there is a real r such that $\omega_1 = \omega_1^{L[r]}$. Let C be the club of ω_1 consisting of all ν such that ν' is countable in $L_\nu[r]$ for every $\nu' < \nu$. Let $E = \{X \in [\gamma]^{\aleph_0} : X = \gamma \cap N$ for some countable $N \preccurlyeq L_\gamma[r], \omega_1 \in N\}$. E includes a club of $[\gamma]^{\aleph_0}$, and so there is some $N \preccurlyeq L_\gamma[r]$ containing ω_1 such that $\alpha := N \cap \omega_1$ is a countable ordinal and such that there is some $\nu \in C$, $\alpha < \nu \leq \beta$, where $L_\beta[r]$ is the transitive collapse of N. Letting π be the collapsing function, $\alpha = \pi(\omega_1)$. But since ν is in C, α must be countable in $L_\nu[r]$ and therefore also in $L_\beta[r]$, contrary to π being an isomorphism. $\Box$

Now let C be the set of all ordinals ν in $\kappa := \omega_1$ such that for every $\nu' < \nu$ and every tree $T \subseteq L_{\nu'}[a]$ in $L[a]$, T has a cofinal branch in $L_\nu[a]$ in case it has such a branch in $L[a]$. Since κ is inaccessible in $L[a]$, C is a club of κ. Fix a κ–tree T in $L[a]$. Let χ be some cardinal above γ. We want to prove that in $L[a]$ there is some cofinal branch through T. By our assumption there is some $N \preccurlyeq L_\chi[a]$ containing T and some $\nu \in C$ such that the transitive collapse of $N \cap L_\gamma[a]$ is $L_\beta[a]$ for some β such that $\alpha < \nu \leq \beta$, where $\alpha := N \cap \kappa$, and such that, letting π be the collapsing function, $\pi(T)$ is the union of all levels of T before α. Since the α-th level of T is nonempty, in $L[a]$ there is some cofinal branch through $\pi(T)$. But then there is some cofinal branch through $\pi(T)$ in $L_\nu[a]$, and obviously also in $L_\beta[a]$. It follows that in $L_\chi[a]$ there is a cofinal branch through T. This shows that κ is weakly compact in $L[a]$. Now we can prove by a similar argument that κ is also a limit of weakly compact cardinals in $L[a]$. $\Box$

Welch has pointed out that from the hypothesis of Fact 2.6 it actually even follows, given any real a, that ω_1 is a completely ineffable[7] cardinal in $L[a]$ and that in V there is a club of ω_1 consisting of completely ineffable cardinals in $L[a]$. It should be also noted that the assumption that there is some $\gamma > \omega_1$ such that $\mathcal{X}^\gamma_{\omega_1, C}$ is a stationary subset of $[\gamma]^{\aleph_0}$ for all clubs $C \subseteq \omega_1$ does not imply the existence of $0^\sharp$ (see Theorem 4.3).

The following fact can be easily proved by an argument as in the above proof.

Fact 2.8 *Suppose there is an ordinal $\gamma > \omega_1$ such that $\bigcup_{X \in E}\{\nu : \omega_1 \cap X \in \nu \leq ot(X)\}$ is a stationary subset of ω_1 for every club $E \subseteq [\gamma]^{\aleph_0}$ (resp., $\bigcup_{X \in E}\{\nu : \omega_1 \cap X \in S, \omega_1 \cap X < \nu \leq ot(X)\}$ is a stationary subset of ω_1 for every stationary $S \subseteq \omega_1$ and every club $E \subseteq [\gamma]^{\aleph_0}$). Then, given any real a and any countable (in the universe) set $\mathcal{Y}$ of stationary subsets of $\kappa := \omega_1$ in $L[a]$ there are stationarily many (resp., club–many) $\alpha < \kappa$ such that in $L[a]$ all $S \in \mathcal{Y}$ reflect at α.*

[7] A regular cardinal κ is completely ineffable iff there is an $\supseteq$–closed collection $\mathcal{A}$ of stationary subsets of κ such that given any $S \in \mathcal{A}$ and any $\chi : [S]^2 \longrightarrow 2$ there is an $S' \subseteq S$, $S' \in \mathcal{A}$, such that $\chi \restriction [S']^2$ is constant (see [A-H-K-Z]).

Similarly it can be proved that if there is an ordinal $\gamma > \omega_1$ such that $\mathcal{X}^{\gamma}_{S,C}$ is a stationary subset of $[\gamma]^{\aleph_0}$ for every stationary $S \subseteq \omega_1$ and every club $C \subseteq \omega_1$, then, for every real a, ω_1 is weakly compact in $L[a]$ and there is a club of ω_1 consisting of weakly compact cardinals of $L[a]$. This is just a particular case of Lemma 4.5.

3 Some exemplifications of $\mathcal{S}$

The first theorem in this section gives as a corollary that the existence of the sharp of some set of ordinals coding H_{ω_2} implies the consistency of $\mathcal{S}$ with ZFC. This theorem is the best upper bound I know for the consistency strength of $\mathcal{S}$.

Theorem 3.1 *Let A be a set of ordinals such that $\omega_1^{L[A]} = \omega_1$ and such that every stationary subset of ω_1 in $L[A]$ is stationary in the universe. If $A^\sharp$ exists, then $\mathcal{S}^\gamma$ holds in $L[A]$ for every Silver indiscernible γ for $L[A]$. In fact, the following holds in $L[A]$ for every such γ: For every club $E \subseteq [\gamma]^{\aleph_0}$ there is a club $C \subseteq \omega_1$ such that $\{ot(X) : X \in E, X \cap \omega_1 = \nu\}$ includes a club of ω_1 for every $\nu \in C$.*

Proof: Let I be the class of Silver indiscernibles for $L[A]$ and let $(\iota_\xi : \xi \leq \omega_1 + \omega)$ be the strictly increasing enumeration of the $\omega_1 + \omega + 1$ first elements of I. By indiscernibility, it will suffice to show that the conclusion holds in $L[A]$ for $\gamma := \iota_{\omega_1}$. Working towards a contradiction, suppose in $L[A]$ there are a stationary $S \subseteq \omega_1$ and $F : [\gamma]^{<\omega} \longrightarrow \gamma$ with the property that for every $\alpha \in S$ there is a stationary $T_\alpha \subseteq \omega_1$ in $L[A]$ such that there is no $X \in [\gamma]^{\aleph_0}$ in $L[A]$ which is closed under F and such that $X \cap \omega_1 = \alpha$ and $ot(X) \in T_\alpha$. In the universe, let θ be a cardinal above γ and let E be the club of all countable $N \preccurlyeq H_\theta$ containing F and A. Since S is really stationary, there is some $N \in E$ such that $\alpha := N \cap \omega_1$ is in S. Now we build a strictly $\subseteq$–increasing and $\subseteq$–continuous sequence $\langle X_\nu : \nu < \omega_1\rangle$ of countable subsets of $L_\theta[A]$ such that

(a) $X_0 = N \cap L_{\iota_{\omega_1+\omega}}[A]$,

(b) for every ν, $X_{\nu+1}$ is the closure, under Skolem terms for $L[A]$ involving only constants for elements of $A \cap X_0$, of $X_\nu \cup \{\bar{\iota}_\nu\}$, where $\bar{\iota}_\nu$ is the least $\iota \in I \cap \gamma$ above $X_\nu \cap \gamma$.

The choice of $\bar{\iota}_\nu$ in (b) makes sense since γ is a limit of uncountable cofinality of elements of I and all X_ν are countable. Notice also that, since N contains A, it also contains $A^\sharp$, and therefore all X_ν contain all ι_{ω_1+n}.

Claim 3.2 *For every $\nu < \omega_1$, $X_\nu \cap \gamma$ is a proper initial segment of $X_{\nu+1} \cap \gamma$.*

Proof: This is a standard argument using the remarkability property of the Silver indiscernibles for $L[A]$ which involves the ω–length tail of indiscernibles above γ. Let ξ be some ordinal in $X_{\nu+1} \cap \sup(X_\nu \cap \gamma)$. Then there is some Skolen term t for $L[A]$ mentioning only constants for members of $A \cap X_0$ and some indiscernibles $\iota_{\zeta_0} < \ldots < \iota_{\zeta_{k-1}} < \bar{\iota}_\nu$ and $\iota_{\omega_1+n_0} < \ldots < \iota_{\omega_1+n_{l-1}}$ (all of them in X_ν) such that

$$\xi = t^{L[A]}(\iota_{\zeta_0}, \ldots \iota_{\zeta_{k-1}}, \bar{\iota}_\nu, \iota_{\omega_1+n_0}, \ldots \iota_{\omega_1+n_{l-1}})$$

But, since ξ is below $\bar{\iota}_\nu$, by the remarkability property it is also equal to $t^{L[A]}(\iota_{\zeta_0}, \ldots \iota_{\zeta_{k-1}}, \iota_{\omega_1}, \ldots \iota_{\omega_1+l})$, which belongs to X_ν. □

Since $(ot(X_\nu \cap \gamma) \,:\, \nu < \omega_1)$ is a strictly increasing and continuous sequence of countable ordinals and T_α is stationary, there is some ν_0 such that $\beta := ot(X_{\nu_0} \cap \gamma) \in T_\alpha$. Since $F \in X_{\nu_0}$, $X := X_{\nu_0} \cap \gamma$ is a countable subset of γ closed under F. But then, since α and β are countable ordinals in $L[A]$, by Lemma 1.4 it follows that in $L[A]$ there is some $Y \in [\gamma]^{\aleph_0}$ closed under F such that $Y \cap \omega_1 = \alpha \in S$ and $ot(Y) = \beta \in T_\alpha$, which is a contradiction. □

By simple indiscernibility arguments it follows that, under the hypotheses of Theorem 3.1, for every Silver indiscernible δ for $L[A]$, the set of ordinals $\gamma < \delta$ such that the conclusion of Theorem 3.1 holds for γ is a stationary subset of δ.

Welch has observed that if A codes H_{ω_2} and $A^\sharp$ exists, then actually $\gamma \longrightarrow (<\!\omega_1)^{<\omega}_{2^{\omega_1}}$ holds in $L[A]$ for every Silver indiscernible γ for $L[A]$. This is due to the fact that all types of countable sequences of indiscernibles are in $L[A]$: Using sequences of Silver indiscernibles for $L[A]$ one can construct in the universe homogeneous sets of the kind specified in the above Erdős property for any structure M in $L[A]$ on some large indiscernible κ. But then, since the model-theoretic type τ of such sequences is in $L[A]$, by an absoluteness argument as in the proof of Theorem 3.1 one can find in $L[A]$ sequences of indiscernibles of arbitrarily large countable order type and with model-theoretic type τ.

Question 3.1 *Does the conclusion for γ in Theorem 3.1 follow from $\mathcal{S}^\gamma$?*

The following result involves a certain game on $\mathcal{I}^+$ corresponding to a given ideal $\mathcal{I}$ over some cardinal, as defined in [G-J-M].

Definition 3.1 *Let κ be an infinite cardinal and let $\mathcal{I}$ be an ideal over κ. $G_\mathcal{I}$ is the following ω–length game with two players I and II, I moving first: I and II alternately choose $\mathcal{I}$–positive subsets S_i of κ such that $S_{i+1} \subseteq S_i$ for all i. I wins if and only if $\bigcap_i S_i$ is empty.*

In [G-J-M] it is shown, for example, that player I fails to have a winning strategy in $G_\mathcal{I}$ if and only if $\mathcal{I}$ is precipitous, that if $\kappa \leq 2^{\aleph_0}$, then player II does not have any winning strategy for $G_\mathcal{I}$, and that player II never has a winning strategy for G_{NS_κ}, where $\kappa \geq \omega_1$ is any regular cardinal.

Also, given κ and $\mathcal{I}$ as in Definition 3.1, let $G'_\mathcal{I}$ be a game exactly as $G_\mathcal{I}$, except that player II moves first. Obviously, if player II has a winning strategy in $G_\mathcal{I}$, then she has one for $G'_\mathcal{I}$.

Simple variants of the proofs in [G-J-M] show that the negative results for player II in the games $G_{\mathcal{I}}$ mentioned above also hold for her in the corresponding games $G'_{\mathcal{I}}$. Also, player I has a winning strategy in $G'_{\mathcal{I}}$ if and only if $\Vdash_{\mathcal{I}^+}$ "The generic ultrapower is ill–founded".

The proof of the following result can be found in [As2].

Lemma 3.3 *Suppose κ is a regular cardinal carrying a κ–complete ideal $\mathcal{I}$ such that player II has a winning strategy in $G'_{\mathcal{I}}$. Then there is a normal κ–complete ideal $\mathcal{J}$ over κ such that player II has a winning strategy in $G'_{\mathcal{J}}$.*

Consider the following generalization of the Strong Chang's Conjecture.

Definition 3.2 *Given an ordinal γ of uncountable cofinality, $CC^*_{\omega_1}(\gamma)$ is the statement that given any club $E \subseteq [\gamma]^{\aleph_0}$ there is a club E' of $[\gamma]^{\aleph_0}$, $E' \subseteq E$ with the property that for every $X \in E'$ there is some $Y \in E'$ such that $X \cap \omega_1 = Y \cap \omega_1$ and X is a proper initial segment of Y.*

Obviously, $CC^*_{\omega_1}(\gamma)$ for γ as in Definition 3.2 implies that every first order structure $\mathcal{M}$ with a countable language and universe γ has an elementary substructure $\mathcal{N}$ of size $\aleph_1$ such that $\mathcal{N} \cap \omega_1$ is countable, and also $\mathcal{S}^\gamma$. In fact it also implies the strong form of $\mathcal{S}^\gamma$ saying that for every club $E \subseteq [\gamma]^{\aleph_0}$ there is a club $C \subseteq \omega_1$ such that $\{ot(X) : X \in E, X \cap \omega_1 = \nu\}$ includes a club for every $\nu \in C$.

Theorem 3.4 *Suppose κ is a cardinal carrying a κ–complete ideal $\mathcal{I}$ such that player II has a winning strategy in $G'_{\mathcal{I}}$. Then, $CC^*_{\omega_1}(\kappa)$.*

The proof of Theorem 3.4, which can be found in [As2], generalizes Woodin's proof that $\mathcal{S}^\kappa$ holds for a measurable κ.

Corollary 3.5 *Suppose there is some cardinal κ carrying a κ–complete ideal $\mathcal{I}$ for which the second player has a winning strategy in $G_{\mathcal{I}}$. If $BPSR$ holds, then so does ψ_{AC}.*

In [G-J-M] it is proved that, after forcing with the Levy collapse of a measurable cardinal to ω_2, ω_2 carries a normal ω_2–complete ideal $\mathcal{I}$ such that player II has a winning strategy in $G_{\mathcal{I}}$. By Theorem 3.4, $\mathcal{S}^{\omega_2}$ holds in this model. However, by the observation of Welch mentioned after Fact 2.4, a much weaker hypothesis suffices to prove the consistency of $\mathcal{S}^{\omega_2}$.

Corollary 3.6 *Suppose γ is a measurable cardinal. Then, $Coll(\omega_2, {<}\gamma^+)$ forces that γ is an ordinal such that $\omega_2 < \gamma < \omega_3$, $cf(\gamma) = \omega_2$ and $\mathcal{S}^\gamma$.*

Proof: Generalizing the proof of [G-J-M], we deduce that $Coll(\omega_2, {<}\gamma)$ forces that there is a normal γ–complete ideal $\mathcal{I}$ over $\omega_3 = \gamma$ such that player II has a winning strategy in $G_{\mathcal{I}}$. Let G be $Coll(\omega_2, {<}\gamma)$–generic over V and let $\mathbb{Q} = Coll(\omega_2, {<}\gamma^+)/G$ in $V[G]$. Work in $V[G]$. We will derive a contradiction from the assumption that there is a $\mathbb{Q}$–term $\dot{F}$ for a function from $[\gamma]^{<\omega}$ into γ, stationary sets S, $T \subseteq \omega_1$ and a condition p in $\mathbb{Q}$ forcing that there is no countable subset X of γ closed under $\dot{F}$ such that $X \cap \omega_1 \in S$ and $ot(X) \in T$ (this is enough since $Coll(\omega_2, {<}\gamma)$ does not add new subsets of ω_1). Letting θ be a large enough cardinal, we obtain from Theorem 3.4 that there is a countable elementary substructure N of H_θ containing p and $\dot{F}$ such that, letting $X = N \cap \gamma$, $X \cap \omega_1 \in S$ and $ot(X) \in T$. Let q be an $(N, \mathbb{Q})$–generic condition extending p. Then, q forces that $X \in [\gamma]^{\aleph_0}$ is closed under $\dot{F}$, contrary to the choice of p. □

Of course, if $\mathcal{S}^{\omega_2}$ fails in the ground model– and this can certainly be always forced by small forcing (see for example [J-S])–, by the Absoluteness Lemma it fails in the $V^{Coll(\omega_2, <\kappa^+)}$ of Corollary 3.6. Hence, it is consistent that there is an ordinal γ such that $\mathcal{S}^\gamma$ holds but $\mathcal{S}^{|\gamma|}$ fails.

Question 3.2 *Can there be an ordinal γ of countable cofinality such that $\mathcal{S}^\gamma$? Of cofinality ω_1?*

The following game appears in [S], XII, 2. It was first considered by Galvin.

Definition 3.3 *Given a cardinal γ, $G_\omega(\gamma, \omega_1)$ is the following game of length ω with two players I and II. At stage n, player I plays a function $F_n : \gamma \longrightarrow \omega_1$ and then player II plays a countable ordinal ν_n. Player II wins if and only if $|\{\xi < \gamma \,:\, F_n(\xi) < \sup_k \nu_k$ for all $n < \omega\}| = \gamma$.*

The following is established by the same argument as in [S], XII, Theorem 2.5 (2).

Fact 3.7 *(Shelah) Let γ be an uncountable regular cardinal. If player II has a winning strategy in $G_\omega(\gamma, \omega_1)$, then $CC^*_{\omega_1}(\gamma)$ holds.*

The first part of the following result is proved in [S], XII, Theorem 2.6.

Fact 3.8 *Let $\gamma \geq \omega_2$ be a regular cardinal and suppose there is a semiproper forcing notion $\mathbb{P}$ such that $\Vdash_{\mathbb{P}} cf(\check{\gamma}) = \omega$. Then, II has a winning strategy in $G_\omega(\gamma, \omega_1)$. In particular, $CC^*_{\omega_1}(\gamma)$ holds and if, in addition, $BPSR$ holds, then ψ_{AC} also does.*

This seems to be a good place to insert the following information on the consistency strength of ψ_{AC}.

Theorem 3.9 *ψ_{AC} is equiconsistent with the existence of an inaccessible limit of measurable cardinals.*

Proof: ψ_{AC} implies CBP ([As-W]), which in turn implies that there is an inner model with an inaccessible limit of measurable cardinals ([D-Do]).

As to the other direction, suppose $(\lambda_\alpha \,:\, \alpha < \kappa)$ is a strictly increasing sequence of measurable cardinals such that $\sup_{\alpha<\kappa} \lambda_\alpha = \kappa$ is inaccessible. Let

$f : \kappa \longrightarrow \kappa \times \kappa$ be a typical bookkeeping function– i.e., f is surjective and, if $f(\alpha) = \langle \beta, \gamma \rangle$, then $\beta \leq \alpha$– and perform the following RCS–iteration $(\mathbb{P}_\alpha : \alpha \leq \kappa)$ based on $(\dot{\mathbb{Q}}_\alpha : \alpha < \kappa)$: Suppose $\mathbb{P}_\alpha$ has been defined and fix an enumeration $(\langle \dot{S}^\alpha_\gamma, \dot{T}^\alpha_\gamma \rangle : \gamma < \kappa)$ such that for all $\mathbb{P}_\alpha$–names $\dot{S}$ and $\dot{T}$ for subsets of ω_1 there is some γ such that $\Vdash_\alpha$ "$\dot{S} = \dot{S}^\alpha_\gamma$ and $\dot{T} = \dot{T}^\alpha_\gamma$". Suppose $f(\alpha) = \langle \beta, \gamma \rangle$. Then let $\dot{\mathbb{Q}}_\alpha$ be such that $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is the trivial forcing unless $\dot{S}^\beta_\gamma$ and $\dot{T}^\beta_\gamma$ are stationary and co-stationary subsets of ω_1, in which case $\dot{\mathbb{Q}}_\alpha$ is the standard poset for shooting a club through $\overline{\mathcal{X}}^{\lambda_{\alpha+1}}_{\dot{S}^\beta_\gamma, \dot{T}^\beta_\gamma} := \{X \in [\lambda_{\alpha+1}]^{\aleph_0} : X \cap \omega_1 \in \dot{S}^\beta_\gamma \text{ iff } ot(X) \in \dot{T}^\beta_\gamma\}$ with countable conditions.[8] In this case, $\dot{\mathbb{Q}}_\alpha$ forces that $\lambda_{\alpha+1}$ is an ordinal less than ω_2 witnessing the conclusion of ψ_{AC} for $(\dot{S}^\beta_\gamma, \dot{T}^\beta_\gamma)$.

By induction, we get that $|\mathbb{P}_\alpha| < \lambda_{\alpha+1}$ for all $\alpha < \kappa$. Hence, after forcing with $\mathbb{P}_\alpha$, $\lambda_{\alpha+1}$ remains measurable and so, by Theorem 3.4, $CC^*_{\omega_1}(\lambda_{\alpha+1})$ holds. In particular, if $f(\alpha) = \langle \beta, \gamma \rangle$ and $\dot{S}^\beta_\gamma$ and $\dot{T}^\beta_\gamma$ are both stationary and co-stationary subsets of ω_1, then given any large enough cardinal θ there are club–many countable elementary substructures N of H_θ for which there is a countable $N' \preccurlyeq H_\theta$ such that $N \subseteq N'$, $N \cap \lambda_{\alpha+1}$ is an initial segment of $N' \cap \lambda_{\alpha+1}$ and $N' \cap \lambda_{\alpha+1} \in \overline{\mathcal{X}}^{\lambda_{\alpha+1}}_{\dot{S}^\beta_\gamma, \dot{T}^\beta_\gamma}$. Since any $(N', \dot{\mathbb{Q}}_\alpha)$–generic condition is $(N, \dot{\mathbb{Q}}_\alpha)$–semigeneric, it follows that $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ is semiproper". By the general theory of RCS–iterations, $\mathbb{P}_\kappa$ is semiproper and, since κ is inaccessible and $|\mathbb{P}_\alpha| < \kappa$ for all $\alpha < \kappa$, it is also κ–c.c. Using this, a standard argument shows that ψ_{AC} holds after forcing with $\mathbb{P}_\kappa$. $\square$

4 Generalizations of $\mathcal{S}$

In this section I consider versions of $\mathcal{S}$ involving more than two parameters.

Definition 4.1 *Let α be a countable ordinal and let $(\gamma_i)_{i<\alpha}$ be a one–to–one sequence of uncountable ordinals. Then, $\mathcal{S}^{(\gamma_i)_{i<\alpha}}$ is the statement that for every sequence $(S_i)_{i<\alpha}$ of stationary subsets of ω_1, $\{X \in [\sup_{i<\alpha} \gamma_i]^{\aleph_0} : ot(X \cap \gamma_i) \in S_i \text{ for all } i < \alpha\}$ is a stationary subset of $[\sup_{i<\alpha} \gamma_i]^{\aleph_0}$.*

$\mathcal{S}(\alpha)$ is the statement that there is a one–to–one sequence $(\gamma_i)_{i<\alpha}$ of uncountable ordinals such that $\mathcal{S}^{(\gamma_i)_{i<\alpha}}$

Hence, for every uncountable ordinal γ, $\mathcal{S}^\gamma$ is the same as $\mathcal{S}^{\langle \omega_1, \gamma \rangle}$.

By slightly modifying the proof of Theorem 3.1, it can be easily seen that a strengthening of $\forall n < \omega\, \mathcal{S}(n)$ holds in the $L[A]$ from that theorem.

[8] Of course, here we are identifying the $\mathbb{P}_\beta$–names $\dot{S}^\beta_\gamma$, $\dot{T}^\beta_\gamma$ with corresponding $\mathbb{P}_\alpha$–names in a natural way (note that $\beta \leq \alpha$).

Theorem 4.1 *Let A be a set of ordinals such that $\omega_1^{L[A]} = \omega_1$ and such that every stationary subset of ω_1 in $L[A]$ is stationary in the universe. If $A^\sharp$ exists, then ZFC + 'There is an inaccessible cardinal κ with the property that for every $n < \omega$ there are inaccessible cardinals $\kappa_0 < \ldots \kappa_{n-1} < \kappa$ such that $\mathcal{S}^{\langle \omega_1, \kappa_0, \ldots \kappa_{n-1}, \kappa \rangle}$' holds in $L[A]$.*

Question 4.1 *Does $\mathcal{S}(\omega)$ hold in the $L[A]$ of Theorem 3.1?*

The strongest– not obviously absurd– version of $\mathcal{S}$ holds in case there are $\aleph_1$–many measurable cardinals, for example.

Theorem 4.2 *Suppose $(\gamma_\xi)_{\xi < \omega_1}$ is a sequence of uncountable regular cardinals such that $CC^*_{\omega_1}(\gamma_\xi)$ holds for each ξ. Then, for every sequence $(S_\xi)_{\xi < \omega_1}$ of stationary subsets of ω_1, every large enough cardinal θ and every countable elementary substructure N of H_θ containing $(\gamma_\xi)_{\xi < \omega_1}$ there is some countable $N' \preccurlyeq H_\theta$ such that $N \subseteq N'$ and, for every $\xi \in N \cap \omega_1$, $N \cap \gamma_\xi$ is an initial segment of $N' \cap \gamma_\xi$ and $ot(N' \cap \gamma_\xi) \in S_\xi$.*

It turns out that the existence of $0^\sharp$ suffices to prove the consistency of the statement resulting from replacing, in the conclusion of Theorem 4.1, stationary subsets of ω_1 by clubs.

Theorem 4.3 *Suppose $0^\sharp$ exists. Then, for every uncountable regular cardinal λ in V, the Levy collapse $Coll(\omega, {<}\lambda)$ forces over L that there is a set modelling ZFC + 'There is an inaccessible κ such that for all $n < \omega$ there are inaccessible $\kappa_0 < \ldots < \kappa_{n-1} < \kappa_n := \kappa$ such that $\{X \in [\kappa]^{\aleph_0} : X \cap \omega_1 \in C, ot(X \cap \kappa_i) \in C$ for all $i \leq n\}$ is a stationary subset of $[\kappa]^{\aleph_0}$ for each club $C \subseteq \omega_1$'.*

Proof: It suffices to prove the conclusion in $L[G]$ for an arbitrary G which is $Coll(\omega, {<}\lambda)$–generic over V. Notice that $\omega_1^{L[G]} = \omega_1^{V[G]} = \lambda$. Let I be the class of Silver indiscernibles for L.

Claim 4.4 *In $V[G]$, $G^\sharp$ exists. Moreover, every ordinal in $I \setminus (\lambda + 1)$ is a Silver indiscernible for $L[G]$.*

Proof: Fix a formula $\varphi(x_0, \ldots x_{n-1})$ and Silver indiscernibles $\iota_0 < \ldots < \iota_{n-1}$ and $\iota'_0 < \ldots < \iota'_{n-1}$ above λ. Suppose $L[G] \models \varphi(\iota_0, \ldots \iota_{n-1})$, Then there is some $p \in G$ forcing this over L. Since $Coll(\omega, {<}\lambda)$ is definable from λ over L, and since there are indiscernibles $\bar{\iota}_0, \ldots \bar{\iota}_{k-1}$ below $\lambda + 1$ such that $p = t^L(\bar{\iota}_0, \ldots \bar{\iota}_{k-1})$ for some Skolem term t, p forces $\varphi(\iota_0, \ldots \iota_{n-1})$ over L if and only if p forces $\varphi(\iota'_0, \ldots \iota'_{n-1})$ over L. Now, $(I \cap \lambda^+) \setminus (\lambda + 1)$ is an uncountable sequence of indiscernibles for $L_{\lambda^+}[G]$. Therefore, $G^\sharp$ exists and every Silver indiscernible for L above λ is a Silver indiscernible for $L[G]$. $\Box$

Let C be a club of λ in $L[G]$, let $n < \omega$, let $\gamma_0 < \ldots \gamma_{n-1} < \overline{\gamma}$ be sufficiently large Silver indiscernibles for $L[G]$ and let $F : [\overline{\gamma}]^{<\omega} \longrightarrow \overline{\gamma}$ be a function in $L[G]$. Since C is a club of ω_1 in $V[G]$, by an argument similar to the proof of Theorem

3.1 in which $L[G]$ plays the role of $L[A]$ and $V[G]$ plays the role of V, we obtain a countable subset of $\overline{\gamma}$ in $L[G]$ closed under F and such that all $ot(X \cap \gamma_i)$ and $ot(X)$ are in C. $\square$

Finally, I will show show that $\forall n < \omega\, \mathcal{S}(n)$ suffices to imply the existence of the sharp of every real. The proof of Lemma 4.5 will be used in the proof of the more general result.

Lemma 4.5 *Suppose $\mathcal{S}(2)$ holds. More generally, suppose there are uncountable ordinals $\gamma_0 < \gamma$ such that $\{X \in [\gamma]^{\aleph_0} : ot(X \cap \gamma_0) \in S$ and $ot(X) \in C\}$ is a stationary subset of $[\gamma]^{\aleph_0}$ for every stationary $S \subseteq \omega_1$ and every club $C \subseteq \omega_1$. Then, for every real a there is a club $D \subseteq \omega_1$ such that for every n, every formula $\varphi(x_0, \dots x_n)$, all $\alpha_0 < \dots \alpha_{n-1} < \omega_1$, and all $\alpha \in D$, $\alpha_{n-1} < \alpha$,*

$$L_{\omega_1}[a] \models \varphi(\alpha_0, \dots \alpha_{n-1}, \alpha) \textit{ iff } L_\gamma[a] \models \varphi(\alpha_0, \dots \alpha_{n-1}, \gamma_0)$$

Proof: Let $\kappa = \omega_1$ and let $C = \{\nu < \kappa : L_\nu[a] \preccurlyeq L_\kappa[a]\}$. C is a club of κ. Fix a formula $\varphi(x_0, \dots x_n)$ and $\alpha_0 < \dots < \alpha_{n-1} < \kappa$. Fix also a stationary $S \subseteq \omega_1$. By our assumption, there is some $X \preccurlyeq L_\gamma[a]$ containing $\alpha_0, \dots \alpha_{n-1}$ and γ_0 such that, letting $L_\nu[a]$ and α be the transitive collapse of X and the image of γ_0 under the collapsing map, respectively, $\alpha \in S$ and $\nu \in C$. Then, $L_\gamma[a] \models \varphi(\alpha_0, \dots \alpha_{n-1}, \gamma_0)$ iff $L_\nu[a] \models \varphi(\alpha_0, \dots \alpha_{n-1}, \alpha)$ iff $L_\kappa[a] \models \varphi(\alpha_0, \dots \alpha_{n-1}, \alpha)$. Since S was an arbitrary stationary set, there is a club $D^\varphi_{\alpha_0, \dots \alpha_{n-1}}$ of ω_1 such that for all $\alpha \in D^\varphi_{\alpha_0, \dots \alpha_{n-1}}$, $L_\kappa[a] \models \varphi(\alpha_0, \dots \alpha_{n-1}, \alpha)$ iff $L_\gamma[a] \models \varphi(\alpha_0, \dots \alpha_{n-1}, \gamma_0)$. For every $\beta < \omega_1$ let $D^{\varphi, \beta} = \bigcap\{D^\varphi_{\alpha_0, \dots \alpha_{n-1}} : \alpha_0 < \dots < \alpha_{n-1} < \beta\}$, and let $D^\varphi = \Delta_{\beta < \omega_1} D^{\varphi, \beta}$. Finally, the intersection of all D^φ is a club of ω_1 with the desired property. $\square$

Question 4.2 *Does the existence of $0^\sharp$ follow from $\mathcal{S}(2)$? Does it follow from the existence of a club $D \subseteq \omega_1$ such that for every n, every formula $\varphi(x_0, \dots x_n)$, and all $\alpha_0 < \dots \alpha_{n-1} < \alpha_n < \alpha_{n+1}$ in D,*

$$L_{\omega_1} \models \varphi(\alpha_0, \dots \alpha_{n-1}, \alpha_n) \textit{ iff } L_{\omega_1} \models \varphi(\alpha_0, \dots \alpha_{n-1}, \alpha_{n+1})\,?$$

Welch has answered the second half of Question 4.2 negatively in case n is fixed and has observed that, in this case, the corresponding statement can be actually forced over L: Suppose $n < \omega$ and κ is a cardinal with the property that for every partition $\chi : [\kappa]^n \longrightarrow \mathcal{P}(\omega)$ there is a stationary $A \subseteq \kappa$ which is homogeneous for χ. In particular there is a stationary $A \subseteq \kappa$ such that $L_\kappa \models \varphi(\alpha_0, \dots \alpha_{n-1})$ iff $L_\kappa \models \varphi(\alpha'_0, \dots \alpha'_{n-1})$ for all $\alpha_0 < \dots < \alpha_{n-1}$ and $\alpha'_0 < \dots < \alpha'_{n-1}$ in A and for every formula $\varphi(x_0, \dots x_{n-1})$. A remains stationary after Levy collapsing κ to become ω_1. Finally, shooting a club through A yields a club D of ω_1 of L_{ω_1}–indiscernibles for formulas with n free variables.[9]

[9]Note that the above argument shows that the existence of a club $D \subseteq \omega_1$ such that $L_{\omega_1} \models \varphi(\alpha)$ iff $L_{\omega_1} \models \varphi(\alpha')$ for all α, $\alpha' \in D$ can be forced just in ZFC.

Now suppose $0^\sharp$ exists and let κ be a Silver indiscernible for L. In the universe there is a filter U[10] on $\mathcal{P}^L(\kappa)$ such that U is amenable to L– in the sense that $f^{-1}(U) \in L$ for every constructible $f : \kappa \longrightarrow \mathcal{P}(\kappa)$– and such that $(L_{(\kappa^+)^L}, \in, U)$ satisfies that U is a normal measure on κ. Hence, U consists of L–stationary subsets of κ, and for any n and any constructible regressive function $f : [\kappa]^n \longrightarrow \kappa$, one can find $\tau < (\kappa^+)^L$ such that $U \cap L_\tau$, which is in L, contains a homogeneous set for f (this can be proved by induction on n, using the amenability of U, by essentially the same standard argument for showing that a measurable cardinal is Ramsey). This shows that, in L, κ has actually the property that for every n and every regressive $f : [\kappa]^n \longrightarrow \kappa$ there is a stationary subset of κ homogeneous for f.

Theorem 4.6 *Let $n \geq 2$ be a natural number and suppose $\omega_1 \leq \gamma_0 < \ldots < \gamma_{n-1}$ are ordinals such that $\{X \in [\gamma_{n-1}]^{\aleph_0} : ot(X \cap \gamma_0) \in S$ and $ot(X \cap \gamma_i) \in C$ for all i, $1 \leq i < n\}$ is a stationary subset of $[\gamma_{n-1}]^{\aleph_0}$ for every stationary $S \subseteq \omega_1$ and every club $C \subseteq \omega_1$. Then, for every real a there is a club $\overline{D} \subseteq \omega_1$ such that for every formula $\varphi(x_0, \ldots x_{n-2})$,*

$$L_{\omega_1}[a] \models \varphi(\alpha_0, \ldots \alpha_{n-2}) \longleftrightarrow \varphi(\alpha'_0, \ldots \alpha'_{n-2})$$

for all $\alpha_0 < \ldots < \alpha_{n-2}$ and $\alpha'_0 < \ldots < \alpha'_{n-2}$ in $\overline{D}$.

Proof: For $n = 2$ this follows from Lemma 4.5, so let us assume $n \geq 3$. As in the proof of that lemma, let C be the club of all $\nu < \kappa := \omega_1$ such that $L_\nu[a] \preccurlyeq L_\kappa[a]$. We know that there is a club $D \subseteq \kappa$ such that $L_\kappa[a] \models \varphi(\alpha_0, \ldots \alpha_{n-3}, \alpha_{n-2}) \longleftrightarrow \varphi(\alpha_0, \ldots \alpha_{n-3}, \alpha_{n-1})$ for each formula $\varphi(x_0, \ldots x_{n-2})$ and for all $\alpha_0 < \ldots < \alpha_{n-1}$ in D.

Claim 4.7 *For every j, $1 \leq j < n-1$, there is a club $D^j \subseteq D$ such that*

$$L_\kappa[a] \models \varphi(\alpha_0, \ldots \alpha_{n-3-j}, \alpha'_0, \ldots \alpha'_j) \longleftrightarrow \varphi(\alpha_0, \ldots \alpha_{n-3-j}, \alpha''_0, \ldots \alpha''_j)$$

for every formula $\varphi(x_0, \ldots x_{n-2})$ and for all $\alpha_0 < \ldots < \alpha_{n-3-j}$, $\alpha'_0 < \ldots < \alpha'_j$ and $\alpha''_0 < \ldots < \alpha''_j$ in D^j such that $\alpha_{n-3} < \alpha'_0$, α''_0.

Proof: By induction on j. For $j = 1$, (fix $\alpha_0 < \ldots < \alpha_{n-4}$ in D and) suppose there are stationary S_0, $S_1 \subseteq D$ such that for all $\alpha'_0 \in S_0$, $\alpha''_0 \in S_1$ and all α'_1 and α''_1 in D such that $\alpha'_0 < \alpha'_1$ and $\alpha''_0 < \alpha''_1$, $L_\kappa[a] \models \varphi(\alpha_0, \ldots \alpha_{n-4}, \alpha'_0, \alpha'_1)$ and $L_\kappa[a] \models \neg\varphi(\alpha_0, \ldots \alpha_{n-4}, \alpha''_0, \alpha''_1)$ for some formula $\varphi(x_0, \ldots x_{n-2})$. Without loss of generality, say that $L_{\gamma_{n-1}}[a] \models \varphi(\alpha_0, \ldots \alpha_{n-4}, \gamma_0, \gamma_1)$. Then, by our hypothesis there is some $X \preccurlyeq L_{\gamma_{n-1}}[a]$ (containing $\alpha_0, \ldots \alpha_{n-4}$) and such that $\alpha''_0 := ot(X \cap \gamma_0) \in S_1$, $\alpha''_1 := ot(X \cap \gamma_1) \in D$ and $\beta := ot(X) \in C$. But then, $L_\kappa[a] \models \varphi(\alpha_0, \ldots \alpha_{n-4}, \alpha''_0, \alpha''_1)$, which contradicts our choice of S_1. Hence, there is a club $D^1_{\alpha_0, \ldots \alpha_{n-4}} \subseteq D$ such that $L_\kappa[a] \models \varphi(\alpha_0, \ldots \alpha_{n-4}, \alpha'_0, \alpha'_1) \longleftrightarrow \varphi(\alpha_0, \ldots \alpha_{n-4}, \alpha''_0, \alpha''_1)$ for every formula $\varphi(x_0, \ldots x_{n-2})$ and all $\alpha'_0 < \alpha'_1$ and

[10] Given an elementary embedding $j : L \longrightarrow L$ with critical point κ, a constructible $X \subseteq \kappa$ is in U iff $\kappa \in j(X)$.

$\alpha''_0 < \alpha''_1$ in $D^1_{\alpha_0,\ldots\alpha_{n-4}}$. For every $\beta < \omega_1$ let $D^{1,\beta} = \bigcap\{D^1_{\alpha_0,\ldots\alpha_{n-4}} : \alpha_0 < \ldots \alpha_{n-4} < \beta\}$ and let $D^1 = \Delta_{\beta<\omega_1} D^{1,\beta}$.

Now suppose that $1 < j < n-1$ and that there is a club D^{j-1} satisfying the claim for $j-1$. Fix $\alpha_0 < \ldots < \alpha_{n-3-j}$ in D^{j-1} and suppose there are stationary S_0, $S_1 \subseteq \omega_1$ such that for all $\alpha'_0 \in S_0$ and $\alpha''_0 \in S_1$ and all $\alpha'_1 < \ldots < \alpha'_j$ and $\alpha''_1 < \ldots < \alpha''_j$ in D^{j-1} such that $\alpha'_0 < \alpha'_1$ and $\alpha''_0 < \alpha''_1$, $L_\kappa[a] \models \varphi(\alpha_0, \ldots \alpha_{n-3-j}, \alpha'_0, \alpha'_1, \ldots \alpha'_j)$ and $L_\kappa \models \neg\varphi(\alpha_0, \ldots \alpha_{n-3-j}, \alpha''_0, \alpha''_1, \ldots \alpha''_j)$. Without loss of generality, say that $L_{\gamma_{n-1}}[a] \models \varphi(\alpha_0, \ldots \alpha_{n-3-j}, \gamma_0, \ldots \gamma_j)$. As in the $j = 1$ case, applying our hypothesis we find $\alpha''_0 < \alpha''_1 < \ldots < \alpha''_j$ such that $\alpha''_0 \in S_1$ and $\alpha''_1, \ldots \alpha''_j \in D^{j-1}$ and $L_\kappa[a] \models \varphi(\alpha_0, \ldots \alpha_{n-3-j}, \alpha''_0, \alpha''_1, \ldots \alpha''_j)$. This contradicts our choice of S_1. Hence, there is a club $D^j_{\alpha_0,\ldots\alpha_{n-3-j}} \subseteq D$ such that for all $\alpha'_0 < \ldots < \alpha'_j$ and $\alpha''_0 < \ldots < \alpha''_j$ in $D^j_{\alpha_0,\ldots,\alpha_{n-3-j}}$, $L_\kappa[a] \models \varphi(\alpha_0, \alpha_{n-3-j}, \alpha'_0, \ldots \alpha'_j)$ iff $L_\kappa[a] \models \varphi(\alpha_0, \ldots \alpha_{n-3-j}, \alpha''_0, \ldots \alpha''_j)$. Arguing as in the $j = 1$ case, we end up with a club $D^j \subseteq D$ that satisfies the Claim for j. □

Finally, $\overline{D} := D^{n-2}$ is as desired. □

Corollary 4.8 *Suppose that for every natural number $n \geq 2$ there are uncountable ordinals $\gamma_0 < \ldots < \gamma_{n-1}$ such that $\{X \in [\gamma_{n-1}]^{\aleph_0} : ot(X \cap \gamma_0) \in S$ and $ot(X \cap \gamma_i) \in C$ for all $i < n\}$ is a stationary subset of $[\gamma_{n-1}]^{\aleph_0}$ for every stationary $S \subseteq \omega_1$ and every club $C \subseteq \omega_1$. Then, $a^\sharp$ exists for every real a. Furthermore, for every real a, a sentence $\varphi(c_0, \ldots c_{n-1})$ belongs to $a^\sharp$ if and only if $L_{\gamma_n}[a] \models \varphi(\gamma_0, \ldots \gamma_{n-1})$, where $(\gamma_i)_{i\leq n}$ is any strictly increasing sequence of uncountable ordinals witnessing the hypothesis for $n + 1$.*

5 Strong forms of BMM and the Club Bounding Principle

Definition 5.1 *Given a stationary set $S \subseteq \omega_1$, the weak Chang's Conjecture at S ($wCC(S)$) is the statement that for every function $f : \omega_1 \longrightarrow \omega_1$ there is some canonical function g and some stationary $T \subseteq S$ such that g dominates f on T.*

Hence, the usual weak Chang's Conjecture (see [Do-Ko]) is $wCC(\omega_1)$. Note that $\Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} j(\check{\omega}_1) = \check{\omega}_2$ holds if and only if $wCC(S)$ holds for all stationary $S \subseteq \omega_1$. The following observation shows that this statement– which is equiconsistent with just the existence of an almost $<\omega_1$–Erdős cardinal– and the much stronger CBP are equivalent modulo BMM.

Lemma 5.1 *Under BMM, CBP holds if and only if $wCC(S)$ holds for every stationary $S \subseteq \omega_1$.*

Proof: One only has to prove the right to left implication, and for this is it enough to prove that given any function $f : \omega_1 \longrightarrow \omega_1$,

$$\{X \in [\omega_2]^{\aleph_0} \,:\, ot(X) > f(X \cap \omega_1)\}$$

is a projective stationary subset of $[\omega_2]^{\aleph_0}$ if $wCC(S)$ holds for all stationary $S \subseteq \omega_1$.

To see this, fix any stationary $S \subseteq \omega_1$ and any club $E \subseteq [\omega_2]^{\aleph_0}$. By $wCC(S)$, there is some stationary $T \subseteq S$, some $\alpha < \omega_2$ and some surjection $\pi : \omega_1 \longrightarrow \alpha$ such that $f(\nu) < ot(\pi``\nu)$ for every $\nu \in T$. Now if $N$ is a countable elementary substructure of $H_{\omega_2}$ containing $\pi$ and such that $\delta := N \cap \omega_1 \in T$ and $N \cap \omega_2 \in E$, $ot(N \cap \omega_2) > ot(N \cap \alpha) = ot(\pi``\delta) > f(\delta)$. $\square$

The following Lemma, which can be easily proved, shows that a principle of generic absoluteness involving Σ_3 sentences for the structure $(H_{\omega_2}, \in)$ implies the corresponding generic absoluteness for Σ_2 sentences for the more expressive structure $(H_{\omega_2}, \in, NS_{\omega_1})$.

Lemma 5.2 *Suppose that for every $a \in H_{\omega_2}$ and every Σ_3 formula $\varphi(x)$, if there is some poset $\mathbb{P}$ preserving stationary subsets of ω_1 such that*

(a) $\Vdash_{\mathbb{P}} H_{\omega_2} \models \varphi(\check{a})$, and

(b) for every $\mathbb{P}$–name $\dot{\mathbb{Q}}$ for a poset preserving stationary subsets of ω_1, $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} H_{\omega_2} \models \varphi(\check{a})$,

then $H_{\omega_2} \models \varphi(a)$.

Then, for every $a \in H_{\omega_2}$ and every Σ_2 formula $\varphi(x)$ for the structure $(H_{\omega_2}, \in, NS_{\omega_1})$, if there is some poset $\mathbb{P}$ preserving stationary subsets of ω_1 such that

(1) $\Vdash_{\mathbb{P}} (H_{\omega_2}, \in, NS_{\omega_1}) \models \varphi(\check{a})$, and

(2) for every $\mathbb{P}$–name $\dot{\mathbb{Q}}$ for a poset preserving stationary subsets of ω_1,

$$\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} (H_{\omega_2}, \in, NS_{\omega_1}) \models \varphi(\check{a}),$$

then $(H_{\omega_2}, \in, NS_{\omega_1}) \models \varphi(a)$.

All forms of generic absoluteness of this kind, meaning that they involve some class of sentences for H_{ω_2} (or $(H_{\omega_2}, \in, NS_{\omega_1})$) with parameters whose truth– once they are forced by some poset preserving stationary subsets of ω_1– is persistent under subsequent forcing extensions preserving stationary subsets of ω_1, are consequences of the following statement:

Suppose $a \in H_{\omega_2}$ and $\varphi(x)$ is a Σ_2 formula. If there is some poset $\mathbb{P}$ preserving stationary subsets of ω_1 such that

(1) $\Vdash_{\mathbb{P}} \varphi(\check{a})$, and

(2) for every $\mathbb{P}$–name $\dot{\mathbb{Q}}$ for a poset preserving stationary subsets of ω_1,

$$\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \varphi(\check{a}),$$

then $\models_2 \varphi(a)$.

This statement– and much more– can be forced, very much like in the standard construction of a model of Martin's Maximum ([Fo-M-S]), by a semiproper poset $\mathbb{P} \subseteq V_\kappa$ whenever κ is a supercompact cardinal such that V_κ is correct about Σ_4 statements with parameters ([As1]).[11] To see that this statement suffices to imply the above forms of generic absoluteness, notice that, given any formula $\varphi(x)$ for $(H_{\omega_2}, \in)$ (or for $(H_{\omega_2}, \in, NS_{\omega_1})$), '$H_{\omega_2} \models \varphi(a)$' ('$(H_{\omega_2}, \in, NS_{\omega_1}) \models \varphi(a)$') can be expressed by $(\exists H)(H = H_{\omega_2} \wedge H \models \varphi(a))$ (by $(\exists H, X)(H = H_{\omega_2} \wedge X = NS_{\omega_1} \wedge (H, \in, X) \models \varphi(a)))$, and these are Σ_2 statements about a.

Theorem 5.3 *Suppose that $(H_{\omega_2}, \in, NS_{\omega_1}) \models \varphi(a)$ holds for every $a \in H_{\omega_2}$ and every Σ_2 formula $\varphi(x)$ for the structure $(H_{\omega_2}, \in, NS_{\omega_1})$ with the property that there is some poset $\mathbb{P}$ preserving stationary subsets of ω_1 such that*

(1) $\Vdash_{\mathbb{P}} (H_{\omega_2}, \in, NS_{\omega_1}) \models \varphi(\check{a})$, and

(2) for every $\mathbb{P}$–name $\dot{\mathbb{Q}}$ for a poset preserving stationary subsets of ω_1,

$$\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} (H_{\omega_2}, \in, NS_{\omega_1}) \models \varphi(\check{a}).$$

Then CBP holds.

Proof: It is enough to prove that if S is any stationary subset of ω_1 and the generic absoluteness in the statement holds, then $wCC(S)$ also holds. Assume on the contrary that there is some $f : \omega_1 \longrightarrow \omega_1$ which dominates every canonical function on some club intersected with S. For every $\nu < \omega_1$ let $e_\nu : \omega \longrightarrow f(\nu)$ be a surjection and let h_n be given, for every $n < \omega$, by $h_n(\nu) = e_\nu(n)$. Then there must be some n such that

$$A = \{\alpha < \omega_2 \,:\, T \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} \alpha = [h_n]_{\dot{G}} \text{ for some stationary } T \subseteq S\}$$

is a stationary subset of ω_2. Pick, for each $\alpha \in A$, some stationary $S_\alpha \subseteq S$ such that $S_\alpha \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} \alpha = [h_n]_{\dot{G}}$. Clearly, the S_α's have pairwise nonstationary intersection. Extend $\mathcal{B} := \{S_\alpha \,:\, \alpha \in A\}$ to a maximal antichain $\mathcal{A}$ of $\mathcal{P}(\omega_1) \backslash NS_{\omega_1}$ below S.

[11] The corresponding statement involving semiproper forcing can be forced from just the existence of a regular κ such that $V_\kappa \preccurlyeq_{\Sigma_4} V$, a large cardinal notion stronger than being reflecting but weaker than being Mahlo (see also [As1]). It is not hard to see that these considerations, together with the fact that $wCC(\omega_1)$ has relatively large consistency strength, show the consistency (with $BSPFA$) of the existence of a function $f : \omega_1 \longrightarrow \omega_1$ such that $\{X \in [\gamma]^{\aleph_0} \,:\, f(X \cap \omega_1) < ot(X)\}$ is nonstationary for every ordinal γ.

Claim 5.4 *Given any $T \in \mathcal{A}\backslash\mathcal{B}$ and any poset $\mathbb{P}$ preserving stationary subsets of ω_1, $\mathbb{P}$ forces that every canonical function is different from h_n on some club intersected with T.*

Proof: Otherwise, applying absoluteness for Σ_1 formulas for the structure $(H_{\omega_2}, \in, NS_{\omega_1})$ to the poset $\mathbb{P}$, there would be some counterexample in the universe, which is a contradiction since $T \cap S_\alpha$ is nonstationary for every $\alpha \in A$. $\square$

Now we apply the stipulated absoluteness to the poset $\mathbb{P}$ for sealing $\mathcal{A}$. More precisely, let $(T_i : i < \omega_2)$ be an enumeration of $\mathcal{A}$ (note that $2^{\aleph_1} = \aleph_2$, so that there is such enumeration in length ω_2). Then, $\mathbb{P}$ is $Coll(\omega_1, \omega_2) * \dot{\mathbb{Q}}$, where, letting $\dot{F}$ be a $Coll(\omega_1, \omega_2)$–name for the generic surjection from ω_1 onto ω_2^V, $\dot{\mathbb{Q}}$ is, in $V^{\mathbb{P}}$, the poset for shooting a club with countable conditions through $\{\nu < \omega_1 : \nu \in \bigcup_{\xi<\nu} T_{\dot{F}(\xi)}\} \cup \{S\}$.

It is a standard fact (see [Fo-M-S]) that, since $\mathcal{A}$ is a maximal antichain of $\mathcal{P}(\omega_1)\backslash NS_{\omega_1}$, $\mathbb{P}$ preserves stationary subsets of ω_1.

$\mathbb{P}$ forces the following statement:

$(*)$: There is an ordinal $\delta < \omega_2$, a sequence $(T'_i : i < \delta)$ of subsets of S, a surjection $F : \omega_1 \longrightarrow \delta$ and a club $C \subseteq \omega_1$ such that for every $\nu \in C \cap S$ there is some $\xi < \nu$ such that $\nu \in T'_{F(\xi)}$ and, furthermore, given any $i < \delta$, either

(1) there is some $\alpha < \delta$ and some canonical function g for α such that $h_n(\nu) = g(\nu)$ for every $\nu \in T'_i$, or else

(2) h_n dominates every canonical function on some club intersected with T'_i.

It is easily seen that $(*)$ can be expressed by means of a Σ_2 sentence for $(H_{\omega_2}, \in, NS_{\omega_1})$ with h_n and S as parameters which, by Claim 5.4, is persistent under forcing extensions of $V^{\mathbb{P}}$ preserving stationary subsets of ω_1. Hence, $(*)$ holds in the ground model. Let δ, $(T'_i : i < \delta)$, F and C be as given by $(*)$ and pick any $\alpha \in A$ above δ. Then there is some ξ such that $S_\alpha \cap T'_{F(\xi)}$ is stationary. But this is a contradiction, since then both (1) and (2) must fail for $F(\xi)$. $\square$

Theorem 5.3 is, as far as I know, the first result assigning consistency strength beyond that of a reflecting cardinal to some principle stated entirely in terms of generic absoluteness involving only parameters from H_{ω_2}.

Although I do not know of any argument for showing large consistency strength just from BMM alone, there is an optimal model separating $BSPFA$ and BMM.

Theorem 5.5 *In L, there is no semiproper poset forcing BMM.*

Proof: Suppose that $\mathbb{P}$ is semiproper in L, G is $\mathbb{P}$–generic over L and $L[G] \models BMM$. The idea of the argument is to force with Namba forcing, so that ω_2 becomes of countable cofinality. In search of a contradiction, we want to express in a Σ_1 way the fact that there is an L–regular cardinal which has countable cofinality. For this it is enough to say that there are some $\delta < \gamma$ such that δ has countable cofinality and such that δ is regular in L_γ, provided we can guarantee in addition that γ is an L–cardinal. This can be achieved using a trick, which I know from Todorčević, to convert the L–regularity of an ordinal η into a Σ_1–statement with η as parameter. That $0^\sharp$ does not exist makes sure that the poset involved in this trick is proper, and actually of the form σ–closed $*$ ccc.[12]

Let Nm denote Namba forcing and work in V_0, where V_0 is $L[G]^{Nm}$. Since $0^\sharp$ does not exist, $\kappa = \aleph_\omega$ is such that $\kappa^+ = (\kappa^+)^L$. In particular, all ordinals between κ and κ^+ are singular in L. Let $\mathbb{Q}$ be the σ–closed collapse of κ^+ to ω_1 with countable conditions. In $V_0^{\mathbb{Q}}$, let D be a club of $(\kappa^+)^{V_0}$ of order type ω_1 consisting of L–singular ordinals. Let $\overline{C} = (C_\alpha\,:\,\alpha$ is a singular ordinal in $L)$ be the canonical constructible $\square$–sequence. In particular, for every L–singular ordinal $\alpha < \kappa^+$,

(a) C_α is a club of α of order type at most κ, and

(b) if β is a limit point of C_α, then β is L–singular and $C_\beta = C_\alpha \cap \beta$.

It follows that, letting for all α, $\beta \in D$, $\beta \prec \alpha$ if and only if $\beta < \alpha$ and β is a limit point of C_α, $T = (D, \prec)$ is a tree.

Claim 5.6 *There is no branch through T of length ω_1 in $V_0^{\mathbb{Q}}$.*

Proof: Since there is no club C of κ^+ in V_0 such that $C \cap \alpha = C_\alpha$ for unboundedly many α below κ– for otherwise there would be some α such that $ot(C_\alpha) > \kappa$– and since the union of every ω_1–branch through T is a club of κ^+, it is enough to see that for every $\mathbb{Q}$–generic G_0 over V_0, the union of an ω_1–branch through T in $V_0[G_0]$ would be in V. Let $G_0 \times G_1$ be $\mathbb{Q} \times \mathbb{Q}$–generic over V_0. Now let b_0 and b_1 be ω_1–branches through T in $V_0[G_0]$ and $V_0[G_0][G_1]$, respectively, $b_0 = \dot{b}_0[G_0]$ and $b_1 = \dot{b}_1[G_0 \times G_1]$ for suitable names $\dot{b}_0$ and $\dot{b}_1$. Since $(\kappa^+)^{V_0}$ has cofinality ω_1 in $V_0[G_0][G_1]$, $\bigcup b_0$ and $\bigcup b_1$ meet at arbitrarily high points below $(\kappa^+)^{V_0}$, and therefore $\bigcup b_0 = \bigcup b_1$. But then, $\bigcup b_0$ is the set of $\alpha < (\kappa^+)^V$ such that some condition in $\mathbb{Q}$ forces over V_0 that α is in $\bigcup b_0$, and therefore it is in V_0.

To see this, suppose $\alpha \in \bigcup \dot{b}_0[G_0]$ but some $q \in Q_0$ forces over V_0 that α is not in $\bigcup \dot{b}_0$. Then, the condition $\langle \emptyset, q\rangle$ forces that α is not in $\bigcup \dot{b}_1$. Hence, if G_1 contains q and is $\mathbb{Q}$–generic over $V[G_0]$, then $\alpha \notin \bigcup \dot{b}_1[G_0 \times G_1] = \bigcup \dot{b}_0[G_0]$, which is a contradiction. Similarly, one can prove that if $\alpha \notin \bigcup b_0$, then $\Vdash_{\mathbb{Q}} \alpha \notin \bigcup \dot{b}_0$. $\square$

Now, if $(T', \prec')$ is any tree without ω_1–branches, then the poset $\mathbb{S}_{T'}$ (consisting of finite functions $p : D \longrightarrow \omega$ such that $p(\alpha) \neq p(\beta)$ for all α and β which

[12] The trick consists in using a proper poset for specializing the tree associated to a certain $\square_\kappa$–sequence. The existence of a proper poset specializing the tree associated to any given $\square_\kappa$–sequence was first proved by Todorčević ([T1]). That this can be done with a σ–closed $*$ ccc poset– and this is the argument presented here– was subsequently proved by Magidor.

are $\prec'$–comparable) for specializing T' has the ccc (see the proof of [J], Lemma 24.2). Hence, $\mathbb{S}_T$ has the ccc, and $\mathbb{Q} * \mathbb{S}_{\dot{T}}$ is a σ–closed $*$ ccc poset adding a club D of κ^+ of order type ω_1 and a function $f : D \longrightarrow \omega$ such that for all $\alpha < \beta$ in D, α and β are singular ordinals in L and, if α is a limit point of C_β, then $f(\alpha) \neq f(\beta)$. But then, in $L[G]$, $Nm * (\dot{\mathbb{Q}} * \mathbb{S}_{\dot{T}})$ preserves stationary subsets of ω_1 and forces that there are

(1) $\delta < \gamma < \omega_2$ such that δ is a regular cardinal in L_γ,

(2) a sequence of length ω cofinal in δ, and

(3) an ω_1–club D of γ and a function $f : D \longrightarrow \omega$ such that for all $\alpha < \beta$ in D, α and β are L–singular and, if α is a limit point of C_β, then $f(\alpha) \neq f(\beta)$.

Since C_β is Δ_1 definable with β as parameter for every L–singular ordinal β, the above statement can be expressed in a Σ_1 way with ω_1 as a parameter. Hence, by BMM in $L[G]$ it is true there and we may fix δ, γ and f witnessing it. But then, γ is a regular cardinal in L, since otherwise f restricted to the set of limit points of $C_\gamma \cap D$ would be a one–to–one function mapping an uncountable set into ω (this is because, by the coherence property of $\overline{C}$, α is a limit point of C_β for all $\alpha < \beta$ which are limit points of C_γ). It follows that δ is actually regular in L. Let $p \in G$ be a condition in $\mathbb{P}$ forcing that δ has countable cofinality. But then, in L, $\mathbb{P} \restriction p$ is a semiproper poset forcing that δ has countable cofinality. Hence, in L, $CC^*_{\omega_1}(\delta)$ holds, and so in particular $0^\sharp$ exists, which of course is a contradiction. $\square$

The following corollary now follows from Theorem 5.5 and the results of [Go-S] mentioned in the introduction.

Corollary 5.7 *If ZFC + $BSPFA$ is consistent, then so is ZFC + $BSPFA$ + $\neg BMM$.*

As I mentioned in the introduction, a model separating $BSPFA$ and BMM had been previously presented in [As-W]. However, the starting assumption used there, namely that of the existence of a cardinal κ with a certain weak Erdős property slightly stronger than $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$, is far from optimal.

An application of the proof of Theorem 5.5 is that the quotable bounded forcing axiom for the class of σ–closed $*$ ccc is as strong, in terms of consistency strength, as $BSPFA$.

Fact 5.8 *Let Γ be the class of σ–closed $*$ ccc posets. Then $BFA(\Gamma)$ implies that ω_2 is reflecting in $L[a]$ for every real a.*

Proof: This is just the conjunction of Todorčević's alternative proof for showing ω_2 reflecting in L from $BPFA$ and the observation of Magidor mentioned in the proof of Theorem 5.5: Let a be a real. Pick $b \in L_{\omega_2}[a]$ and suppose $L \models \varphi(b)$, where $\varphi(x)$ is a Σ_2 formula. Let κ be a singular cardinal such that L_κ contains a witness for $\varphi(b)$. We can assume that $a^\sharp$ does not exist, since it is easy to see that every Silver indiscernible for $L[a]$ is reflecting in $L[a]$. Hence, the proof of Theorem 5.5 shows that an application of $BFA(\Gamma)$ implies the existence of an $L[a]$–regular $\gamma < \omega_2$ such that $L_\gamma[a] \models \varphi(b)$. Since $\varphi(x)$ is Σ_2 and γ is regular in $L[a]$, $L_{\omega_2}[a] \models \varphi(b)$. $\Box$

Notice that, by the proof of Theorem 5.3, $\neg CBP$ implies that there is an $h : \omega_1 \longrightarrow \omega_1$ and a stationary $A \subseteq \omega_2$ such that, given any $\alpha \in A$, $S \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} \alpha = [h]_{\dot{G}}$ for some stationary $S \subseteq \omega_1$. Actually, the following equivalence is easily proved.

Fact 5.9 *CBP fails if and only if there are $h_n : \omega_1 \longrightarrow \omega_1$ $(n < \omega)$ with the property that for every $\alpha < \omega_2$ there is some n and some stationary $S \subseteq \omega_1$ such that $S \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} \alpha = [h_n]_{\dot{G}}$.*

The above characterization of $\neg CBP$ motivates considering the following notion.

Definition 5.2 *Let $h : \omega_1 \longrightarrow \omega_1$. h guesses all canonical functions if and only if for every $\alpha < \omega_2$ there is some stationary $S \subseteq \omega_1$ such that $S \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}} \alpha = [h]_{\dot{G}}$.*

So, the existence of a function guessing all canonical functions is a strong form of the failure of CBP.

Fact 5.10 *$\diamondsuit_{\omega_1}$ implies that there is a function guessing all canonical functions.*

Proof: Let $(X_\nu : \nu < \omega_1)$ be a $\diamondsuit_{\omega_1}$–sequence. Given any $\nu < \omega_1$, suppose X_ν codes, in some fixed way, a directed system $\mathcal{D} = (Y_\xi, j_{\xi,\xi'} : \xi \leq \xi' < \nu)$ such that each Y_ξ is a subset of ν and $(j_{\xi,\xi'} : \xi \leq \xi' < \nu)$ is a commuting family of strictly increasing functions, where $j_{\xi,\xi'} : Y_\xi \longrightarrow Y_{\xi'}$ and $j_{\xi,\xi} = id_{Y_\xi}$ for all $\xi \leq \xi'$. If the direct limit Y of $\mathcal{D}$ is well–founded, then let $h(\nu)$ be the order type of Y.

To see that this function h guesses all canonical functions, pick $\alpha < \omega_2$, let $\pi : \omega_1 \longrightarrow \alpha$ be a surjection and let $\mathcal{D} = (Y_\xi, j_{\xi,\xi'} : \xi \leq \xi' < \omega_1)$ be a directed system such that for all $\xi < \xi' < \omega_1$, the diagram

$$\begin{array}{ccc} \pi``\xi & \xrightarrow{id} & \pi``\xi' \\ \big\downarrow & & \big\downarrow \\ Y_\xi & \xrightarrow{j_{\xi,\xi'}} & Y_{\xi'} \end{array}$$

commutes (where the downward arrows represent isomorphisms) and let A be a subset of ω_1 coding $\mathcal{D}$ in such a way that there is a club $C \subseteq \omega_1$ such that $A \cap \nu$ codes $(Y_\xi, j_{\xi,\xi'} : \xi \leq \xi' < \nu)$ for all ν in C. $S = \{\nu \in C : A \cap \nu = X_\nu\}$ is stationary and, for all $\nu \in C$, $h(\nu)$ is the order type of the direct limit of $(Y_\xi, j_{\xi,\xi'} : \xi \leq \xi' < \nu)$, which by construction is easily seen to be equal to the order type of $\pi``\nu$. Hence, modulo S, h equals a canonical function for α. $\Box$

Fact 5.11 *BPSR implies that there is no function guessing all canonical functions.*

Proof: Given a function $h : \omega_1 \longrightarrow \omega_1$, it is enough to see that

$$\{X \in [\omega_2]^{\aleph_0} \,:\, ot(X) \neq h(X \cap \omega_1)\}$$

is a projective stationary subset of $[\omega_2]^{\aleph_0}$. So fix a stationary $S \subseteq \omega_1$ and a function $F : [\omega_2]^{<\omega} \longrightarrow \omega_2$. Pick α, $\omega_1 < \alpha < \omega_2$ such that α is closed under F. Let E be a club of $[\alpha]^{\aleph_0}$ such that for every $X \in E$ there is some Y closed under F such that $\alpha \in Y$ and $Y \cap \alpha = X$. Pick any X in E such that $X \cap \omega_1 \in S$ and X is closed under F. By the choice of E there is some $Y \in [\omega_2]^{\aleph_0}$ such that Y is closed under F, $X \subseteq Y$, $Y \cap \omega_1 = X \cap \omega_1 \in S$, and $ot(X) < ot(Y)$. But then, either $ot(X) \neq h(X \cap \omega_1)$ or $ot(Y) \neq h(Y \cap \omega_1)$. □

Question 5.1 *Does the existence of a function guessing all canonical functions follow from $\neg CBP$?* [13]

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[13] Added in proof: It does not (by results in [As3]).

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